Montes Taurus J. Pure Appl. Math. 3 (3), 112–121, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

A Survey on Nikodým and Vitali-Hahn-Saks Properties

Salvador López-Alfonso [a], Manuel López-Pellicer [b], Jos Mas [c]

[a]Departamento Construcciones Arquitectnicas, Universitat Politcnica de Valncia, 46022 Valencia, Spain
[b]Professor Emeritus, Universitat Politcnica de Valncia and IUMPA, 46022 Valencia, Spain
[c]Universitat Politcnica de Valncia, Instituto de Matemtica Multidisciplinar, 46022 Valencia, Spain

Abstract

Let $ba(\mathcal{A})$ be the Banach space of the real (or complex) finitely additive measures of bounded variation defined on an algebra $\mathcal{A}$ of subsets of Ω and endowed with the variation norm. A subset $\mathcal{B}$ of $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$ if each $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$ and $\mathcal{B}$ is a strong Nikodým set for $ba(\mathcal{A})$ if each increasing covering $(\mathcal{B}_m)_{m=1}^{\infty}$ of $\mathcal{B}$ contains a $\mathcal{B}_n$ which is a Nikodým set for $ba(\mathcal{A})$. If, additionally, the Nikodým subset $\mathcal{B}$ verifies that the sequential $\mathcal{B}$-pointwise convergence in $ba(\mathcal{A})$ implies weak convergence then $\mathcal{B}$ has the Vitali-Hahn-Saks property, (VHS) in brief, and $\mathcal{B}$ has the strong (VHS) property if for each increasing covering $(\mathcal{B}_m)_{m=1}^{\infty}$ of $\mathcal{B}$ there exists $\mathcal{B}_q$ that has (VHS) property.

Motivated by Valdivia result that every σ-algebra has strong Nikodým property and by his 2013 open question concerning that if Nikodým property in an algebra of subsets implies strong Nikodým property we survey this Valdivia theorem and we get that in a strong Nikodým set the (VHS) property implies the strong (VHS) property.

Keywords: Bounded set, Algebra and σ-algebra of subsets, Bounded finitely additive scalar measure, Nikodým and strong Nikodým property, Vitali-Hahn-Saks and strong Vitali-Hahn-Saks property

2010 MSC: 28A60, 46G10

1. Introduction

For an algebra $\mathcal{A}$ of subsets of a set Ω we denote by $L(\mathcal{A})$ the normed real or complex space generated by the characteristic functions $e(A)$ of the sets $A \in \mathcal{A}$ and endowed with the supremum norm $\|\cdot\|_\infty$, which dual is the Banach space $ba(\mathcal{A})$ of bounded finitely additive measures on $\mathcal{A}$ endowed with the variation norm $|\cdot|(\Omega)$, or $|\cdot|$ in brief, [2, Theorem 1.13]). As it is usual for each $\mu \in ba(\mathcal{A})$ and $C \in \mathcal{A}$ the value $\mu(C)$ of the measure μ in C defines the value $\mu(e(C))$ of the linear form μ in $e(C)$, $|\mu|(C)$ is the variation of μ on C and defines a seminorm in $ba(\mathcal{A})$ such that for each finite partition $\{C_i : C_i \in \mathcal{A}, 1 \leqslant i \leqslant n\}$ of C we have $|\mu|(C) = \Sigma_i |\mu|(C_i)$. Polar sets, [9, Chapter IV, §20, 8 Polarity, where are named absolute polar sets], are considered in the dual pair $< L(\mathcal{A}), ba(\mathcal{A}) >$ and the polar of a set M is denoted by M°. For each family of sets $\mathcal{B}$ contained in the algebra $\mathcal{A}$ the topology $\tau_s(\mathcal{B})$ in $ba(\mathcal{A})$ is the topology of pointwise convergence in $\mathcal{B}$ and the weak* topology in $ba(\mathcal{A})$ is $\tau_s(\mathcal{A})$. The completion of $L(\mathcal{A})$ endowed with the supremum norm $\|\cdot\|_\infty$ is the space $\widehat{L(\mathcal{A})}$ of bounded $\mathcal{A}$-measurable functions.

†*Article ID:* MTJPAM-D-20-00058

Email addresses: salloal@csa.upv.es (Salvador López-Alfonso), mlopezpe@mat.upv.es (Manuel López-Pellicer), jmasm@imm.upv.es (Jos Mas)

Received:31 December 2020, Accepted:11 January 2021, Published:25 April 2021

⋆*Corresponding Author:* Manuel López-Pellicer

The convex (absolutely convex) hull of a subset B of a vector space E is denoted by co(B) (absco(B)) and the seminorm in span B defined by $\inf\{|\lambda| : x \in \lambda(\operatorname{absco} B)\}$, for each $x \in \operatorname{span} B$, is the gauge of absco B. The gauge of absco$(\{\chi_C : C \in \mathcal{A}\})$ is a norm in $L(\mathcal{A})$ equivalent to the supremum norm $\|\cdot\|_\infty$, [17, Propositions 1 and 2 for an inductive proof]. The dual norm of the gauge of absco$(\{\chi_C : C \in \mathcal{A}\})$ is the $\mathcal{A}$-supremum norm. For each $B \in \mathcal{A}$ we have that in $ba(\mathcal{A})$ the seminorms variation on B and supremum of modulus on $\{C \in \mathcal{A} : C \subset B\}$ are equivalent.

For a σ-algebra Σ of subsets of a set Ω Nikodým boundedness theorem (see [1, page 80, named as Nikodym-Grothendieck Boundedness Theorem]) states that a Σ-pointwise bounded subset M of $ba(\Sigma)$ is bounded in $ba(\Sigma)$. Then a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a *Nikodým set for* $ba(\mathcal{A})$, or that $\mathcal{B}$ has (N) property, if each $\mathcal{B}$-pointwise bounded subset M of $ba(A)$ is bounded in $ba(A)$ (see [16, Definition 2.4], [18, Definition 1] and [13, see uniform bounded deciding property]), where we may suppose that M is weak* closed and absolutely convex and also we may suppose that M is countable. Nikodým boundedness theorem says that if Σ is a σ-algebra then Σ is a Nikodým set for $ba(\Sigma)$.

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a *strong Nikodým set for* $ba(\mathcal{A})$, or that $\mathcal{B}$ has strong (N) or (sN) property in brief, if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ is a Nikodým set for $ba(\mathcal{A})$. Valdivia obtained in [17, Theorem 2] that for a σ-*algebra* Σ the set Σ has (sN) property and in [18, Problem 1] he asks if for an algebra $\mathcal{A}$ it is true that (N) property implies (sN) property. A partial solution of this still open is provided in [4, Theorem 3.3]. Previous results, more strong properties and examples are provided in [5], [8, Theorem 2], [10], [12, Theorem 3] and [14].

An algebra of sets $\mathcal{A}$ has property (G) if the space $\widehat{L(\mathcal{A})}$ is a Grothendieck space, i.e., in the dual pair $\left\langle \widehat{L(\mathcal{A})}, ba(\mathcal{A}) \right\rangle$ the sequential weak* convergence in $ba(\mathcal{A})$ implies weak convergence, or, in brief, $\mathcal{A}$ has property (G) if the space $\widehat{L(\mathcal{A})}$ is a Grothendieck space, see [16, Introduction] where it is stated that each σ-algebra has property (G). Then an algebra of sets $\mathcal{A}$ has property (G) if, and only if, each bounded sequence $(\mu_n)_{n=1}^\infty$ of $ba(\mathcal{A})$ such that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, for each $A \in \mathcal{A}$ with $\mu \in ba(\mathcal{A})$, verifies that the sequence $(\mu_n)_{n=1}^\infty$ converges weakly to μ. This equivalence follows easily from Banach-Steinhaus theorem that says that the condition $\lim_{n\to\infty} \mu_n(f) = \mu(f)$, for each $f \in \widehat{L(\mathcal{A})}$, implies that the sequence $(\mu_n)_{n=1}^\infty$ is bounded in $ba(\mathcal{A})$, and from the following direct Claim 1.1.

Claim 1.1. Let E be a Banach space and let $(\mu_n)_{n=1}^\infty$ a bounded sequence in its dual E^* endowed with the polar norm. If $\mu \in E^*$ and $(\mu_n)_{n=1}^\infty$ converges pointwise to μ in the subset F of E, then this sequence $(\mu_n)_{n=1}^\infty$ converges pointwise to μ in the closure $\overline{F}$ of F.

Proof. Clearly for $\epsilon > 0$ and $v \in \overline{F}$ there exists $f \in F$ such that $\|v - f\| < \epsilon(2(1 + |\mu| + \sup_n |\mu_n|)^{-1}$, and then, by the pointwise convergence condition, there exists n_ϵ such that $|(\mu_n - \mu)(f)| < 2^{-1}\epsilon$, for every $n > n_\epsilon$. Hence for $n > n_\epsilon$ we have that

$$|(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| < \frac{\epsilon(|\mu| + \sup_n |\mu_n|)}{2(1 + |\mu| + \sup_n |\mu_n|} + \frac{\epsilon}{2} \leq \epsilon,$$

so $|(\mu_n - \mu)(v)| \leq |(\mu_n - \mu)(v - f)| + |(\mu_n - \mu)(f)| < \epsilon$. This proves that $(\mu_n)_{n=1}^\infty$ converges pointwise to μ in $\overline{F}$. □

In [16, Introduction] it is stated that each σ-algebra Σ verifies the Vitali-Hahn-Saks theorem, i.e., every sequence $(\mu_n)_{n=1}^\infty$ of $ba(\Sigma)$ such that

$$\lim_{n\to\infty} \mu_n(B) = \mu(B), \text{ for every } B \in \Sigma,$$

is uniformly exhaustive, i.e., for each sequence $(B_j)_{j=1}^\infty$ of pairwise disjoint subsets of Σ the $\lim_{j\to\infty} \mu_n(B_j)$ is 0, uniformly in $n \in \mathbb{N}$. An algebra $\mathcal{A}$ has (VHS) property if it verifies the thesis of Vitali-Hahn-Saks theorem or, equivalently, if $\mathcal{A}$ has properties (N) and (G), see [16, 2.5. Theorem] or [7, Theorem 4.2]. Therefore $\mathcal{A}$ has (VHS) property if and only $\mathcal{A}$ has property (N) and for if each sequence $(\mu_n)_{n=1}^\infty$ and μ in $ba(\mathcal{A})$ such that $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, for every $A \in \mathcal{A}$, we have that the sequence $(\mu_n)_{n=1}^\infty$ converges weakly to μ. The last equivalence follows easily from the observation that property (N) and $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, for every $A \in \mathcal{A}$, imply that the sequence $(\mu_n)_{n=1}^\infty$ is bounded and from the Claim 1.1. This last characterization and the (sN) property suggest the following definition.

Definition 1.2. A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets has (VHS) property if $\mathcal{B}$ is a Nikodým set for $ba(A)$ and each sequence $(\mu_n)_{n=1}^\infty$ and μ, both in $ba(\mathcal{A})$, such that $\lim_{n\to\infty} \mu_n(B) = \mu(B)$, for each $B \in \mathcal{B}$, we have that $(\mu_n)_{n=1}^\infty$ converges weakly to μ. $\mathcal{B}$ has the strong (VHS) property if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that $\mathcal{B}_p$ has (VHS).

In the next section we provide a proof of Valdivia theorem stating that for each σ-algebra Σ the set Σ has the strong Nikodým property. This proof simplify the proof contained in [11], where was given a proof of this theorem following [17] and independent of the theory of barrelled spaces that are locally convex spaces that verify the thesis of Banach-Steinhaus theorem and which main properties may be found in [3], [6] and [15], among others. For the sake of completeness we will give all the proofs, with references to the proofs given in [11, 3 Revisiting Valdivia theorem on Nikodým sets].

In the last section we will prove that if a subset $\mathcal{B}$ of an algebra of sets $\mathcal{A}$ has (VHS) property then $\mathcal{B}$ has the strong (VHS) property if and only if $\mathcal{B}$ has the strong (N) property. Therefore a positive solution of the mentioned Valdivia open problem [18, Problem 1] imply a positive solution for the corresponding problem for the (VHS) property, i.e., that (VHS) property for an algebra $\mathcal{A}$ imply strong (VHS) property in $\mathcal{A}$.

2. Valdivia theorem on Nikodým sets

A natural internal characterization of Nikodým sets for $ba(\mathcal{A})$ is provided in the next Proposition 2.1.

Proposition 2.1. *Let $\mathcal{A}$ be an algebra of sets and let $\mathcal{B}$ be a subset of $\mathcal{A}$. $\mathcal{B}$ has property (N) if and only if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $p \in \mathbb{N}$ such that*

$$\overline{\text{absco}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})}$$

is a neighborhood of zero in $L(\mathcal{A})$.

Proof. If $\mathcal{B}$ is a not a Nikodým set for $ba(\mathcal{A})$ there exists a subset C in $ba(\mathcal{A})$ which is $\mathcal{B}$-pointwise bounded and C is unbounded in $ba(\mathcal{A})$. Then the sets $\mathcal{B}_n = \{A \in \mathcal{B} : \sup_{\mu \in C} |\mu(A)| \leq n\}$, $n \in \mathbb{N}$, are an increasing covering of $\mathcal{B}$ and C° is not a neighborhood of zero in $L(\mathcal{A})$. By the definition of $\mathcal{B}_n$ we have that for each $n \in \mathbb{N}$

$$\overline{\text{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} \subset nC^\circ.$$

Hence so $\overline{\text{absco}\{e(A) : A \in \mathcal{B}_m\}}^{L(\mathcal{A})}$ is not a neighborhood of zero in $L(\mathcal{A})$ for each $n \in \mathbb{N}$.

Conversely, if there exists an increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ such that

$$\overline{\text{absco}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})}$$

is not a neighborhood of zero in $L(\mathcal{A})$ for every $n \in \mathbb{N}$, then the polar sets $\{e(A) : A \in \mathcal{B}_n\}^\circ$ are unbounded, so there exists $\mu_n \in \{e(A) : A \in \mathcal{B}_n\}^\circ$ such that $|\mu_n| \geq n$, for each $n \in \mathbb{N}$, hence $\{\mu_n : n \in \mathbb{N}\}$ is an unbounded subset of $ba(\mathcal{A})$. If $A \in \mathcal{B}$ there exists $q_A \in \mathbb{N}$ such that $A \in \mathcal{B}_n$ for each $n \geq q_A$, hence $|\mu_n(e(A))| \leq 1$ for $n \geq q_A$, and we get that $\{|\mu_n(e(A))| : n \in \mathbb{N}\}$ is bounded. Hence $\mathcal{B}$ does not have property (N). □

The fact that $\overline{\text{absco}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})}$ is a neighborhood of zero in $L(\mathcal{A})$ implies that $\overline{\text{span}\{e(A) : A \in \mathcal{B}_p\}}^{L(\mathcal{A})} = L(\mathcal{A})$ and $\overline{\text{absco}\{e(A) : A \in \mathcal{B}_p\}}^{\widehat{L(\mathcal{A})}}$ is a neighborhood of zero in $\widehat{L(\mathcal{A})}$.

Lemma 2.2. *Let M be an unbounded, weak*-closed and absolutely convex subset of $ba(\mathcal{A})$ such that $\overline{\text{span}\, M^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$. For each finite subset Q of $\mathcal{A}$ we have that $M \cap \{e(A) : A \in \mathrm{Q}\}^\circ$ is unbounded in $ba(\mathcal{A})$, i.e.,*

$$\sup_{\mu \in M \cap \{e(A): A \in \mathrm{Q}\}^\circ} \{|\mu|(\Omega)\} = \infty. \tag{2.1}$$

Proof. The set $B = M^\circ$ is a closed absolutely convex subset of the normed space $L(\mathcal{A})$ such that $\overline{\text{span}\, B}^{L(\mathcal{A})} = L(\mathcal{A})$ and B is not a zero neighborhood in $L(\mathcal{A})$. It is direct to prove that for each finite subset C of $L(\mathcal{A})$ there exists a subset D of C such that *span* $B \cap$ *span* $D = \{0\}$ and the gauges defined by $\text{absco}(B \cup C)$ and $\text{absco}(B \cup D)$ are equivalents. Hence $\text{absco}(B \cup C)$is not a zero neighborhood in $L(\mathcal{A})$. In particular, the set $\text{absco}(B \cup \{e(A) : A \in Q\})$ is not a zero neighborhood in $L(\mathcal{A})$ and then its polar set

$$\{\text{absco}(B \cup \{e(A) : A \in Q\})\}^\circ = M^{\circ\circ} \cap \{e(A) : A \in Q\}^\circ$$

is an unbounded subset of $ba(\mathcal{A})$ and as $M = M^{\circ\circ}$ we get (2.1). □

Proposition 2.3. *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of a subset of $\mathcal{B}$ of an algebra $\mathcal{A}$ such that $\mathcal{B}$ is a Nikodým set for $ba(\mathcal{A})$ and for every $n \in \mathbb{N}$ the set $\mathcal{B}_n$ is not a Nikodým for $ba(\mathcal{A})$. Then there exists $p \in \mathbb{N}$ such that for each $n \geq p$ there exists a subset M_n in $ba(\mathcal{A})$ that it is $\mathcal{B}_n$-pointwise bounded, absolutely convex, weak*-closed and such that for each finite subset* Q *of $\mathcal{A}$ the intersection $M_n \cap \{e(A) : A \in \mathrm{Q}\}^\circ$ is unbounded in $ba(\mathcal{A})$.*

Proof. By Proposition 2.1 there exists $p \in \mathbb{N}$ such that for each $n \geq p$

$$\overline{\operatorname{span}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} = L(\mathcal{A}). \tag{2.2}$$

As $\mathcal{B}_n$ is not a Nikodým for $ba(\mathcal{A})$ there exists an unbounded, weak*-closed and absolutely convex subset of M_n in $ba(\mathcal{A})$ which is unbounded in $ba(\mathcal{A})$ and M_n is pointwise bounded in $\{e(A) : A \in \mathcal{B}_n\}$. The pointwise boundedness imply that $\{e(A) : A \in \mathcal{B}_n\} \subset \operatorname{span} M_n^\circ$, hence for each $n \geq p$ we have by (2.2) that

$$L(\mathcal{A}) = \overline{\operatorname{span}\{e(A) : A \in \mathcal{B}_n\}}^{L(\mathcal{A})} \subset \overline{\operatorname{span} M_n^\circ}^{L(\mathcal{A})} \subset L(\mathcal{A}). \tag{2.3}$$

From 2.3 we deduce that $\overline{\operatorname{span} M_n^\circ}^{L(\mathcal{A})} = L(\mathcal{A})$, for each $n \geq p$, and the Proposition follows from Lemma 2.2. □

Claim 2.4. Let B be an element of an algebra $\mathcal{A}$, let M be a subset of $ba(\mathcal{A})$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} \{|\mu|(B)\} = \infty \tag{2.4}$$

if $\{B_1, B_2, \ldots, B_q\}$ is a finite partition of B by elements of $\mathcal{A}$ there exist j, $1 \leq j \leq q$, such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} \left\{|\mu|(B_j)\right\} = \infty. \tag{2.5}$$

Proof. If this Claim were not true there exists a finite subset Q_i of $\mathcal{A}$, $1 \leq i \leq q$, such that $\sup_{\mu \in M \cap \{e(A):A \in Q_i\}^\circ} \{|\mu|(B_i)\} < \infty$. Then for $Q = \cup\{Q_i : 1 \leq i \leq q\}$ the first member of (2.4) is the first member of the inequality

$$\sum_{i=1}^{q} \sup_{\mu \in M \cap \{e(A):A \in Q\}^\circ} \{|\mu|(B_i)\} \leq \sum_{i=1}^{q} \sup_{\mu \in M \cap \{e(A):A \in Q_i\}^\circ} \{|\mu|(B_i)\} < \infty,$$

and from this contradiction with (2.4) follows (2.5). □

Lemma 2.5. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω, $A \in \mathcal{A}$ and M a weak*-closed and absolutely convex subset of $ba(\mathcal{A})$ such that for each finite subset* Q *of $\mathcal{A}$*

$$\sup_{\mu \in M \cap \{e(D):D \in \mathrm{Q}\}^\circ} |\mu|(A) = \infty.$$

For each $(p, \alpha) \in (\mathbb{N} \backslash \{0, 1\}) \times \mathbb{R}^+$ and each finite subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists a partition $\{A_i : A_i \in \mathcal{A}, 1 \leq i \leq p\}$ of A and a subset $\{\mu_i : 1 \leq i \leq p\}$ of M such that

$$|\mu_i(e(A_i))| > \alpha \quad \textit{and} \quad \Sigma_{j=1}^{n} \left|\mu_i(e(B_j))\right| \leq 1, \quad \textit{for } 1 \leq i \leq p \tag{2.6}$$

Proof. First we claim that if for each finite subset Q of $\mathcal{A}$ we have that

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A) = \infty, \tag{2.7}$$

then for each $\alpha \in \mathbb{R}^+$ and each subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists $(\mu_1, A_1) \in M \times \mathcal{A}$, $A_1 \subset A$ such that

$$|\mu_1(e(A_1))| > \alpha, \quad |\mu_1(e(A \backslash A_1))| > \alpha, \quad \Sigma_{j=1}^{n} \left|\mu_1(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu|(A \backslash A_1) = \infty.$$

In fact, by (2.7) with $Q = \{A, B_1, \cdots, B_n\}$ there exists $(\nu_1, P_{11}) \in (M \cap \{e(D) : D \in Q\}^\circ) \times \mathcal{A}$, with $P_{11} \subset A$ such that

$$|\nu_1(P_{11})| > n(\alpha + 1), \quad |\nu_1(A)| \leq 1 \quad \text{and} \quad \left|\nu_1(B_j)\right| \leq 1, \text{for } 1 \leq j \leq n.$$

Let $P_{12} := A \backslash P_{11}$ and $\mu_1 = n^{-1}\nu_1$. The measure $\mu_1 \in M$ and verifies that

$$|\mu_1(P_{11})| > \alpha + 1, \ \ |\mu_1(A)| \leq 1, \ \ \Sigma_{j=1}^n \left|\mu_1(e(B_j))\right| \leq 1,$$

hence

$$|\mu_1(P_{12})| = |\mu_1(A) - \mu_1(P_{11})| \geq |\mu_1(P_{11})| - |\mu_1(A)| > \alpha.$$

By Claim 2.4 it is verified at least one of the inequalities

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} \{|\mu| (P_{11})\} = \infty, \text{for each finite subset } Q \in \mathcal{A}$$

or

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} \{|\mu| (P_{12})\} = \infty, \text{for each finite subset } Q \in \mathcal{A}.$$

In the first we define $A_1 := P_{12}$ and in the second we take $A_1 := P_{11}$ to get our claim.

Hence by this claim just proved there exists in A a partition $\{A_1, A \backslash A_1\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_1 \in M$ such that

$$|\mu_1(e(A_1))| > \alpha, \quad |\mu_1(e(A \backslash A_1))| > \alpha, \ \ \Sigma_{j=1}^n \left|\mu_1(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu| (A \backslash A_1) = \infty.$$

If we apply the claim to $A \backslash A_1$ we get in $A \backslash A_1$ a partition $\{A_2, A \backslash (A_1 \cup A_2)\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_2 \in M$ such that

$$|\mu_2(e(A_2))| > \alpha, \quad |\mu_2(e(A \backslash (A_1 \cup A_2)))| > \alpha, \ \ \Sigma_{j=1}^n \left|\mu_2(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M \cap \{e(D):D \in Q\}^\circ} |\mu| (A \backslash (A_1 \cup A_2)) = \infty.$$

Following this method we get in $A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-2})$ a partition $\{A_{p-1}, A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-2} \cup A_{p-1})\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_{p-1} \in M$ such that$\left|\mu_{p-1}(e(A_{p-1}))\right| > \alpha$,$\left|\mu_{p-1}(e(A \backslash (A_1 \cup \cdots \cup A_{p-1})))\right| > \alpha$ and $\Sigma_{j=1}^n \left|\mu_{p-1}(e(B_j))\right| \leq 1$.

To finish the proof we define $A_p := A \backslash (A_1 \cup A_2 \cup \cdots \cup A_{p-2} \cup A_{p-1})$ and $\mu_p := \mu_{p-1}$.

The property given in Lemma 2.5 has the following corollary. □

Corollary 2.6. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω, $A \in \mathcal{A}$ and $M_n, n \in 1, 2, ...$, a weak*-closed and absolutely convex subset of $ba(\mathcal{A})$ such that for each finite subset* Q *of $\mathcal{A}$*

$$\sup_{\mu \in M_n \cap \{e(D):D \in \mathrm{Q}\}^\circ} |\mu| (A) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n. For each $\alpha \in \mathbb{R}^+$ and each finite subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists in A a partition $\{A_1, A \backslash A_1\} \in \mathcal{A} \times \mathcal{A}$ and a measure $\mu_1 \in M_{n_1}$ such that

$$|\mu_1(e(A_1))| > \alpha \quad \text{and} \quad \Sigma_{j=1}^n \left|\mu_1(e(B_j))\right| \leq 1$$

and for each finite subset Q *of $\mathcal{A}$*

$$\sup_{\mu \in M_n \cap \{e(D):D \in \mathrm{Q}\}^\circ} |\mu| (A \backslash A_1) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n.

Proof. By Lemma 2.5 for each $(p+2,\alpha) \in (\mathbb{N}\backslash\{0,1\}) \times \mathbb{R}^+$ and for the subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists a partition $\{D_i : D_i \in \mathcal{A}, 1 \leq i \leq p+2\}$ of A and a subset $\{\upsilon_i : 1 \leq i \leq p+2\}$ of M_{n_1} such that

$$|\upsilon_i(e(D_i))| > \alpha \quad \text{and} \quad \Sigma_{j=1}^n \left|\upsilon_i(e(B_j))\right| \leq 1, \quad \text{for } 1 \leq i \leq p+2.$$

From Claim 2.4 and for each $1 \leq j \leq p$ there exists $i_j \in \{1, 2, \cdots, p+2\}$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_{n_j} \cap \{e(D): D \in Q\}^\circ} |\mu| (D_{n_{i_j}}) = \infty$$

and also there exists $i_0 \in \{1, 2, \cdots, p+2\}$ such that for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu| (D_{n_{i_0}}) = \infty$$

for infinite values of n. Let us suppose that $i^* \in \{1, 2, \cdots, p+2\} \backslash \{i_m : m = 0, 1, \cdots, p\}$. To finish this proof let $\mu_1 := \upsilon_{i^*}$ and $A_1 := D_{i^*}$. Then

$$|\mu_1(e(A_1))| = |\upsilon_{i^*}(e(D_{i^*}))| > \alpha \text{ and } \Sigma_{j=1}^n \left|\mu_1(e(B_j))\right| = \Sigma_{j=1}^n \left|\upsilon_{i^*}(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu| (A \backslash A_1) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n, because $A \backslash A_1 = A \backslash D_{i^*}$ contains $\cup\{D_{n_{i_j}} : 0 \leq j \leq p\}$. □

Remark 2.7. Corollary 2.6 works without the finite subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$. Then we get that $|\mu_1(e(A_1))| > \alpha$ and that for each finite subset Q of $\mathcal{A}$ the set $M_n \cap \{e(D) : D \in Q\}^\circ$ is unbounded in $ba(\mathcal{A})$ for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n.

Proposition 2.8. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω, $A \in \mathcal{A}$ and $M_n, n \in 1, 2, ...$ a weak*-closed and absolutely convex subset of $ba(\mathcal{A})$ such that for each finite subset* Q *of $\mathcal{A}$*

$$\sup_{\mu \in M_n \cap \{e(D): D \in \mathrm{Q}\}^\circ} |\mu| (A) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n. For each $(p,\alpha) \in (\mathbb{N}\backslash\{0,1\}) \times \mathbb{R}^+$ and each subset $\{B_i : 1 \leq i \leq n\}$ of $\mathcal{A}$ there exists a partition $\{A_i : A_i \in \mathcal{A}, 1 \leq i \leq p+1\}$ of A and $\mu_i \in M_{n_i}$, $1 \leq i \leq p$, such that

$$|\mu_i(e(A_i))| > \alpha, \quad \Sigma_{j=1}^n \left|\mu_i(e(B_j))\right| \leq 1, \quad \text{for } 1 \leq i \leq p$$

and for each finite subset Q *of $\mathcal{A}$*

$$\sup_{\mu \in M_n \cap \{e(D): D \in \mathrm{Q}\}^\circ} |\mu| (A_{p+1}) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n.

Proof. Corollary 2.6 provides in A a subset $A_1 \in \mathcal{A}$ and $\mu_1 \in M_{n_1}$ such that

$$|\mu_1(e(A_1))| > \alpha \quad \text{and} \quad \Sigma_{j=1}^n \left|\mu_1(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu \in M_n \cap \{e(D): D \in Q\}^\circ} |\mu| (A \backslash A_1) = \infty$$

for $n = n_1, n_2, \cdots, n_p$ and for an infinity of values of n. If we apply again the Corollary 2.6 to $A \backslash A_1$ we get $A_2 \in \mathcal{A}$, $A_2 \subset A \backslash A_1$, and $\mu_2 \in M_{n_2}$ such that

$$|\mu_2(e(A_2))| > \alpha, \quad \Sigma_{j=1}^n \left|\mu_2(e(B_j))\right| \leq 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(A\backslash(A_1\cup A_2)) = \infty$$

for $n = n_1, n_2,\cdots, n_p$ and for an infinity of values of n.

Following this method, for each $1 \le i \le p-1$ we get in A the pairwise disjoint subsets $A_i \in \mathcal{A}$ and in $ba(\mathcal{A})$ the measures $\mu_i \in M_{n_i}$, $1 \le i \le p-1$, such that

$$|\mu_i(e(A_i))| > \alpha, \quad \Sigma_{j=1}^n \left|\mu_i(e(B_j))\right| \le 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-1})) = \infty$$

for $n = n_1, n_2,\cdots, n_p$ and for an infinity of values of n. The Corollary 2.6 applied to $A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-1})$ provides $A_p \in \mathcal{A}$, $A_p \subset A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-1})$, and $\mu_p \in M_{n_p}$ such that

$$\left|\mu_i(e(A_p))\right| > \alpha, \quad \Sigma_{j=1}^n \left|\mu_i(e(B_j))\right| \le 1$$

and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-1}\cup A_p)) = \infty$$

for $n = n_1, n_2,\cdots, n_p$ and for an infinity of values of n. With $A_{p+1} := A\backslash(A_1\cup A_2\cup\cdots\cup A_{p-1}\cup A_p)$ the proof is done. □

Proposition 2.9. *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of a σ-algebra Σ of subsets of a set Ω. If $\mathcal{B}_n$ is not a Nikodým set for $ba(\Sigma)$ for each $n \in \mathbb{N}$ then for each $(i, j) \in \mathbb{N}^2$, such that $1 \le i \le j$, there exists $A_{ij} \in \Sigma$ and $\mu_{ij} \in ba(\Sigma)$ such that the sets A_{ij} are pairwise disjoint, for each natural number i the set of measures $\{\mu_{ij} : j \in \mathbb{N}, j \ge i\}$ is pointwise bounded in $\mathcal{B}_i$ and*

$$\left|\mu_{ij}(e(A_{ij}))\right| > j, \quad \Sigma_{1\le k\le m<j} \left|\mu_{ij}(e(A_{km}))\right| \le 1.$$

Proof. By Nikodým boundedness theorem Σ is a Nikodým set for $ba(\Sigma)$, hence by Proposition 2.3 there exists $p \in \mathbb{N}$ such that for each $n \ge p$ there exists in $ba(\mathcal{A})$ an absolutely convex and weak*-closed subset M_n that it is pointwise bounded in $\mathcal{B}_n$ and for each finite subset Q of $\mathcal{A}$

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(A) = \infty.$$

We may suppose that $p = 1$, deleting the sets $\mathcal{B}_n$, $n \le p-1$ and renumbering the subindex n. The proof will be obtained by induction on j.

For $j = 1$, the Corollary 2.6 in the case considered in the Remark 2.7 with $\mathcal{A} = \Sigma$, $n = n_1 = 1$ and $\alpha = 1$ provides a measure $\mu_{11} \in M_{n_1}$ and $A_{11} \in \Sigma$ such that

$$|\mu_{11}(e(A_{11}))| > 1$$

and for each finite subset Q of Σ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(\Omega\backslash A_{11}) = \infty,$$

for $n = n_1$ and for the elements n of an infinity subset N_1 of $\mathbb{N}\backslash\{n_1\}$. Then let $n_2 = \min\{n : n \in N_1\}$.

By Proposition 2.8 with $\mathcal{A} = \Sigma$, $A = \Omega\backslash A_{11}$, $n \in \{n_1, n_2\} \cup (N_1\backslash\{n_2\})$, $p = \alpha = 2$ and with $\{B_i : 1 \le i \le n\}$ equal to $\{A_{11}\}$ we obtain two measures $\mu_{i2} \in M_{n_i}$, $i = 1, 2$, and two disjoints elements of Σ, A_{12} and A_{22}, contained in $\Omega\backslash A_{11}$ such that

$$|\mu_{i2}(e(A_{i2}))| > 2, \quad |\mu_{i2}(e(A_{11}))| \le 1, \quad \text{for } 1 \le i \le 2$$

and for each finite subset Q of Σ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(\Omega\backslash(A_{11}\cup A_{12}\cup A_{22}) = \infty,$$

for $n\in\{n_1,n_2\}\cup N_2$, where N_2 is an infinite subset of $N_1\backslash\{n_2\}$. Then we define $n_3=\min\{n : n\in N_2\}$.

Let's suppose that the step j produces the measures $\mu_{ij}\in M_{n_i}$ and the pairwise disjoints elements A_{ij}, $1\le i\le j$, contained in $\Omega\backslash(\cup\{A_{km} : 1\le k\le m<j\})$ with $A_{ij}\in\Sigma$ such that

$$\left|\mu_{ij}(e(A_{ij}))\right| > j,\quad \Sigma_{1\le k\le m<j}\left|\mu_{ij}(e(A_{km}))\right|\le 1,\quad \text{for } 1\le i\le j$$

and for each finite subset Q of Σ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(\Omega\backslash(\cup\{A_{km} : 1\le k\le m\le j\})) = \infty$$

for $n=\{n_1,n_2,\cdots,n_j\}\cup N_j$, with N_j an infinity subset of $N_{j-1}\backslash\{n_j\}$.

Then we define $n_{j+1}=\min\{n : n\in N_j\}$ and from Proposition 2.8 with $\mathcal{A}=\Sigma$, $A=\Omega\backslash(\cup\{A_{km} : 1\le k\le m\le j\})$, $n\in\{n_1,n_2,\cdots,n_j,n_{j+1}\}\cup(N_j\backslash\{n_{j+1}\}$, $p=\alpha=j+1$ and with $\{B_i : 1\le i\le n\}$ equal to $\{A_{km} : 1\le k\le m\le j\}$ we obtain the measures $\mu_{i,j+1}\in M_{n_i}$ and the pairwise disjoints elements $A_{i,j+1}$ of Σ, $1\le i\le j+1$, such that each $A_{i,j+1}$ is contained in $\Omega\backslash(\cup\{A_{km} : 1\le k\le m\le j\})$, and for $1\le i\le j+1$

$$\left|\mu_{i,j+1}(e(A_{i,j+1}))\right| > j+1,\qquad \Sigma_{1\le k\le m<j+1}\left|\mu_{i,j+1}(e(A_{km}))\right|\le 1$$

and for each finite subset Q of Σ we have that

$$\sup_{\mu\in M_n\cap\{e(D):D\in Q\}^\circ} |\mu|\,(\Omega\backslash(\cup\{A_{km} : 1\le k\le m\le j+1\})) = \infty,$$

for $n=\{n_1,n_2,\cdots,n_j,n_{j+1}\}\cup N_{j+1}$, where N_{j+1} is an infinity subset $N_j\backslash\{n_{j+1}\}$. To finish the induction we define $n_{j+2}=\min\{n : n\in N_{j+1}\}$. □

Theorem 2.10. *Let $\{\mathcal{B}_n : n\in\mathbb{N}\}$ be an increasing covering of a σ-algebra Σ of subsets of a set Ω. There exists a $q\in\mathbb{N}$ such that $\mathcal{B}_n$ is a Nikodým set for $ba(\Sigma)$ for each $n\ge q$.*

Proof. Let's proceed by contradiction and suppose that every $\mathcal{B}_n$ is not a Nikodým set for $ba(\Sigma)$. By Proposition 2.9 for each $(i,j)\in\mathbb{N}^2$, such that $1\le i\le j$, there exists $A_{ij}\in\Sigma$ and $\mu_{ij}\in ba(\Sigma)$ such that

$$\left|\mu_{ij}(e(A_{ij}))\right| > j,\quad \Sigma_{1\le k\le m<j}\left|\mu_{ij}(e(A_{km}))\right|\le 1,$$

the sets A_{ij} are pairwise disjoint and the set of measures $\{\mu_{ij} : j\in\mathbb{N},\ j\ge i\}$ is pointwise bounded in $\mathcal{B}_i$, for each $i\in\mathbb{N}$.

We claim that there exists a sequence $(i_n,j_n)_{n\in\mathbb{N}}$ such that $(i_n)_{n\in\mathbb{N}}$ is the sequence of the first components of the sequence obtained when the elements of $\mathbb{N}^2$ are ordered by the diagonal order, i.e.,

$$(i_1,i_2,i_3,i_4,i_5,i_6,i_7,\cdots)=(1,1,2,1,2,3,1,\cdots),$$

and $(j_n)_{n\in\mathbb{N}}$ is a strict increasing sequence such that for each $n\in\mathbb{N}$

$$\left|\mu_{i_n,j_n}\right|(\cup\{A_{i_m,j_m} : m>n\})\le 1.$$

Let $(i_1,j_1):=(1,1)$, suppose that $\left|\mu_{i_1,j_1}\right|\le k_1$ and split the set $\{j\in\mathbb{N} : j>1\}$ in k_1 infinite subsets $N_{11},\cdots,N_{1k_1}$. At least one of this subsets, named N_1, verifies that

$$\left|\mu_{i_1,j_1}\right|(\cup\{A_{i,j} : i\le j,\ j\in N_1\})\le 1,$$

because

$$k_1\ge\left|\mu_{i_1,j_1}\right|=\Sigma_{1\le r\le k_1}\left|\mu_{i_1,j_1}\right|(\cup\{A_{i,j} : i\le j,\ j\in N_{1r}\}).$$

Then we define $j_2 := \inf\{n : n \in N_1\}$. Suppose that we have obtained the natural number j_n and the infinite subset N_n of $\mathbb{N}$ such that

$$\left|\mu_{i_n,j_n}\right|(\cup\{A_{i,j} : i \leq j, j \in N_n\}) \leq 1.$$

Then we define $j_{n+1} = \inf\{n : n \in N_n\}$ and if $\left|\mu_{i_{n+1},j_{n+1}}\right| \leq k_{n+1}$ we split the set $\{j \in N_n : j > j_{n+1}\}$ in k_{n+1} infinite subsets $N_{n+1,1}, \cdots, N_{n+1,k_{n+1}}$. At least one of this subsets, named N_{n+1} verifies that

$$\left|\mu_{i_{n+1},j_{n+1}}\right|(\cup\{A_{i,j} : i \leq j, j \in N_{n+1}\}) \leq 1$$

because

$$k_{n+1} \geq \left|\mu_{i_{n+1},j_{n+1}}\right| = \Sigma_{1\leq r\leq k_{n+1}} \left|\mu_{i_{n+1},j_{n+1}}\right|(\cup\{A_{i,j} : i \leq j, j \in N_{n+1,r}\}).$$

As $A = \cup\{A_{i_m,j_m} : m \in \mathbb{N}\} \in \Sigma$ there exists $r \in \mathbb{N}$ such that $A \in \mathcal{B}_r$. By construction there exists an increasing sequence $(m_s : s \in \mathbb{N})$ such that each $i_{m_s} = r$, $s \in \mathbb{N}$. Therefore the set of measures $\{\mu_{i_{m_s},j_{m_s}} : s \in \mathbb{N}\} = \{\mu_{r,j_{m_s}} : s \in \mathbb{N}\}$ is pointwise bounded in $\mathcal{B}_r$ and, in particular

$$\sup\{\left|\mu_{i_{m_s},j_{m_s}}(A)\right| : s \in \mathbb{N}\} = \sup\{\left|\mu_{r,j_{m_s}}(A)\right| : s \in \mathbb{N}\} < \infty. \tag{2.8}$$

But from

$$\left|\mu_{i_{m_s},j_{m_s}}(A)\right| = \left|\mu_{i_{m_s},j_{m_s}}\left(\bigcup_{m\in\mathbb{N}} A_{i_m,j_m}\right)\right| \geq \left|\mu_{i_{m_s},j_{m_s}}(A_{i_{m_s},j_{m_s}})\right| - \sum_{1\leq k\leq m<j_{m_s}} \left|\mu_{i_{m_s},j_{m_s}}(A_{km})\right| - \left|\mu_{i_{m_s},j_{m_s}}\right|\left(\bigcup_{m>j_{m_s}} A_{i_m,j_m}\right) > j_{m_s} - 2$$

we get that $\lim_{s\to\infty} \left|\mu_{i_{m_s},j_{m_s}}(A)\right| = \infty$, in contradiction with (2.8). □

3. Strong (VHS) property

Theorem 3.1. *Let $\mathcal{B}$ be a subset of an algebra of sets $\mathcal{A}$. $\mathcal{B}$ has the strong (VHS) property if and only if $\mathcal{B}$ has the (VHS) and the strong (N) properties.*

Proof. Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of $\mathcal{B}$ and suppose that $\mathcal{B}$ has the (VHS) and strong (N) properties. It is obvious that $\mathcal{A}$ has the (VHS) and then $\mathcal{A}$ has property (G). By Proposition 2.1 there exists $p \in \mathbb{N}$ such that such that $\mathcal{B}_p$ has (N) property and $\overline{\text{absco}\{e(A) : A \in \mathcal{B}_p\}}^{\widehat{L(\mathcal{A})}}$ is a neighborhood of 0 in $\widehat{L(\mathcal{A})}$. Let $(\mu_n)_{n=1}^\infty$ be a sequence in $ba(\mathcal{A})$ and $\mu \in ba(\mathcal{A})$ such that $\lim_{n\to\infty} \mu_n(B) = \mu(B)$, for each $B \in \mathcal{B}_p$. Then $\lim_{n\to\infty} \mu_n(f) = \mu(f)$, for each $f \in \text{absco}\{e(A) : A \in \mathcal{B}_p\}$. As $\mathcal{B}_p$ is a Nikodým set for $ba(\mathcal{A})$ then $\{\mu_n : n \in \mathbb{N}\}$ is a bounded subset of $ba(\mathcal{A})$. Then Claim 1.1 implies that

$$\lim_{n\to\infty} \mu_n(f) = \mu(f), \text{ for each } f \in \overline{\text{absco}\{e(A) : A \in \mathcal{B}_p\}}^{\widehat{L(\mathcal{A})}}.$$

So, also $\lim_{n\to\infty} \mu_n(f) = \mu(f)$, for each $f \in \widehat{L(\mathcal{A})}$. Therefore, as $\mathcal{A}$ has property (G) of $\mathcal{A}$ it follows that $(\mu_n)_{n=1}^\infty$ converges weakly to μ, that is, $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for each $f \in (ba(\mathcal{A}))^*$, hence $\mathcal{B}_p$ has (VHS) property and we get that $\mathcal{B}$ has the strong (VHS) property.

The converse is trivial. □

Corollary 3.2. *An algebra of sets $\mathcal{A}$ has the strong (VHS) property if and only if $\mathcal{A}$ has the (VHS) and the strong (N) properties.*

Therefore, the Valdivia open question that if in an algebra $\mathcal{A}$ it is true that (N) property implies (sN) property [18, Problem 1] is equivalent to the following problem.

Problem 3.3. Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be an increasing covering of an algebra $\mathcal{A}$ with (VHS) property. We do not know if there exists a natural number p such that $\mathcal{A}_p$ has (VHS) property.

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

Research supported for the second named author by Grant PGC2018-094431-B-I00 of Ministry of Science, Innovation & Universities of Spain.

References

[1] J. Diestel, *Sequences and Series in Banach Spaces*, Number 92 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1984.

[2] J. Diestel and J. J. Uhl, *Vector Measures*, Number 15 in Mathematical Surveys and Monographs, American Mathematical Society, Providence, 1977.

[3] J. C. Ferrando, *Strong barrelledness properties in certain $l_0^\infty(\mathcal{A})$ spaces*, J. Math. Anal. Appl. **190**, 194-202, 1995.

[4] J. C. Ferrando, S. López-Alfonso and M. López-Pellicer, *On Nikodým and Rainwater sets for ba(R) and a problem of M. Valdivia*, Filomat **33 (8)**, 2409-2416, 2019.

[5] J. C. Ferrando and M. López-Pellicer, *Strong barrelledness properties in $l_0^\infty(x,\mathcal{A})$ and bounded finite additive measures*, J. Math. Anal. Appl. **287**, 727-736, 1990.

[6] J. C. Ferrando, M. López-Pellicer and L. M. Sánchez Ruiz, *Metrizable Barrelled Spaces*, Number 332 in Pitman Research Notes in Mathematics Series, Longman, Harlow, 1995.

[7] J. C. Ferrando and L. M. Sánchez Ruiz, *A survey on recent advances on the Nikodým boundedness theorem and spaces of simple functions*, Rocky Mount. J. Math. **34**, 139-172, 2004.

[8] J. Kakol and M. López-Pellicer, *On Valdivia strong version of Nikodým boundedness property*, J. Math. Anal. Appl. **446**, 1-17, 2017.

[9] G. Köthe, *Topological Vector Spaces I and II*, Springer, Berlin, 1979.

[10] S. López-Alfonso, *On Schachermayer and Valdivia results in algebras of Jordan measurable sets*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110**, 799-808, 2016.

[11] S. López-Alfonso, *Vitali–Hahn–Saks property in coverings of sets algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115**, Paper No 17, 2021.

[12] S. López-Alfonso, J. Mas and S. Moll, *Nikodým boundedness property for webs in σ-algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110**, 711-722, 2016.

[13] S. López-Alfonso and S. Moll, *The uniform bounded deciding property and the separable quotient problem*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**, 1223-1230, 2019.

[14] M. López-Pellicer, *Webs and bounded finitely additive measures*, J. Math. Anal. Appl. **210**, 257-267, 1997.

[15] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, Number 131 in North-Holland Mathematics Studies, Notas de Matemática, North-Holland Publishing Co., Amsterdam, 1987.

[16] W. Schachermayer, *On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras*, Dissertationes Math. (Rozprawy Mat.) **214**, 33 pp., 1982.

[17] M. Valdivia, *On certain barrelled normed spaces*, Ann. Inst. Fourier (Grenoble) **29**, 39-56, 1979.

[18] M. Valdivia, *On Nikodým boundedness property*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107**, 355-372, 2013.