Acoustics in 2D Spaces of Constant Curvature

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Abstract Maximally symmetric spaces play a vital rôle in modelling various physical phenomena. The simplest representative is the 2-sphere $\mathbb{S}^2$, having constant positive curvature. By embedding it into $(2+1)$D spacetime with Lorentzian signature it becomes the prototype of homogeneous and isotropic spacetime of constant curvature with constant scale factor: the Einstein cylinder $\mathbb{R}\times\mathbb{S}^2$. This work outlines a variational approach on how to model acoustic wave propagation on this particular curved spacetime. On the Einstein cylinder, the analytical solutions of the wave equation for the acoustic potential are shown to reduce to solutions of a differential equation of Sturm-Liouville type and simple harmonic time and angular dependence. Moreover, we discuss the implementation of such an underlying curved spacetime within an acoustic metamaterial—an artificially engineered material with remarkable properties exceeding the possibilities found in nature.

1 Introduction

The principal aim of metamaterial research is the theoretical design and conception of artificial materials followed by its industrial engineering. Metamaterials possess remarkable properties which by far exceed the ones found in nature. This makes

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them not only attractive for fundamental research but they will certainly provide useful future applications in all sectors of living.

The theoretical framework for designing advanced devices with acoustic metamaterials belongs to so-called analogue models of gravity [18]. Its idea is to model physical phenomena as close as possible to general relativity without giving up too much of its underlying differential-geometric structure. Obviously classical acoustics is not a relativistic theory—in acoustics there is no such equivalent as a constant speed of light, independent of the observer's reference frame—but certain features can be adapted and carried over with the prime objective to emulate acoustic wave propagation in curved spacetime.

By implementing specific spacetimes in acoustics, transformation acoustics [2, 4, 11, 16] can be used, for example, to design perfect acoustic lenses with unlimited resolution (see *e.g.* [15]), to construct ships and submarines invisible to sonar detection, to improve concert halls, and to devise applications involving acoustic cloaking [2, 5, 14].

Spaces of constant curvature are maximally symmetric spaces (see *e.g.* [19]) which explains their importance in a variety of important applications in physics and engineering, such as *e.g.* the description of uncharged, perfect relativistic fluids [8] and standard cosmological models [6]. Moreover, in the past years, quantum mechanical phenomena in spaces of constant curvature have attracted the focus of intense investigation [12], raising critical fundamental questions beyond their possible experimental verification. However, the simulation of acoustic phenomena [4] in such spaces has so far been vastly neglected.

Among the most significant spaces of constant curvature are the n-spheres embedded in $(n+1)$-dimensional Euclidean space, denoted by $\mathbb{S}^n(K)$, where $K > 0$ is the curvature. The corresponding m-dimensional spacetime consists of the Lorentzian manifold $\mathbb{R} \times \mathbb{S}^{m-1}(K)$. It is the simplest non-Euclidean geometry with elliptic geometry, that is, a homogeneous and isotropic spacetime of Robertson-Walker type with positive curvature and constant scale factor. In this sense, it may be considered as the counterpart of flat Minkowski space for the sphere.

In 2012, we proposed a general framework for transformation acoustics [14] using a variational principle for the acoustic potential in order to model acoustic wave propagation with a curved background space. We also obtained the general constitutive equations, linking a chosen spacetime with the relevant physical parameters for the acoustic metamaterial. So far several applications of interest already have been studied [14–17]. Here we shall examine and implement in transformation acoustics the static spacetime $\mathbb{R} \times \mathbb{S}^2$—sometimes called the *Einstein cylinder* [3].

In the following, we demonstrate how this approach yields a partial differential equation for the acoustic potential, which leads to a classical Sturm-Liouville problem for the radial isotropic coordinates that can be tackled analytically. This will make further investigations and expected wave predictions possible. We also comment on the design and implementation of such spacetime with suitable acoustic metadevices.

2 Variational Principle and Constitutive Equations

Hamilton's or Fermat's variational principle are powerful methods in classical mechanics and optics to describe in a very concise manner the laws which govern the physical phenomena for these respective fields. In this formalism the solutions which extremize the postulated action integral yield the equations of motion that completely determine the system in question. Variational principles owe much of their elegance due to their coordinate-frame independence and because symmetry properties are easily revealed by Noether's theorem. Moreover, in this approach a physical law manifests itself in a self-adjoint differential operator acting on the physical fields in question [9]. As a consequence the equations of motion are typically separable partial differential equations which include a Sturm-Liouville problem for one of the variables (see also [7]). This dramatically simplifies the analytical or semi-analytical treatment of these solutions.

Hamilton's variational principle in transformation acoustics for a smooth spacetime M (with Lorentzian metric $\mathbf{g}$ so that $g = \det \mathbf{g} < 0$) only requires to postulate the explicit form of the Lagrangian density function $\mathscr{L} : M \times TP \to \mathbb{R}$, where P is the ambient space defined by the acoustic potential $\phi : M \to \mathbb{R}$. In a fixed laboratory frame the gradient[1] satisfies [14]

$$\phi_{;\mu} = \phi_{,\mu} = \begin{pmatrix} p/c\rho_0 \\ -\mathbf{v} \end{pmatrix}, \tag{1}$$

where in the acoustic metamaterial $\mathbf{v}$ denotes the local fluid velocity, p the acoustic pressure, and ρ_0 its density. As usual, $c > 0$ is the time-independent acoustic wave speed [10].

In general, the Lagrangian may be a function of x^μ, ϕ, and $\phi_{,\mu}$ as well as higher-order derivatives. However, physical symmetry constraints as energy-momentum conservation, locality and free-wave propagation restrict the acoustic Lagrangian to take the following simplest possible form [14], containing only a covariant kinetic term:

$$\mathscr{L}(\phi_{,\mu}) = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}. \tag{2}$$

Thus, the associated action integral is given by

$$\mathscr{A}[\phi] = \int d\mathrm{vol}_g \mathscr{L}(\phi_{,\mu}) \tag{3}$$

and its functional derivative with respect to the field variable ϕ must vanish [14]:

$$\frac{\delta}{\delta\phi}\mathscr{A}[\phi] = 0 \quad \Rightarrow \quad \frac{\delta}{\delta\phi}\int_\Omega d\mathrm{vol}_g\, g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 0. \tag{4}$$

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[1] Greek tensor indices indicate the full range of spacetime values, whereas Latin will only refer to the spatial values. Comma and semicolon are standard notation for partial and covariant derivatives, respectively. For scalars, partial and covariant derivative are identical.

The integration Ω domain is a bounded, closed set of spacetime and $d\mathrm{vol}_g = \sqrt{-g}\,dx^0 \wedge \ldots \wedge dx^3$ is the invariant volume element (see *e.g.* [13, pp. 14]). Eq. (4) produces the Euler-Lagrange equation for the acoustic potential, which is sufficient to fully describe the dynamics of the acoustic system with underlying spacetime metric **g**.

The constitutive equations describe how to precisely implement the acoustic system in the laboratory—this is *physical space* and denoted by unbarred quantities. *Virtual space*, on the other hand, is the space with known acoustic wave propagation and denoted by barred quantities. The relation of mass-density tensor $\rho = \rho_{ij}\,dx^i \otimes dx^j$ and bulk modulus κ in both transformation spaces are determined by [14]

$$\kappa = \frac{\sqrt{-g}}{\sqrt{-\bar{g}}}\,\bar{\kappa}, \qquad \rho_0 \rho^{ij} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\,\bar{g}^{ij}, \tag{5}$$

where without loss of generality $\bar{\rho}/\rho_0 \equiv 1$. Moreover, for simplicity $\bar{\kappa}$ may be set to unity,

2.1 The Einstein Cylinder and Its Acoustic Implementation

The spacetime line element for the *Einstein cylinder* $\mathbb{R} \times \mathbb{S}^2$ may be written in terms of the solid angle $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$, with constant $a > 0$, so that

$$ds^2 = -c^2 dt^2 + a^2 d\Omega^2. \tag{6}$$

It will be convenient to use isotropic radial coordinates, taking $r = a\sin\vartheta$, in which the line element takes the form:

$$ds^2 = -(c\,dt) \otimes (c\,dt) + \frac{dr}{\sqrt{1 - r^2/a^2}} \otimes \frac{dr}{\sqrt{1 - r^2/a^2}} + (r d\varphi) \otimes (r d\varphi). \tag{7}$$

Then, we may identify the nonholonomic (noncoordinate) basis 1-forms as $\theta^0 = c\,dt, \theta^1 = dr/\sqrt{1 - r^2/a^2}$, and $\theta^2 = r\,d\varphi$. Cartan's structure equations readily yield the only independent curvature 2-form in this frame as $\hat{\Omega}^1{}_2 = \hat{R}^1{}_{212}\,\theta^1 \wedge \theta^2$, where the only independent component of the Riemann tensor is $\hat{R}^1{}_{212} = 1/a^2 = -\hat{R}^2{}_{112}$. All other components vanish.

From this the only non-zero components of the Ricci tensor in the nonholonomic frame are computed as $\hat{R}_{11} = \hat{R}_{22} = 1/a^2$. Therefore the associated Einstein tensor also has only the following non-zero components

$$G_{00} = \hat{G}_{00} = \hat{R}_{00} - \frac{1}{a^2}\,\eta_{00} = \frac{1}{a^2}, \tag{8}$$

a result which equally holds for both frames, nonholonomic or coordinate frame. Comparing $G_{\mu\nu}$ with the stress-energy tensor of a perfect fluid (being shear-free and isotropic), we conclude that an observer falling along a geodesic in spacetime

$\mathbb{R}\times\mathbb{S}^2$ is pressure-free and only has constant mass-energy density $\rho_0 = c^2/8\pi G a^2$, where here c is the speed of light and G is the gravitational constant. It is just this mass-energy distribution which generates $\mathbb{R}\times\mathbb{S}^2$.

For the implementation of the equivalent acoustic space and its relevant physical parameters, we only require the components of metric $\mathbf{g}$ implicit in Eq. (7), namely

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \dfrac{1}{1-r^2/a^2} & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{9}$$

Substituting this metric into the constitutive equations, Eqs. (5), immediately gives the prescription

$$\kappa = \frac{1}{\sqrt{1-r^2/a^2}}, \qquad \rho_0\rho^{ij} = \sqrt{1-r^2/a^2}\begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}. \qquad 0<r<a. \tag{10}$$

2.2 Acoustic Wave Propagation on the Einstein Cylinder

The variational principle, Eq. (4), with underlying metric, Eq. (9), generates the acoustic wave equation as geodesics for field ϕ on the Einstein cylinder $\mathbb{R}\times\mathbb{S}^2$. The associated Euler-Lagrange equation is

$$\Delta_{\mathbb{R}\times\mathbb{S}^2}\phi = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\sqrt{1-r^2/a^2}}{r}\frac{\partial}{\partial r}\left(r\sqrt{1-r^2/a^2}\,\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial\varphi^2} = 0, \tag{11}$$

where $\Delta_{\mathbb{R}\times\mathbb{S}^2}$ is the Laplace-Beltrami operator on manifold $\mathbb{R}\times\mathbb{S}^2$. For the derivation in the general case with pseudo-Riemannian manifold M see [14].

The solution $\phi(t,r,\varphi)$ for wave equation, Eq. (11), displays a harmonic dependence in the time variable t and for azimuthal angle φ. All of the non-trivial behaviour is contained in the radial dependence, as expected. Employing the standard technique, separation of variables [7] yields a solution of the general form

$$\phi(t,r,\varphi) = \phi_1(r)\left[A\cos\left(\sqrt{\lambda}ct\right) + B\sin\left(\sqrt{\lambda}ct\right)\right]\left[C\cos\left(\sqrt{\mu}\,\varphi\right) + D\sin(\sqrt{\mu}\,\varphi)\right] \tag{12}$$

with $$\left(1-\frac{r^2}{a^2}\right)\phi_1'' + \left(\frac{1}{r}-\frac{2r}{a^2}\right)\phi_1' + (\lambda-\mu)\phi_1 = 0, \tag{13}$$

where $\phi_1 = \phi_1(r)$ is the radial function, $\lambda,\mu > 0$ are the harmonic eigenvalues, and $A,\dots,D$ are integration constants. The general analytic solutions of Eq. (13) can be expressed in terms of a hypergeometric series and Meijer-G functions. (see *e.g.* [1]). In practice, however, a semi-numerical approach is advisable, where Eq. (13) is assessed numerically. Of significant interest are also certain limiting cases; *e.g* note that for $r \to a$ the r-dependent solution approaches $\phi_1(r) \sim e^{a(\lambda-\mu)r}$.

3 Conclusions

The acoustic analogue of the Einstein cylinder is a particularly intriguing model for transformation physics as it was the first and simplest cosmological model to be formulated within Einstein's geometric gravity.

Whereas in gravity the Einstein-cylinder world requires space to be filled with a uniform static mass-energy distribution, we have shown that its acoustic pendant requires for its metamaterial implementation a bulk modulus and isotropic density which display a specific radial dependence, *viz.* Eq. (5).

We have also outlined how a covariant variational principle gives rise to the wave equation for the acoustic potential which provides the complete description of acoustic free-wave phenomena with the underlying spacetime of an Einstein cylinder. Although fully analytic solutions are available in terms of hypergeometric series and Meijer-G functions, it might be more practicable to numerically approximate the radial dependence of the potential via the derived second-order linear differential equation of Sturm-Liouville type. Time and azimuthal angular dependence are harmonic and pose no difficulties.

The variational spacetime approach to transformation acoustics supplies a powerful tool for the study and design of acoustic metadevices. It may help to open up new research pathways in this field, overcoming challenges in the engineering of acoustic phenomena with curved background spacetimes.

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