International Journal of Production Management and Engineering

https://doi.org/10.4995/ijpme.2022.16736
Received: 2021-11-25 Accepted: 2022-01-24

Comparative Study of Whale Optimization Algorithm and Flower Pollination Algorithm to Solve Workers Assignment Problem

Huthaifa Al-Khazraji

[a] Control and System Engineering Department, University of Technology-Iraq, Baghdad, Iraq.
60141@uotechnology.edu.iq

Abstract:

Many important problems in engineering management can be formulated as Resource Assignment Problem (RAP). The Workers Assignment Problem (WAP) is considered as a sub-class of RAP which aims to find an optimal assignment of workers to a number of tasks in order to optimize certain objectives. WAP is an NP-hard combinatorial optimization problem. Due to its importance, several algorithms have been developed to solve it. In this paper, it is considered that a manager is required to provide a training course to his workers in order to improve their level of skill or experience to have a sustainable competitive advantage in the industry. The training cost of each worker to perform a particular job is different. The WAP is to find the best assignment of workers to training courses such that the total training cost is minimized. Two metaheuristic optimizations named Whale Optimization Algorithm (WOA) and Flower Pollination Algorithm (FPA) are utilized to final the optimal solution that reduces the total cost. MATLAB Software is used to perform the simulation of the two proposed methods into WAP. The computational results for a set of randomly generated problems of various sizes show that the FPA is able to find good quality solutions.

Key words:

Servitization, Resource Assignment Problem, Workers Assignment Problem, Metaheuristic Optimization, Whale Optimization Algorithm, Flower Pollination Algorithm.

1. Introduction

With the increase in competition in the global market, industrial companies are forced to improve their manufacturing processes via cutting costs and increasing process efficiency (Ostadi et al., 2021). In this direction, it's become necessary for decision-makers to find the best strategies that utilize their resources in order to have the best performance (Lin and Chiu, 2018). The problem of finding the best utilization of resources in the industrial companies is named a Resource Assignment Problem (RAP). Among many varieties of resources, human resources play a significant role in the success of the industrial organization if they are well allocated to different services or systems, with an aim to maximize or minimize certain objectives related to performance and productivity (Bouajaja and Dridi, 2017). Therefore, Worker Assignment Problem (WAP) is defined as a process of assigning workers among various tasks for maximization of the profit (or efficiency) or minimization of the cost (or time). A WAP is the most widely used in the context of industrial and engineering management such as in production planning and maintenance management (Krokhmal and Pardalos, 2009).

For some industrial processes, the load needs to be distributed among workers in such a way that the time is minimized or/and the efficiency is increased. Moreover, it can be noticed that there are particular workers who can perform some of the jobs with less

To cite this article: Al-Khazraji, H. (2022). Comparative Study of Whale Optimization Algorithm and Flower Pollination Algorithm to Solve Workers Assignment Problem. *International Journal of Production Management and Engineering, 10*(1), 91-98. https://doi.org/10.4995/ijpme.2022.16736

time or more efficient way than others due to their experience or skills (Mahmoud, 2009). Caron et al. (1999) considered the case of WAP with additional constrain that an unassigned worker cannot be given a certain job unless the unassigned worker has the qualification to perform that particular job. However, if the manager focuses only on the workers which have skills or experiences to perform the tasks without having the right blanching of distrusting the loads among all workers, a worker fatigue problem might be happed in the long term (Yadav et al., 2020; Demiral, 2017). Besides, intangible resources represented by the skill and experience of the workers are valuable and scarce. Moreover, the experiences that the workers gained in the past are not enough to have a sustainable competitive advantage in the present (Ruiz et al., 2020). Therefore, a manager has to provide a training course continuously to his workers in order to improve their level of skill or experience to have a sustainable competitive advantage in the industry.

The problem of WAP that is considered in this paper can be stated as follows: if the manager needs to assign n of jobs to n of workers where the training cost of each worker to perform that particular job is different. The problem is to find the best assignment of workers to training courses such that the total training cost is minimized. The problem is NP-hard problems with enormous search spaces (Ammar et al., 2013).

Pentico (2007) reviewed the mathematical model for most of the variations of the Assignment Problem (AP) which is the general form of WAP. Bouajaja and Dridi (2017) presented a comprehensive review study on the numerous approaches that applied to solve varieties classes of WAP in different application areas. The General AP (GAP) can be formulated as integer linear programming. Different methods such as exact, heuristic and metaheuristic are developed to solve the problem. Kuhn (1955) developed the well-known Hungarian method to solve the general AP. In Ross and Soland (1975), a Branch and Bound (B&B) technique is proposed to solve the general AP. Xuezhi and Xuehua (1996) described how AP can be solved using dynamic programming. Exact methods such as the Hungarian method, B&B technique and dynamic programming are only effective in certain problems with a small size of decision variables. Therefore, larger-sized problems are often solved by using heuristic and metaheuristic to obtain high-quality solutions with reasonable computational time (Bouajaja and Dridi, 2017).

In terms of heuristic methods, Cattrysse et al. (1994) proposed a column generation heuristic method where the problem was formulated as a set partitioning problem. On other hand, among many metaheuristic methods, Ant Colony Optimization (ACO) was the most approach that is utilized to solve the problem. For example, Wang et al. (2009) presented a detailed procedure to apply ACO to WAP where the objective was to maximize efficiency. Demiral (2017) implemented ACO for a set of randomly generated WAP. Three objectives of WAP (minimization cost, maximization sales and maximization profit) were investigated in the study. Statistical analysis based on mean, standard deviation and variance was performed to help decision-makers to select the best objective based on their perspective. In the same direction, Suliman (2019) examined the performance of ACO in comparison with the traditional Hungarian method for solving the WAP with the size of 3×3 in terms of running time, number of iteration and quality of solutions. Besides ACO, Chu and Beasley (1997) presented a genetic algorithm (GA) for solving the AP. Jia and Gong (2008) solved the multi-objective WAP using Multi-Objective Particle Swarm Optimization (MOPSO). In this paper, two metaheuristic optimizations named Whale Optimization Algorithm (WOA) and Flower Pollination Algorithm (FPA) are utilized final the optimal solution that reduces the total cost. MATLAB Software is used to perform the simulation of the two proposed methods into WAP.

2. Training Course

Training can be defined as a process of developing programs to ensure that employees are provided the right skills that are needed to achieve better positive in the market (Halawi and Haydar, 2018). The importance of providing training courses to the workforce of the industrial companies appears to be a recognition strategy toward gaining a competitive advantage in the global market competition (Sharma, 2014). Workforces are required continuous training courses to have sustainability professional qualifications to cope with the recent advanced technology within industry 4.0. According to Walsh and Volini (2017), 80% of human resource managers reported that workforce training is one of the biggest problems to improve the effectiveness and competitiveness of industrial companies.

In this direction, industrial companies are compelled to provide workforce training strategies continuously to increase productivity. The present paper consideres that a manager is required to assign n of jobs to n of workers where the training cost of each worker to perform a particular job is different. The problem is to find the best assignment of workers to training courses such that the total training cost is minimized.

3. Mathematical Model

Consider there are (n) of workers (w) are required to be assigned to (n) of training courses (s) as shown in Figure 1. Each worker $w_i(i=1,2,\ldots,n)$ can be assign to any course $s_j(j=1,2,\ldots,n)$ with different cost ($c_{i,j}$). The problem is to find the best assignment of workers to courses such that the total cost of training is minimized. The number of the workers was assumed equal to the number of courses in this study. The WAP is formulated as (Krokhmal and Pardalos, 2009):

$$Min\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}x_{ij} \quad (1)$$

$s.t$

$$\sum_{i=1}^{n}x_{ij}=1 \ (j=1,2,\ldots,n) \quad (2)$$

$$\sum_{j=1}^{n}x_{ij}=1 \ (i=1,2,\ldots,n) \quad (3)$$

where x_{ij} has two values, either 1 if the worker i is assigned to job j, otherwise is zero. The constraint in Equation (2) satisfies that each training course is assign to a worker and the constraint in Equation (3) satisfies that each worker is assign to a training course.

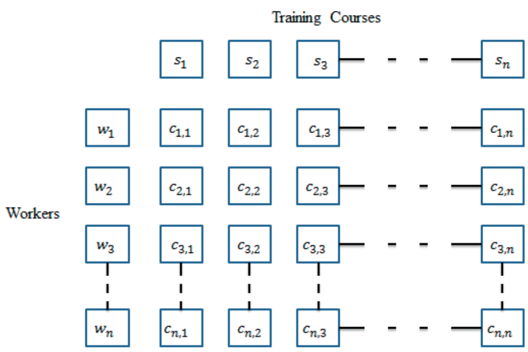

Figure 1. Workers Assignment Problem.

4. Solution Approach

Bio-inspired algorithms are considered powerful in solving NP-hard combinatorial optimization problems (Yang, 2009). Therefore, two algorithms which are inspired from the biological systems in nature are proposed to final the optimal solution that reduces the total cost. These two algorithms are Whale Optimization Algorithm (WOA) and Flower Pollination Algorithm (FPA). The next subsections explain these two algorithms.

4.1. Whale Optimization Algorithm

Whale Optimization Algorithm (WOA) is a population-based swarm optimization algorithm. It was developed by Mirjalili and Lewis in 2016. WOA mimics the bubble-net hunting behavior of humpback whales. The mathematical model of this algorithm consists of two processes named exploitation and exploration.

In the exploitation process, the position of the humpback whale is updated based on the location of the prey using a bubble-net attacking strategy. In this strategy, the movement towards the prey is performed by two mechanisms (Satapathy et al., 2018). The first one is the shrinking encircling mechanism. This behavior is represented by the following equations (Mirjalili and Lewis, 2016):

$$Q=2r_1 \quad (4)$$

$$A=2ar_2-a \quad (5)$$

$$D=Q\cdot p^*(t)-p(t) \quad (6)$$

$$p(t+1)=p^*(t)-A\cdot D \quad (7)$$

where

- a Coefficient value linearly decreased from 2 to 0 for each iteration
- Q Coefficient value calculated as given in Equation (4)
- A Coefficient value calculated as given in Equation (5)
- D Coefficient value calculated as given in Equation (6)
- r_1,r_2 Random value between [0,1]
- t Current iteration

- p^* Position of the prey
- p Position of the whale

The second mechanism is named spiral updating position where in this strategy the whale moves towards the prey in a helix-shaped movement. This behavior is represented by the following equations (Mirjalili and Lewis, 2016):

$$D' = |\, p^*(t) - p(t) \,| \tag{8}$$

$$p(t+1) = D' e^{bl} \cos(2\pi l) + p^*(t) \tag{9}$$

where

- D' Coefficient value calculated as given in Equation (8)
- b Constant used to define the shape of the logarithmic spiral
- l Random value between [0,1]

To model the changes between these two strategies, it was assumed that there is a probability of 50% to select between the shrinking encircling strategy and the spiral one to update the position of the current whale in the simulation of the algorithm. The mathematical model of this scenario can be formulated by selecting a random value (*Rand*), then if the value of the random value >50 the movement of the individual will be performed based on Equation (7), otherwise wiii be performed based on Equation (9) (Mirjalili and Lewis, 2016).

In the exploration process, the position of the humpback whale is updated randomly. The basic idea of this strategy is to ensure the search space explored globally. This strategy is represented by the following equations (Mirjalili and Lewis, 2016):

$$D = |\, Q \cdot p^{rand}(t) - p(t) \,| \tag{10}$$

$$p(t+1) = p^{rand}(p) + A \cdot D \tag{11}$$

where

- p^{rand} Random position chosen from the current population

The pseudo code of FPA is illustrated in Figure 2.

1. Input
- ✓ Objective function (fitness function), population size (N), coefficient value a, number of iteration T

2. Initialization
- ✓ Initialize population
- ✓ Evaluate Objective function
- ✓ Find p*

3. Loop:
- ✓ For $t = 1{:}T$
- ✓ For $i = 1{:}\ N$
- ✓ Update Q as in Eq. (4), A as in Eq. (5) and select random value θ
- ✓ If $\theta > 0.5$
  - ➢ If |A| < 1:Update the position of the current whale based on Eq. (7)
  - ➢ If |A) > 1:Update the position of the current whale based on Eq. (9)
- ✓ Else
  - ➢ Update the position of the current whale based on Eq. (11)
- ✓ Perform greedy selection and update p"
- ✓ If there is no convergence of the current solution & if t>T go to Loop

4. Print the optimal solution

Figure 2. The pseudo code of WOA.

4.2. Flower Pollination Algorithm

Flower Pollination Algorithm (FPA) is a swarm-based meta-heuristic optimization developed by Yang in 2012. The FPA mimics the pollination phenomena in the flower. The main idea of the pollination in flower is to transfer the pollen from the male into the female. This process can be classified based on the way that pollen is transferred into biotic and abiotic. In the biotic, the pollinator can be animal or insect, whereas in the abiotic, the pollinator is the wind and diffusion in water (Abdel-Basset and Shawky, 2019). The optimization procedure of the FPA starts with randomly initialized a population of N of flowers within the search space as given in Equation (12):

$$y_i = y_l + Rand^*(y_u - y_l) \tag{12}$$

where

- i Counter (i=1,2,3,…)
- $Rand$ Initial solution
- y_i Random value between [0,1]
- y_l Lower bound of the search space
- y_u Upper bound of the search space

In the FPA, there are two ways to search of the optimum value. These are the global pollination and local pollination (Yang, 2012). In the global pollination stage, the movement of each individual in the population is directed by the one that has the best cost function found yet. This can be represented mathematically as:

$$y_i^{new} = y_i^{old} + \sigma^*(y_i^{old} - y_g) \quad (13)$$

where

- y_i^{new} New solution
- y_i^{old} Old solution
- y_g The current best solution
- σ The step size

The step size σ can be set fixed or follow a random steps such as Lèvy flight. In terms of local pollination, the algorithm selects two solutions randomly, and then the new solution is generated based on the following (Yang, 2012):

$$y_i^{new} = y_i^{old} + \varepsilon^*(y_j - y_k) \quad (14)$$

where

- y_j A solution chosen randomly
- y_k A solution chosen randomly
- ε Random value between [0,1]

The pseudo code of FPA is illustrated in Figure 3

1. Input
- ✓ Objective function (fitness function), population size (N), switch probability (p), number of iteration (T)

2. Initialization
- ✓ Initialize population N flowers based on Eq. (12)
- ✓ Evaluate objective function and assign yz

3. Loop:
- ✓ For t = 1: T
- ✓ For i = 1: N
- ✓ If rand < p
  - ➢ Generate a step size (σ)
  - ➢ Generate a new solution based on Eq. (13) (global pollination)
- ✓ Else
  - ➢ Choose two solutions randomly among all solutions
  - ➢ Generate a new solution based on Eq. (14) (local pollination)
- ✓ Perform greedy selection and update y_g
- ✓ If there is no convergence of the current solution & if t > T go to Loop

4. Print the optimal solution

Figure 3. The pseudo code of FPA

5. Simulation Study

For evaluating the performance of the two algorithms (WOA and FPA) to solve WAP, a set of different size randomly generated problems have been used. Three sizes (*n*=5,10,15) of WAP are considered to perform the evaluation as given in the Appendix I. MATLAB Software is used to perform the simulation. MATLAB becomes a powerful tool in wide applications in engineering, economics and management. It can handle different computational algorithms with a reasonable time. On other words, different algorithms could be tested and evaluated with less time. As a consequence of using simulation, more knowledge and insight can be gained to enhance the solution of WAP. All simulations were conducted on a computer with Intel(R) Core(TM) CPU i7-4500 and 8 GB RAM. The WOA and FPA parameters are presented in Table 1. For justify the comparison between the WOA and FPA, the size of the population and the number of iteration are set equally. The coefficient value (*a*) is set to 2 as recommended by the (Mirjalili and Lewis, 2016). In the same way, the value of the switch probability (*p*) is set to 0.5 as recommended by the Yang (2012).

Both algorithms were run 10 times and the statistical data such as the average (Avg.), the maximum (Max), the minimum (Min) and the standard deviation (Std.) were recorded for each algorithm. Table 2 presents the statistical data of the experiments for the three problem size of WAP using WOA and FPA.

Table 1. WOA and FPA algorithms parameters.

Parameters	WOA	FPA
Number of population (*N*)	50	50
Number of iteration (*T*)	100	100
Coefficient value (*a*)	-	2
Switch probability (*p*)	0.5	-

Table 2. Statistics for solving WAP using WOA and FPA

Size	Method	Avg.	Min	Max	Std.
5	WOA	30	30	30	0
	FPA	30	30	30	0
10	WOA	64.6	63	67	1.175
	FPA	62.2	61	63	0.78
15	WOA	104.1	103	105	0.738
	FPA	101.8	101	102	0.422

It can be noticed form Table 2 that in general for small size problem, both algorithms are obtained a good solution results. However, Table 2 show that if the size of the problem increased (i.e. n=10 and n=15), the FPA is recommended to solve the WAP. FPA shows better performance in terms of obtaining a less average value, less minimum value, less maximum value and less standard deviation.

6. Conclusions

The Worker Assignment Problem (WAP) is an important problem faced by manufacturing companies. WAP is an NP-hard combinatorial optimization problem. Human resources play a significant role in the overall performance of manufacturing companies. Workforce training is one of the biggest problems in the industrial companies to improve effectiveness and competitiveness. The paper considers the problem of assigning workers to training courses in order to improve the level of skill or experience of the worker to have a sustainable competitive advantage in the industry. The training cost to perform a particular job of each worker is different. The WAP is to find the best assignment of workers to training courses such that the total training cost is minimized. Two metaheuristic optimizations named Whale Optimization Algorithm (WOA) and Flower Pollination Algorithm (FPA) are utilized to final the optimal solution that reduces the total cost. The simulations results reveal that for a small size problem, both algorithms are obtained a good solution result. However, for a large size problem (i.e. n=10 and n=15), the FPA is recommended to solve the WAP. FPA shows better performance in terms of obtaining a less average value, less minimum value, less maximum value and less standard deviation.

References

Abdel-Basset, M., & Shawky, L. A. (2019). Flower pollination algorithm: a comprehensive review. *Artificial Intelligence Review*, *52*(4), 2533-2557. https://doi.org/10.1007/s10462-018-9624-4

Ammar, A., Pierreval, H., & Elkosentini, S. (2013). Workers assignment problems in manufacturing systems: A literature analysis. In *Proceedings of 2013 international conference on industrial engineering and systems management (IESM)* (pp. 1-7). IEEE.

Bouajaja, S., & Dridi, N. (2017). A survey on human resource allocation problem and its applications. *Operational Research*, *17*(2), 339-369. https://doi.org/10.1007/s12351-016-0247-8

Caron, G., Hansen, P., & Jaumard, B. (1999). The assignment problem with seniority and job priority constraints. *Operations Research*, *47*(3), 449-453. https://doi.org/10.1287/opre.47.3.449

Cattrysse, D. G., Salomon, M., & Van Wassenhove, L. N. (1994). A set partitioning heuristic for the generalized assignment problem. *European Journal of Operational Research*, *72*(1), 167-174. https://doi.org/10.1016/0377-2217(94)90338-7

Chu, P. C., & Beasley, J. E. (1997). A genetic algorithm for the generalised assignment problem. *Computers & Operations Research*, *24*(1), 17-23. https://doi.org/10.1016/S0305-0548(96)00032-9

Demiral, M. F. (2017). Ant Colony Optimization for a Variety of Classic Assignment Problems. In *International Turkish World Engineering and Science Congress, Antalya*.

Halawi, A., & Haydar, N. (2018). Effects of Training on Employee Performance: A Case Study of Bonjus and Khatib & Alami Companies. *International Humanities Studies*, *5*(2).

Jia, Z., & Gong, L. (2008). Multi-criteria human resource allocation for optimization problems using multi-objective particle swarm optimization algorithm. In *2008 International Conference on Computer Science and Software Engineering*, *1*, 1187-1190. IEEE. https://doi.org/10.1109/CSSE.2008.1506

Koleva, N., & Andreev, O. (2018, June). Aspects of Training in the Field of Operations Management with Respect to Industry 4.0. In *2018 International Conference on High Technology for Sustainable Development (HiTech)* (pp. 1-3). IEEE. https://doi.org/10.1109/HiTech.2018.8566581

Krokhmal, P. A., & Pardalos, P. M. (2009). Random assignment problems. *European Journal of Operational Research*, *194*(1), 1-17. https://doi.org/10.1016/j.ejor.2007.11.062

Kuhn, H. W. (1955). The Hungarian method for the assignment problem. *Naval research logistics quarterly*, *2*(1-2), 83-97. https://doi.org/10.1002/nav.3800020109

Lin, J. T., & Chiu, C. C. (2018). A hybrid particle swarm optimization with local search for stochastic resource allocation problem. *Journal of Intelligent Manufacturing*, *29*(3), 481-495. https://doi.org/10.1007/s10845-015-1124-7

Mahmoud, K. I. (2009). Split Assignment With Transportation Model for Job-Shop Loading (Case Study). *Journal of Engineering*, *15*(2).

Mirjalili, S., & Lewis, A. (2016). The whale optimization algorithm. *Advances in engineering software*, *95*, 51-67. https://doi.org/10.1016/j.advengsoft.2016.01.008

Pentico, D. W. (2007). Assignment problems: A golden anniversary survey. *European Journal of Operational Research*, *176*(2), 774-793. https://doi.org/10.1016/j.ejor.2005.09.014

Ross, G. T., & Soland, R. M. (1975). A branch and bound algorithm for the generalized assignment problem. *Mathematical programming*, *8*(1), 91-103. https://doi.org/10.1007/BF01580430

Ruiz, M., Igartua, J. I., Mindeguia, M., & Orobengoa, M. (2020). Understanding and representation of organizational training programs and their evaluation. *International Journal of Production Management and Engineering*, *8*(2), 99-109. https://doi.org/10.4995/ijpme.2020.12271

Satapathy, P., Mishra, S. P., Sahu, B. K., Debnath, M. K., & Mohanty, P. K. (2018, April). Design and implementation of whale optimization algorithm based PIDF controller for AGC problem in unified system. In *International Conference on Soft Computing Systems* (pp. 837-846). Springer, Singapore. https://doi.org/10.1007/978-981-13-1936-5_85

Sharma, H. (2014). Importance and performance of managerial training in Indian companies-an empirical study. *The Journal of Management Development*, *33*(2), 75-89. https://doi.org/10.1108/JMD-11-2013-0144

Suliman, A. S. A. (2019). Using ant colony algorithm to find the optimal assignment. *AL-Anbar University journal of Economic and Administration Sciences*, *11*(25).

Ostadi, B., Taghizadeh Yazdi, M., & Mohammadi Balani, A. (2021). Process Capability Studies in an Automated Flexible Assembly Process: A Case Study in an Automotive Industry. *Iranian Journal of Management Studies*, *14*(1), 1-37.

Walsh, B. & Volini, E. (2017). *Rewriting the rules for the digital age.* Deloitte University Press. New York.

Wang, Z., Li, S., Wang, Y., & Li, S. (2009, August). The research of task assignment based on ant colony algorithm. In *2009 International Conference on Mechatronics and Automation* (pp. 2334-2339). IEEE.

Xuezhi, Q., & Xuehua, W. (1996). Dynamic programming model of a sort of optimal assignment problem [J]. *Mathematics In Practice and Theory*, *3*.

Yadav, N., Banerjee, K., & Bali, V. (2020). A survey on fatigue detection of workers using machine learning. *International Journal of E-Health and Medical Communications (IJEHMC)*, *11*(3), 1-8. https://doi.org/10.4018/IJEHMC.2020070101

Yang, X. S. (2009). Firefly algorithms for multimodal optimization. In *International symposium on stochastic algorithms* (pp. 169-178). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-04944-6_14

Yang, X. S. (2012). Flower pollination algorithm for global optimization. In *International conference on unconventional computing and natural computation* (pp. 240-249). Springer, Berlin, Heidelberg.

Appendix I

This appendix presents the three randomly generated WAP that were used in this study.

Size	Example ($c_{i,j}$)	Optimal Assignment ($x_{i,j}$)	Minimum Cost
5	$\begin{bmatrix} 10 & 7 & 8 & 6 & 7 \\ 8 & 8 & 6 & 9 & 8 \\ 8 & 6 & 9 & 10 & 7 \\ 6 & 7 & 7 & 8 & 10 \\ 8 & 7 & 9 & 7 & 6 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	30
10	$\begin{bmatrix} 10 & 7 & 8 & 6 & 8 & 8 & 9 & 7 & 9 & 11 \\ 8 & 8 & 7 & 9 & 8 & 7 & 8 & 9 & 10 & 6 \\ 8 & 7 & 9 & 10 & 6 & 7 & 8 & 7 & 7 & 7 \\ 9 & 7 & 7 & 8 & 10 & 6 & 9 & 7 & 10 & 9 \\ 8 & 7 & 6 & 7 & 8 & 9 & 7 & 10 & 11 & 8 \\ 10 & 6 & 8 & 8 & 9 & 8 & 9 & 7 & 9 & 11 \\ 8 & 8 & 7 & 9 & 8 & 7 & 9 & 6 & 10 & 10 \\ 6 & 7 & 9 & 10 & 7 & 7 & 8 & 7 & 9 & 7 \\ 10 & 7 & 8 & 10 & 8 & 8 & 9 & 7 & 6 & 11 \\ 8 & 8 & 7 & 9 & 8 & 7 & 6 & 9 & 10 & 10 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$	60
15	$\begin{bmatrix} 8 & 8 & 7 & 9 & 8 & 7 & 9 & 6 & 10 & 10 & 9 & 8 & 7 & 7 & 9 \\ 8 & 8 & 7 & 9 & 8 & 7 & 6 & 9 & 10 & 10 & 8 & 7 & 9 & 8 & 7 \\ 11 & 10 & 7 & 8 & 7 & 9 & 7 & 8 & 9 & 7 & 6 & 11 & 8 & 7 & 8 \\ 10 & 6 & 9 & 7 & 8 & 7 & 7 & 8 & 10 & 8 & 9 & 7 & 10 & 9 & 7 \\ 8 & 8 & 7 & 9 & 8 & 7 & 8 & 9 & 10 & 6 & 8 & 7 & 9 & 8 & 11 \\ 8 & 8 & 7 & 9 & 6 & 7 & 10 & 9 & 10 & 10 & 7 & 8 & 9 & 11 & 7 \\ 10 & 8 & 8 & 8 & 9 & 8 & 9 & 7 & 9 & 11 & 8 & 6 & 7 & 9 & 8 \\ 6 & 7 & 9 & 10 & 7 & 7 & 8 & 7 & 9 & 7 & 10 & 7 & 7 & 8 & 11 \\ 10 & 7 & 8 & 10 & 8 & 8 & 9 & 7 & 8 & 11 & 8 & 9 & 8 & 11 & 6 \\ 8 & 7 & 7 & 8 & 10 & 6 & 9 & 7 & 10 & 9 & 8 & 10 & 8 & 8 & 7 \\ 9 & 7 & 9 & 6 & 8 & 7 & 8 & 7 & 7 & 7 & 9 & 11 & 8 & 7 & 8 \\ 10 & 9 & 8 & 7 & 8 & 7 & 7 & 8 & 10 & 8 & 9 & 7 & 6 & 7 & 7 \\ 9 & 6 & 7 & 8 & 9 & 10 & 8 & 7 & 9 & 9 & 10 & 11 & 7 & 7 & 8 \\ 10 & 7 & 8 & 10 & 8 & 8 & 9 & 7 & 6 & 11 & 7 & 9 & 7 & 11 & 10 \\ 8 & 8 & 7 & 9 & 9 & 7 & 9 & 10 & 11 & 7 & 9 & 9 & 10 & 6 & 11 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 1 & 0 \end{bmatrix}$	90