Approximating a Special Class of Linear Fourth-Order Ordinary Differential Problems

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Abstract Differential matrix models are an essential ingredient of many important scientific and engineering applications. In this work, we propose a procedure to approximate the solutions of special linear fourth-order matrix differential problems of the type $Y^{(4)}(x) = A(x)Y(x) + B(x)$ with higher-order matrix splines. An example is included.

1 Introduction

This work presents a spline method for the approximation of a special class of fourth-order ordinary differential equations of the following form

$$Y^{(4)} = A(x)Y(x) + B(x), \quad A(x) \in \mathbb{C}^{r\times r}, \quad Y(x) \in \mathbb{C}^{r\times q}, \quad x \in [a,b], \tag{1}$$

with initial conditions $Y(a) = Y_a, Y'(a) = Y'_a, Y''(a) = Y''_a$ and $Y^{(3)}(a) = Y^{(3)}_a$, in which the first, second, and third derivatives do not appear explicitly. Routinely, fourth-order ordinary differential equations are transformed to a first-order system

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of ordinary differential equations with the hindsight that standard numerical methods may be applied. However, this technique comes with an increase of the computational cost due to the increase of size of the problem. On an alternative pathway, integration methods have attracted the attention of several authors for solving higher-order matrix equations of type (1). These direct methods have demonstrated favourable features in accuracy and speed, see *e.g.* [2,4,6,7] and references therein. Here we elaborate an algorithm to deal with matrix differential equations of the fourth order (1). Throughout this work we will adopt the conventional notation for norms and matrix splines as in previous publications, see e.g. [1]. We denote by $I_{r\times r}$ and $\|A\|$ the identity matrix of dimension r and any multiplicative norm of the matrix A, respectively.

This paper is organized as follows: Section 2 introduces the proposed method and describes all the algorithmic details of the procedure. Section 3 concludes the discussion with a numerical example.

2 Description of the Method

Let us consider the fourth-order matrix problem

$$Y^{(4)}(x) = A(x)Y(x) + B(x), \quad a \le x \le b, \tag{2}$$

where $Y(x) \in \mathbb{C}^{r\times q}$ is the unknown matrix with initial conditions $Y(a) = Y_a, Y'(a) = Y'_a, Y''(a) = Y''_a, Y^{(3)}(a) = Y_a^{(3)} \in \mathbb{C}^{r\times q}$. The matrix-valued functions $A : [a,b] \to \mathbb{C}^{r\times r}$ and $B : [a,b] \to \mathbb{C}^{r\times q}$ are of differentiability class $A, B \in \mathscr{C}^s(I), s \ge 1, I = (a,b)$. We consider the following partition of the interval $[a,b]$:

$$\Delta_{[a,b]} = \left\{a = x_0 < x_1 < \ldots < x_n = b\right\},\ x_k = a + kh,\ k = 0,1,\ldots,n, \tag{3}$$

where n is a positive integer with the corresponding step size $h = (b-a)/n$. We will construct in each subinterval $[a+kh, a+(k+1)h]$ a matrix spline $S(x)$ of order $m \in \mathbb{N}$ with $4 \le m \le s$, where s is the order of the differentiability class of A and B. This will approximate the solution of problem (2) so that $S(x) \in \mathscr{C}^4([a,b])$. In the first interval $[a, a+h]$, we define the matrix spline as

$$\begin{aligned} S_{\big|_{[a,a+h]}}(x) &= Y(a) + Y'(a)(x-a) + \frac{1}{2!}Y''(a)(x-a)^2 + \frac{1}{3!}Y^{(3)}(a)(x-a)^3 \\ &\quad + \cdots + \frac{1}{(m-1)!}Y^{(m-1)}(a)(x-a)^{m-1} + \frac{1}{m!}A_0(x-a)^m, \end{aligned} \tag{4}$$

where $A_0 \in \mathbb{C}^{r\times q}$ is a matrix parameter to be determined. It is straightforward to check

$$S_{\big|_{[a,a+h]}}(a) = Y_a, \quad S'_{\big|_{[a,a+h]}}(a) = Y'_a, \quad S''_{\big|_{[a,a+h]}}(a) = Y''_a, \quad S'''_{\big|_{[a,a+h]}}(a) = Y'''_a,$$

and

$$S^{(4)}_{\big|_{[a,a+h]}}(a) = Y^{(4)}(a) = A(a)Y(a) + B(a)$$

and therefore the spline (4) satisfies the differential equation (2) at $x = a$.

Next we must obtain the values $Y^{(5)}(a), Y^{(6)}(a), \ldots, Y^{(m-1)}(a)$, and A_0 in order to determine the matrix spline (4). It is straightforward to obtain

$$Y^{(5)}(x) = A'(x)Y(x) + A(x)Y'(x) + B'(x) = g_1\left(x, Y(x), Y'(x)\right), \tag{5}$$

where $g_1 \in \mathscr{C}^{s-1}(I)$. This enables us to evaluate $Y^{(5)}(a)$ as

$$Y^{(5)}(a) = g_1\left(a, Y(a), Y'(a)\right) = g_1\left(a, Y_a, Y'_a\right),$$

using (5). Similarly, we can assume that $A, B \in \mathscr{C}^s(T)$ for $s \geq 2$. Then, the second derivatives of A and B exist and are continuous. This gives the sixth derivative $Y^{(6)}(x)$:

$$\begin{aligned} Y^{(6)}(x) &= A''(x)Y(x) + 2A'(x)Y'(x) + A(x)Y''(x) + B''(x) \\ &= g_2\left(x, Y(x), Y'(x), Y''(x)\right) \in \mathscr{C}^{s-2}(I). \end{aligned} \tag{6}$$

Finally we can compute $Y^{(6)}(a) = g_2(a, Y(a), Y'(a), Y''(a)) = g_2(a, Y_a, Y'_a, Y''_a)$ using (6). For all higher-order derivatives $Y^{(7)}(x), \ldots, Y^{(m-1)}(x)$ we proceed analogously to obtain

$$\left.\begin{aligned} Y^{(7)}(x) &= g_3(x, Y(x), Y'(x), Y''(x), Y'''(x)) \in \mathscr{C}^{s-3}(I) \\ &\vdots \\ Y^{(m-1)}(x) &= g_{m-5}\left(x, Y(x), Y'(x), \ldots, Y^{(m-5)}(x)\right) \in \mathscr{C}^{s-(m-5)}(I) \end{aligned}\right\}. \tag{7}$$

These derivatives can be easily computed with standard computer algebra systems. Substituting $x = a$ in (7), one gets $Y^{(7)}(a), \ldots, Y^{(m-1)}(a)$. Now all matrix parameters of the spline which were to be determined are known, except for A_0. To determine A_0, we suppose that (4) is a solution of problem (2) at $x = a + h$, which gives

$$S^{(4)}_{\big|_{[a,a+h]}}(a+h) = A(a+h)\, S_{\big|_{[a,a+h]}}(a+h) + B(a+h). \tag{8}$$

Next, we obtain from (8) the matrix equation with only one unknown A_0:

$$\begin{aligned} A_0 = \frac{(m-4)!}{h^{m-4}} \Bigg[& A(a+h)\left(Y(a) + Y'(a)h + \cdots + \frac{h^{m-1}}{(m-1)!}Y^{(m-1)}(a) + \frac{h^m}{m!}A_0\right) \\ & + B(a+h) - Y^{(4)}(a) - Y^{(5)}(a)h - \cdots - \frac{1}{(m-5)!}Y^{(m-1)}(a)h^{m-5}\Bigg]. \end{aligned} \tag{9}$$

Assuming that the matrix equation (9) has only one solution A_0, the matrix spline (4) is totally determined in the interval $[a,a+h]$. In the next interval $[a+h,a+2h]$, the matrix spline takes the form

$$S_{\big|_{[a+h,a+2h]}}(x) = \sum_{i=0}^{3} \frac{S^{(i)}_{\big|_{[a,a+h]}}(a+h)}{i!}(x-(a+h))^i + \sum_{j=4}^{m-1} \frac{\overline{Y^{(j)}(a+h)}}{j!}(x-(a+h))^j + \frac{A_1}{m!}(x-(a+h))^m, \tag{10}$$

where

$$\begin{aligned} \overline{Y^{(4)}(a+h)} &= A(a+h)\,S_{\big|_{[a,a+h]}}(a+h) + B(a+h), \\ \overline{Y^{(5)}(a+h)} &= g_1\left(a+h, S_{\big|_{[a,a+h]}}(a+h), S'_{\big|_{[a,a+h]}}(a+h)\right), \\ &\vdots \\ \overline{Y^{(m-1)}(a+h)} &= g_{m-5}\left(a+h, S_{\big|_{[a,a+h]}}(a+h), \ldots, S^{(m-5)}_{\big|_{[a,a+h]}}(a+h)\right). \end{aligned} \tag{11}$$

The matrix spline $S(x)$ defined by (4) and (10) is of differentiability class $\mathscr{C}^4([a,a+2h])$. By construction, spline (10) satisfies the differential equation (2) at $x=a+h$, and all of its coefficients are determined with the exception of $A_1 \in \mathbb{C}^{r\times q}$. The value of A_1 can be found by taking the spline (10) as a solution of (2) at point $x=a+2h$:

$$S^{(4)}_{\big|_{[a+h,a+2h]}}(a+2h) = A(a+2h)\,S_{\big|_{[a+h,a+2h]}}(a+2h) + B(a+2h).$$

An expansion yields the matrix equation with the only unknown A_1:

$$A_1 = \frac{(m-4)!}{h^{m-4}}\left[A(a+2h)\left(\sum_{i=0}^{3} \frac{S^{(i)}_{\big|_{[a,a+h]}}(a+h)}{i!}h^i + \sum_{j=4}^{m-1} \frac{\overline{Y^{(j)}(a+h)}}{j!}h^j + \frac{A_1 h^m}{m!}\right) + B(a+2h) - \overline{Y^{(4)}(a+h)} - \overline{Y^{(5)}(a+h)}h - \cdots - \frac{h^{m-5}}{(m-5)!}\overline{Y^{(m-5)}(a+h)}\right]. \tag{12}$$

Assume the matrix equation (12) has only one solution A_1. This way the spline is totally determined within interval $[a+h,a+2h]$. Iterating this process, we can construct the matrix spline approximation taking $[a+(k-1)h,a+kh]$ as the last subinterval. For the subsequent subinterval $[a+kh,a+(k+1)h]$, we define the corresponding matrix spline as

$$S_{\big|_{[a+kh,a+(k+1)h]}}(x) = \sum_{i=0}^{3} \frac{S^{(i)}_{\big|_{[a+(k-1)h,a+kh]}}(a+kh)}{i!}(x-(a+kh))^i + \sum_{j=4}^{m-1} \frac{\overline{Y^{(j)}(a+kh)}}{j!}(x-(a+kh))^j + \frac{A_k}{m!}(x-(a+kh))^m,$$

where

$$\begin{aligned}
\overline{Y^{(4)}(a+kh)} &= A\,(a+kh)\,S_{\big|_{[a+(k-1)h,a+kh]}}(a+kh) + B\,(a+kh)\,,\\
\overline{Y^{(5)}(a+kh)} &= g_1\left(a+kh, S_{\big|_{[a+(k-1)h,a+kh]}}(a+kh), S'_{\big|_{[a+(k-1)h,a+kh]}}(a+kh)\right),\\
&\vdots\\
\overline{Y^{(m-1)}(a+kh)} &= g_{m-5}\left(a+kh, S_{\big|_{[a+(k-1)h,a+kh]}}(a+kh),\ldots, S^{(m-5)}_{\big|_{[a+(k-1)h,a+kh]}}(a+kh)\right).
\end{aligned}$$

With this definition, the matrix spline $S(x) \in \mathscr{C}^4\left([a, a+(k+1)h]\right)$ fulfills the differential equation (2) at point $x = a+kh$. As an additional requirement, we assume that $S_{\big|_{[a+kh,a+(k+1)h]}}(x)$ satisfies (2) at point $x = a+(k+1)h$:

$$S^{(4)}_{\big|_{[a+kh,a+(k+1)h]}}(a+(k+1)h) = A\,(a+(k+1)h)\,S_{\big|_{[a+kh,a+(k+1)h]}}(a+(k+1)h) + B\,(a+(k+1)h)\,,$$

and expanding this expression gives

$$A_k = \frac{(m-4)!}{h^{m-4}}\left[A(a+(k+1)h)\left(\sum_{i=0}^{3} \frac{S^{(i)}_{\big|_{[a+(k-1)h,a+kh]}}(a+kh)}{i!}h^i + \sum_{j=4}^{m-1} \frac{\overline{Y^{(j)}(a+kh)}}{j!}h^j + \frac{A_k}{m!}h^m\right) + B(a+(k+1)h) - \overline{Y^{(4)}(a+kh)} - \cdots - \frac{h^{m-5}}{(m-5)!}\overline{Y^{(m-1)}(a+kh)}\right]. \quad (13)$$

Observe that the final result (13) relates directly to equations (9) and (12), when setting $k=0$ and $k=1$. We will demonstrate that these equations have a unique solution. We rewrite equation (13) in the form

$$\left(I_{r\times r} - \frac{h^4 A(a+(k+1)h)}{m(m-1)(m-2)(m-3)}\right)A_k$$

$$= \frac{(m-4)!}{h^{m-4}}\left[A(a+(k+1)h)\left(\sum_{i=0}^{3} \frac{S^{(i)}_{\big|_{[a+(k-1)h,a+kh]}}(a+kh)}{i!}h^i + \sum_{j=4}^{m-1} \frac{\overline{Y^{(j)}(a+kh)}}{j!}h^j\right)\right.$$

$$+ B(a+(k+1)h) - \overline{Y^{(4)}(a+kh)} - \cdots - \frac{h^{m-5}}{(m-5)!}\overline{Y^{(m-1)}(a+kh)}\Bigg]. \qquad (14)$$

Note also that solubility of equation (14) is guaranteed by showing that the coefficient matrix $\left(I_{r\times r} - \frac{h^4 A(a+(k+1)h)}{m(m-1)(m-2)(m-3)}\right)$ is invertible, for $k = 0,1,\ldots,n-1$. To see this, let us denote

$$M = \max\{\|A(x)\|\,;\, x \in [a,b]\}. \qquad (15)$$

Then, one obtains

$$\left\| I_{r\times r} - \left(I_{r\times r} - \frac{h^4 A(a+(k+1)h)}{m(m-1)(m-2)(m-3)}\right)\right\| \leq \frac{h^4 M}{m(m-1)(m-2)(m-3)}.$$

Taking

$$0 < h < \sqrt[4]{\frac{m(m-1)(m-2)(m-3)}{M}} \qquad (16)$$

according to Lemma 2.3.3 in [3, p.58], it follows that coefficient matrix

$$\left(I_{r\times r} - \frac{h^4 A(a+(k+1)h)}{m(m-1)(m-2)(m-3)}\right)$$

is invertible for $0 \leq k \leq n-1$. Therefore equation (13) has unique solutions A_k for $k = 0,1,\ldots,n-1$, and the matrix spline is completely determined. In summary, we have proved the following theorem:

Theorem 1. *For the fourth-order matrix differential equation (2), let M be the constant defined by (15). We also consider the partition (3) with step size h satisfies (16). Then, the matrix spline $S(x)$ of order $m, 4 \leq m \leq s$ exists in each subinterval $[a+kh, a+(k+1)h]$, $k = 0,1,\ldots,n-1$, as defined in the previous construction and is of class $\mathscr{C}^4([a,b])$.*

Observe that the so constructed splines have a global error of $O(h^{m-1})$, which follows from an analysis similar to Loscalzo and Talbot's work [5].

3 Numerical Example

A suitable benchmark for testing our method is the following scalar problem:

$$y^{(4)}(x) = \left(x^4 - 6x^2 + 3\right) y(x), y(0) = 1, y'(0) = y^{(3)}(0) = 0, y''(0) = -1, 0 \leq x \leq 1 \qquad (17)$$

whose known solution is $y(x) = e^{x^2/2}$. Following [2] and (15), we determine constant $M = \max_{x\in[0,1]}\left\{x^4 - 6x^2 + 3\right\} \leq 3$. For splines of the sixth order ($m = 6$), according to theorem 1 we obtain $h < 3.30975$. We choose $n = 10$ partitions and $h = 0.1$, then condition (16) holds. For the symbolic solutions of the algebraic equations which arise from the algorithm, we use software *Mathematica*. The results are summarized in Table 1. In Table 2, we evaluated the difference between the estimates of our numerical approach and the exact solution. The maximum of these errors are indicated for each subinterval.

Table 1 Approximation for problem (17).

Interval	APPROXIMATION
$[0,0.1]$	$1.-0.5x^2+0.125x^4-0.0207122x^6$
$[0.1,0.2]$	$1.-3.5462\times10^{-8}x-0.499999x^2-0.0000166527x^3+0.125143x^4-0.0006445x^5-0.019517x^6$
$[0.2,0.3]$	$1.-3.196534\times10^{-6}x-0.499955x^2-0.000336338x^3+0.126447x^4-0.00346416x^5-0.0169916x^6$
$[0.3,0.4]$	$1.-0.0000427757x-0.499603x^2-0.00200073x^3+0.130856x^4-0.00967378x^5-0.013358x^6$
$[0.4,0.5]$	$1.00002-0.000260775x-0.498173x^2-0.00699231x^3+0.140637x^4-0.0198766x^5-0.00893112x^6$
$[0.5,0.6]$	$1.00008-0.00102047x-0.494226x^2-0.0179163x^3+0.157619x^4-0.0339389x^5-0.00408475x^6$
$[0.6,0.7]$	$1.00027-0.00297411x-0.48582x^2-0.0371872x^3+0.182447x^4-0.0509828x^5+0.000786428x^6$
$[0.7,0.8]$	$1.00072-0.0069686x-0.471151x^2-0.0658931x^3+0.214024x^4-0.0694961x^5+0.00530606x^6$
$[0.8,0.9]$	$1.00159-0.0136859x-0.449633x^2-0.102634x^3+0.249291x^4-0.0875411x^5+0.00915112x^6$
$[0.9,1.0]$	$1.00296-0.0229897x-0.423195x^2-0.142683x^3+0.2834x^4-0.103027x^5+0.0120793x^6$

Table 2 Approximation error for problem (17).

Interval	$[0,0.1]$	$[0.1,0.2]$	$[0.2,0.3]$	$[0.3,0.4]$	$[0.4,0.5]$
Max. error	9.5148×10^{-11}	3.0342×10^{-9}	2.3307×10^{-8}	9.9968×10^{-8}	3.0769×10^{-7}
Interval	$[0.5,0.6]$	$[0.6,0.7]$	$[0.7,0.8]$	$[0.8,0.9]$	$[0.9,1.0]$
Max. error	7.6477×10^{-7}	1.6374×10^{-6}	3.1402×10^{-6}	5.5327×10^{-6}	9.1126×10^{-6}

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