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Research Article

Common Fixed Point Results on Generalized Weak Compatible Mapping in Quasi-Partial b-Metric Space

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The focus of this paper is to acquaint with generalized condition (B) in a quasi-partial b-metric space and to establish coincidence and common fixed point theorems for weakly compatible pairs of mapping. Additionally, with the background of quasi-partial b-metric space, the outcomes obtained are exemplified to prove the existence and uniqueness of fixed point.

1. Introduction

In the early years of 20^{th} century, the French mathematician Fréchet [1] commenced the concept of metric space, and due to its consequences and practicable implementations, the idea has been enlarged, upgraded, and generalized in different directions. In 1922, Banach [2] introduced the very important Banach contraction principle which holds a remarkable position in the field on nonlinear analysis. One such generalization was established by Künzi et al. [3] known as quasi-partial metric space by Karapinar et al. [4, 5]. In 1993, Czerwik [6] introduced the concept of b-metric space. Later, Gupta and Gautam [7, 8] generalized quasi-partial metric space to quasi-partial b-metric space and proved some fixed point results for such spaces. Several authors [9–18] have already proved the fixed point theorem in metric space, partial metric space [19], quasi-partial metric space, quasi-partial b-metric space [7], and many different spaces. After these classical results, some researchers [20–25] introduced the distinctive concepts and used fixed point theorems to demonstrate the uniqueness of a solution of the equations in different metric spaces such as multivalued contractive type mappings, Reich–Rus–Cirić and Hardy–Rogers contraction mappings, and Chatterjea and cyclic Chatterjea contraction.

In this paper, we have introduced the generalized condition (B) in quasi-partial b-metric space to obtain coincidence and common fixed points. Moreover, some examples are given to exemplify the concept followed up with pictographic grid.

2. Preliminaries

Let us recall some definition.

Definition 1 (see [19]). A partial metric space on a nonempty set X is a function $M\colon X \times X \longrightarrow \mathbb{R}^+$ satisfying

(1) $M(\tau, \upsilon) = M(\upsilon, \tau)$ (symmetry)
(2) if $0 < M(\tau, \tau) = M(\tau, \upsilon) = M(\upsilon, \upsilon)$, then $\tau = \upsilon$ (indistancy implies equality)
(3) $M(\upsilon, \upsilon) \leq M(\tau, \upsilon)$, then $\tau = \upsilon$ (small self-distances)
(4) $M(\tau, \Upsilon) + M(\upsilon, \upsilon) \leq M(\tau, \upsilon) + M(\upsilon, \Upsilon)$ (triangularity)

for all $\tau, \upsilon, \Upsilon \in X$.

Definition 2 (see [4]). A quasi-partial metric on a nonempty set X is a function $q: X \times X \longrightarrow \mathbb{R}^+$ satisfying

(1) If $q(\tau,\tau) = q(\tau,\upsilon) = q(\upsilon,\upsilon)$, then $\tau = \upsilon$ (indistancy implies equality)

(2) $q(\tau,\tau) \leq q(\tau,\upsilon)$ (small self-distances)

(3) $q(\tau,\tau) \leq q(\upsilon,\tau)$ (small self-distances)

(4) $q(\tau,\upsilon) + q(\Upsilon,\Upsilon) \leq q(\tau,\Upsilon) + q(\Upsilon,\upsilon)$ (triangularity)

for all $\tau,\upsilon,\Upsilon \in X$.

Definition 3 (see [20]). A quasi-partial b-metric on a nonempty set X is a function $qp_b: X \times X \longrightarrow \mathbb{R}^+$ such that for some real number $\rho \geq 1$

(1) If $qp_b(\tau,\tau) = qp_b(\tau,\upsilon) = qp_b(\upsilon,\upsilon)$, then $\tau = \upsilon$ (indistancy implies equality)

(2) $qp_b(\tau,\tau) \leq qp_b(\tau,\upsilon)$ (small self-distances)

(3) $qp_b(\tau,\tau) \leq qp_b(\upsilon,\tau)$ (small self-distances)

(4) $qp_b(\tau,\upsilon) + qp_b(\Upsilon,\Upsilon) \leq \rho\{qp_b(\tau,\Upsilon) + qp_b(\upsilon,\Upsilon)\}$ (triangularity)

for all $\tau,\upsilon,\Upsilon \in X$. The infimum over all reals $\rho \geq 1$ satisfying condition (30) is called the coefficient of (X, qp_b) and represented by $R(X, qp_b)$.

Lemma 1 (see [6]). *Let (X, qp_b) be a quasi-partial b-metric space. Then the following hold*:

(1) If $qp_b(\tau,\upsilon) = 0$, then $\tau = \upsilon$

(2) If $\tau \neq \upsilon$, then $qp_b(\tau,\upsilon) > 0$ and $qp_b(\upsilon,\tau) > 0$

Definition 4 (see [6]). Let (X, qp_b) be a quasi-partial b-metric. Then

(1) A sequence $\{\tau_n\} \subset X$ converges to $\tau \in X$ if and only if

$$qp_b(\tau,\tau) = \lim_{n,m\longrightarrow\infty} qp_b(\tau,\tau_n) = \lim_{n,m\longrightarrow\infty} qp_b(\tau_n,\tau). \tag{1}$$

(2) A sequence $\{\tau_n\} \subset X$ is called a Cauchy sequence if and only if

$$\begin{aligned} &\lim_{n,m\longrightarrow\infty} qp_b(\tau_n,\tau_m),\\ &\lim_{n,m\longrightarrow\infty} qp_b(\tau_m,\tau_n)\ \text{exist}. \end{aligned} \tag{2}$$

(3) The quasi-partial b-metric space (X, qp_b) is said to be complete if every Cauchy sequence $\{\tau_n\} \subset X$ converges with respect to x_{qp_b} to a point $\tau \in X$ such that

$$qp_b(\tau,\tau) = \lim_{n,m\longrightarrow\infty} qp_b(\tau_n,\tau_m) = \lim_{n,m\longrightarrow\infty} qp_b(\tau_m,\tau_n). \tag{3}$$

(4) A mapping $f: X \longrightarrow X$ is said to be continuous at $\tau_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(\tau_0,\delta)) \subset B(f(\tau_0),\varepsilon)$.

Lemma 2 (see [6]). *Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.*

Definition 5 (see [26]). A self-mapping P on a metric space (X, d) satisfies condition (B), if there exist $\delta \in [0,1]$ and $\omega > 0$ such that for all $\tau,\upsilon \in X$, we have,

$$d(P\tau,P\upsilon) \leq \delta\, d(P\tau,P\upsilon) + \omega \min\{d(P\tau,P\upsilon), d(\upsilon,P\upsilon), d(\tau,P\upsilon), d(\upsilon,P\tau)\}. \tag{4}$$

Following Babu et al. [26], Abbas et al. [27] and Abbas and Illic [28] extended the concept of condition (B) to a pair of mappings. Abbas et al. [27] called it generalized condition (B), and Abbas and Illic [28] called it generalized almost A-contraction.

Definition 6 (see [27]). Let P and Q be two self-mappings on a metric space (X, d). The mapping Q satisfies generalized condition (B) associated with P if there exist $\delta \in (0,1)$ and $\omega \geq 0$ with $\rho \geq 1$ such that

$$\begin{aligned} d(Q_\tau,Q\upsilon) \leq{}& \delta \max\left\{d(P\tau,P\upsilon), d(P\tau,Q\tau), d(P\upsilon,P\upsilon), \frac{1}{2\rho}\{d(P\tau,Q\upsilon) + d(P\upsilon,Q\tau)\}\right\}\\ &+ \omega \min\{d(P\tau,Q\tau), d(P\upsilon,Q\upsilon), d(P\tau,Q\upsilon), d(P\upsilon,Q\tau)\}. \end{aligned} \tag{5}$$

Clearly condition (B) implies generalized condition (B).

Definition 7 (see [29]). Let P and Q be self-mappings on a set X. A point $x \in X$ is called a coincidence point of P and Q if $Px = Qx = w$, where w is called a point of coincidence of P and Q.

Definition 8 (see [30]). Let X be a nonempty set. Two mappings $P, Q: X \longrightarrow X$ are said to be weakly compatible if they commute at their coincidence point, that is, if $Pu = Qu$ for some $u \in X$, then $PQu = QPu$.

3. Main Results

Definition 9. Let P and R be two self-mappings on a quasi-partial b-metric space (X, qp_b).

The mapping R satisfies generalized condition (B) associated with P (R is a generalized almost P contraction) if there exist $\delta \in (0,1)$, $\rho \geq 1$, and $M \geq 0$ such that for all $\tau, \upsilon \in X$, we have

$$
\begin{aligned}
qp_b(R\tau, R\upsilon) \leq \delta \max\left\{qp_b(P\tau, P\upsilon), qp_b(P\tau, R\tau), qp_b(P\upsilon, R\upsilon), \frac{1}{2\rho}\left(qp_b(R\tau, P\upsilon) + qp_b(P\tau, R\upsilon)\right)\right\} \\
+ M \min\{qp_b(P\tau, R\tau), qp_b(P\upsilon, R\upsilon), qp_b(P\tau, R\upsilon), qp_b(P\upsilon, R\tau)\}.
\end{aligned}
\tag{6}
$$

Definition 10. Let P, Q, R, S be four self-mappings on a quasi-partial b-metric space (X, qp_b).

The pair of mapping (P, R) satisfies generalized condition (B) associated with (Q, S) ((P, R) is generalized almost (Q, S) contraction) if there exist $\delta \in (0,1)$, $\rho \geq 1$ and $M \geq 0$ such that for all $\tau, \upsilon \in X$, we have

$$
\begin{aligned}
qp_b(R\tau, S\upsilon) \leq \delta \max\left\{qp_b(P\tau, Q\upsilon), qp_b(P\tau, R\tau), qp_b(Q\upsilon, S\upsilon), \frac{1}{2\rho}\left(qp_b(R\tau, Q\upsilon) + qp_b(P\tau, S\upsilon)\right)\right\} \\
+ M \min\{qp_b(P\tau, R\tau), qp_b(Q\upsilon, S\upsilon), qp_b(P\tau, S\upsilon), qp_b(Q\upsilon, R\tau)\}.
\end{aligned}
\tag{7}
$$

Theorem 1. *Let P, Q, R, S be four self-mappings on quasi-partial b-metric space (X, qp_b) and if we take the mappings in pair as (P, R) associated with (Q, S) for all $\tau, \upsilon \in X$, $\delta \in (0,1)$, and $M \geq 0$, $\rho \geq 1$ and*

(1) $RX \subset QX$ and $SX \subset PX$

(2) PX or QX is closed

(3) $(1/\rho)(\delta + 2M) < 1$

then the pairs (P, R) and (Q, S) have a coincidence point. Also P, Q, R, S have a unique common fixed point, providing that pairs (P, R) and (Q, S) are weakly compatible.

Proof. Let $\tau^* \in X$. Since $RX \subset QX$ there exists $\tau_0 \in X$ such that $\upsilon_0 = Q\tau_0 = R\tau^*$. Suppose there exists a point $\upsilon_1 \in S\tau_0$ corresponding to the point υ_0. Also since $SX \subset PX$ there exist $\tau_1 \in X$ such that $\upsilon_1 = P\tau_1 = S\tau_0$. Going this way we get a sequence $\{\upsilon_n\} \in X$ as

$$
\begin{aligned}
\upsilon_{2m+1} &= Q\tau_{2m+1} = R\tau_{2m}, \\
\upsilon_{2m+2} &= P\tau_{2m+2} = R\tau_{2m+1}, \\
qp_b(\upsilon_{2m+1}, \upsilon_{2m+2}) &= qp_b(R\tau_{2m}, S\tau_{2m+1}) \\
&\leq \delta \max\{qp_b(P\tau_{2m}, Q\tau_{2m+1}), qp_b(P\tau_{2m}, R\tau_{2m}), qp_b(Q\tau_{2m+1}, S\tau_{2m+1}), \\
&\quad \frac{1}{2\rho}\left(qp_b(R\tau_{2m}, Q\tau_{2m+1}) + qp_b(P\tau_{2m}, S\tau_{2m+1})\right)\Big\} + \\
&\quad M \min\{qp_b(P\tau_{2m}, R\tau_{2m}), qp_b(Q\tau_{2m+1}, S\tau_{2m+1}), \\
&\quad qp_b(P\tau_{2m}, R\tau_{2m+1}), qp_b(Q\tau_{2m+1}, R\tau_{2m})\} \\
&\leq \delta \max\{qp_b(\upsilon_{2m}, \upsilon_{2m+1}), qp_b(\upsilon_{2m}, \upsilon_{2m+1}), qp_b(\upsilon_{2m+1}, \upsilon_{2m+2}), \\
&\quad \frac{1}{2\rho}\left(qp_b(\upsilon_{2m+1}, \upsilon_{2m+1}) + qp_b(\upsilon_{2m}, \upsilon_{2m+2})\right)\Big\} + \\
&\quad M \min\{qp_b(\upsilon_{2m}, \upsilon_{2m+1}), qp_b(\upsilon_{2m+1}, \upsilon_{2m+2}) \\
&= \delta \max\{qp_b(\upsilon_{2m}, \upsilon_{2m+1}), qp_b(\upsilon_{2m+1}, \upsilon_{2m+2})\} + \\
&\quad M \min\{qp_b(\upsilon_{2m}, \upsilon_{2m+2}), qp_b(\upsilon_{2m+1}, \upsilon_{2m+1})\}.
\end{aligned}
\tag{8}
$$

This condition gives 4 cases.

Case 1.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m}, v_{2m+1}). \tag{9}$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m}, v_{2m+2}). \tag{10}$$

which implies

$$\begin{aligned}\rho qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m}, v_{2m+1}) + Mqp_b(v_{2m}, v_{2m+2})\\ &\leq (\delta + M)qp_b(v_{2m}, v_{2m+1}) + \rho Mqp_b(v_{2m+1}, v_{2m+2})\\ &\leq \frac{\delta + M}{(1-M)\rho}(qp_b(v_{2m}, v_{2m+1})).\end{aligned} \tag{11}$$

Let $\mu_1 = ((\delta + M)/(1 - M)\rho)$, $((\delta + 2M)/\rho) < 1$ and $M \geq 0$ then $\mu_1 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_1 qp_b(v_{2m}, v_{2m+1})$.

Case 2.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m}, v_{2m+1}). \tag{12}$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m+1}, v_{2m+1}). \tag{13}$$

which implies,

$$\begin{aligned}qp_b(v_{2m+1}, v_{2m+2}) &\leq \frac{1}{\rho}\{\delta qp_b(v_{2m}, v_{2m+1})\} + Mqp_b(v_{2m}, v_{2m+1})\\ &\leq \frac{\delta + M}{\rho} qp_b(v_{2m}, v_{2m+1}).\end{aligned} \tag{14}$$

Let $\mu_2 = ((\delta + M)/\rho)$, $((\delta + 2M)/\rho) < 1$ then $\mu_2 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_2 qp_b(v_{2m}, v_{2m+1})$.

Case 3.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m+1}, v_{2m+2}). \tag{15}$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m}, v_{2m+2}). \tag{16}$$

which implies

$$\begin{aligned}qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m+1}, v_{2m+2}) + Mqp_b(v_{2m}, v_{2m+2})\\ &\leq \frac{M}{\rho(1 - \delta - M)} qp_b(v_{2m}, v_{2m+1}).\end{aligned} \tag{17}$$

Let $\mu_3 = (M/(\rho(1 - \delta - M)))$, $((\delta + 2M)/\rho) < 1$ then $\mu_3 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_3 qp_b(v_{2m}, v_{2m+1})$.

Case 4.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m+1}, v_{2m+2}). \tag{18}$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m+1}, v_{2m+1}). \tag{19}$$

which implies

$$\begin{aligned}qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m+1}, v_{2m+2}) + Mqp_b(v_{2m+1}, v_{2m+1})\\ &\leq \frac{M}{\rho(1-\delta)} qp_b(v_{2m}, v_{2m+1}).\end{aligned} \tag{20}$$

Let $\mu_4 = (M/(\rho(1 - \delta)))$, $((\delta + 2M)/\rho) < 1$ then $\mu_4 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_4 qp_b(v_{2m+1}, v_{2m+2})$.

Choose $\mu = \max\{\mu_1, \mu_2, \mu_3, \mu_4\} \Rightarrow 0 < \mu < 1$.

$$\Rightarrow qp_b(v_{2m+1}, v_{2m+2}) \leq \mu qp_b(v_{2m}, v_{2m+1}). \tag{21}$$

Using mathematical induction,

$$qp_b(v_m, v_{m+1}) \leq \mu^m qp_b(v^*, v_0), \tag{22}$$

which tends to 0 as m tends to ∞

So, $\{v_m\}$ and its subsequence is convergent

Let PX be closed. Therefore, $\tau \in PX$, that is, there exists $\Upsilon \in X$ such that $\tau = P\Upsilon$, and we need to show $\tau = R\Upsilon$

By definition,

$$qp_b(R\Upsilon, \tau) \leq \frac{\delta + M}{\rho} qp_b(R\Upsilon, \tau), \tag{23}$$

which is a contradiction. Hence,

$$qp_b(R\Upsilon, \tau) = 0 \Rightarrow R\Upsilon = \tau. \tag{24}$$

So, $P\Upsilon = R\Upsilon$, that is, P and R have a coincidence point.

Similarly, Q and S have a coincidence point.

If we also assume QX is closed, then (P, R) and (Q, S) have a coincidence point.

Since (P, R) and (Q, S) are weakly compatible, we can prove there exists a common fixed point for P, Q, R, S by contradiction.

Example 1. Let $X = [0, 4]$ equipped with quasi-partial b-metric $qp_b(\tau, \upsilon) = |\tau - \upsilon| + |\tau|$. Let P, Q, R, S be self-mappings on quasi-partial b-metric defined by

$$P\tau = \begin{cases} \dfrac{\tau}{2}, & \tau \in [0, 2], \\ \dfrac{5}{4}, & \tau \in (2, 4], \end{cases}$$
$$Q\tau = \begin{cases} \dfrac{3\tau}{2}, & \tau \in [0, 2], \\ \dfrac{3}{2}, & \tau \in (2, 4] \end{cases}$$
$$R\tau = \begin{cases} \dfrac{\tau}{6}, & \tau \in [0, 2], \\ \dfrac{1}{2}, & \tau \in (2, 4] \end{cases} \tag{25}$$
$$S\tau = \begin{cases} \dfrac{\tau}{4}, & \tau \in [0, 2] \\ \dfrac{1}{4}, & \tau \in (2, 4]. \end{cases}$$

Here,

$$PX = \left[0, \frac{1}{2}\right] \cup \left\{\frac{5}{4}\right\}$$
$$QX = \left[0, \frac{3}{2}\right],$$
$$SX = \left[0, \frac{1}{4}\right] \subset PX \tag{26}$$
$$RX = \left[0, \frac{1}{6}\right] \cup \left\{\frac{1}{2}\right\}.$$

The point 0 is a coincidence point of these mapping. Furthermore, $PR0 = RP0 = 0$ and $SQ0 = QS0 = 0$, that is, the two pairs (P, R) and (Q, S) are weakly compatible.

Case 1. For $\tau, \upsilon \in [0, 2]$, we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{\tau}{6} - \frac{\upsilon}{4}\right| + \left|\frac{\tau}{6}\right|\right\} \leq \frac{8}{5}\left\{\left|\frac{\tau}{2} - \frac{3\upsilon}{2}\right| + \left|\frac{\tau}{2}\right|\right\}. \tag{27}$$

Dominance of right-hand side of equation (27) is easily visually checked in Figure 1. Thus the inequality required in Definition 10 holds for $\tau, \upsilon \in [0, 2]$.

Case 2. For $\tau \in [0, 2]$, $\upsilon \in [2, 4]$ we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{\tau}{6} - \frac{1}{4}\right| + \left|\frac{\tau}{6}\right|\right\} \leq \frac{8}{5}\left\{\left|\frac{3}{2} - \frac{1}{4}\right| + \left|\frac{3}{2}\right|\right\}. \tag{28}$$

Dominance of right-hand side of equation (28) is easily visually checked in Figure 2. Thus the inequality required in Definition 10 holds for $\tau \in [0, 2], \upsilon \in [2, 4]$.

Case 3. For $\tau \in [2, 4]$, $\upsilon \in [0, 2]$ we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{1}{2} - \frac{\upsilon}{4}\right| + \left|\frac{1}{2}\right|\right\} \leq \frac{8}{5}\left\{\left|\frac{5}{4} - \frac{3\upsilon}{2}\right| + \left|\frac{5}{4}\right|\right\}. \tag{29}$$

Dominance of right-hand side of equation (29) is easily visually checked in Figure 3. Thus the inequality required in Definition 10 holds for $\tau \in [2, 4], \upsilon \in [0, 2]$.

Case 4. For $\tau, \upsilon \in [2, 4]$, we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{1}{2} - \frac{1}{4}\right| + \left|\frac{1}{2}\right|\right\} \leq \frac{8}{5}\left\{\left|\frac{3}{2} - \frac{1}{4}\right| + \left|\frac{3}{2}\right|\right\}. \tag{30}$$

Dominance of right-hand side of equation (30) is easily visually checked in Figure 4. Thus the inequality required in Definition 10 holds for $\tau, \upsilon \in [2, 4]$.

As a result, all postulates of Theorem 1 are satisfied ($\delta = (4/5), \rho = 2 \geq 1$, and $M = 0$) and 0 is a unique common fixed point of P, Q, R, S.

If $P = Q$ and $R = S$, we get a corollary.

Corollary 1. *Let P and S be self-mappings on quasi-partial b-metric space (X, qp_b). If for all $\tau, \upsilon \in X$, P satisfies the following conditions*:

(1)$SX \subset PX$

(2) PX is closed

(3) $((\delta + 2M)/\rho) < 1$

then P and S have a coincidence point. Also P and S have a common fixed point if (P, S) are weakly compatible.

Proof. Taking $P = Q$ and $R = S$ in Theorem 1, the above result can be obtained.

Theorem 2. *Let P, Q, R, S be self-mappings on a quasi-partial b-metric space (X, qp_b). If the pair (P, R) is associated with (Q, S) and satisfies*

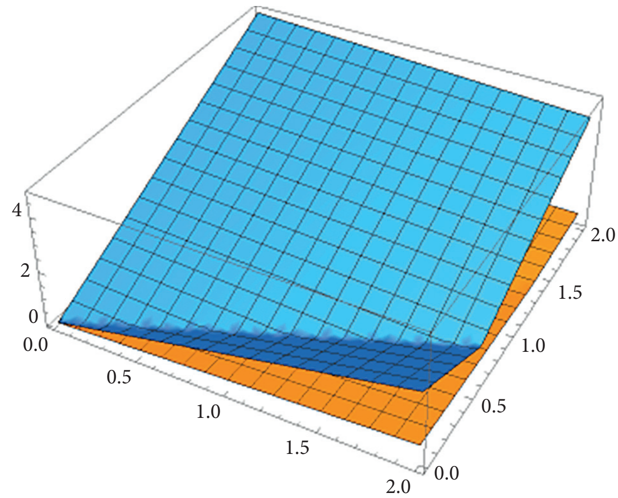

Figure 1: Dominance of right-hand side of equation (27) is visually checked for $\tau, \upsilon \in [0, 2]$.

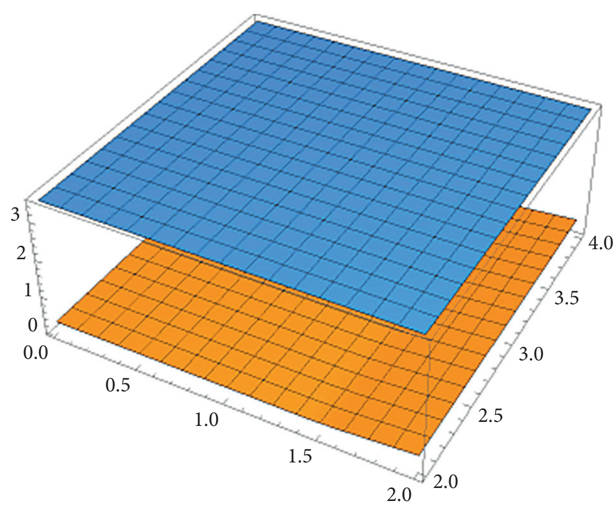

Figure 2: Dominance of right-hand side of equation (28) is visually checked for $\tau \in [0, 2], \upsilon \in [2, 4]$.

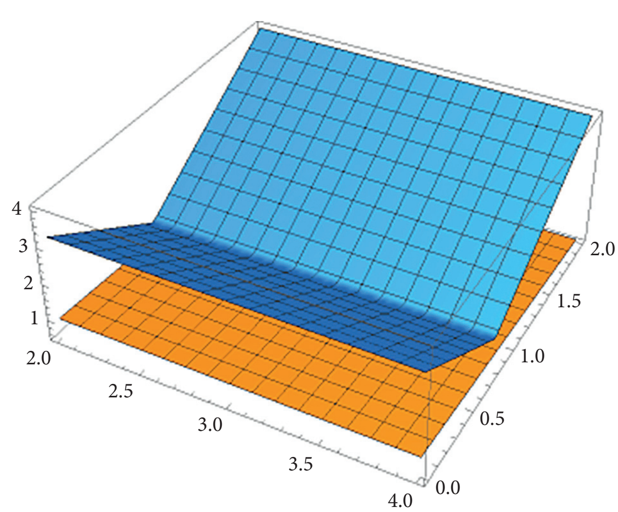

Figure 3: Dominance of right-hand side of equation (29) is visually checked for $\tau \in [2, 4], \upsilon \in [0, 2]$.

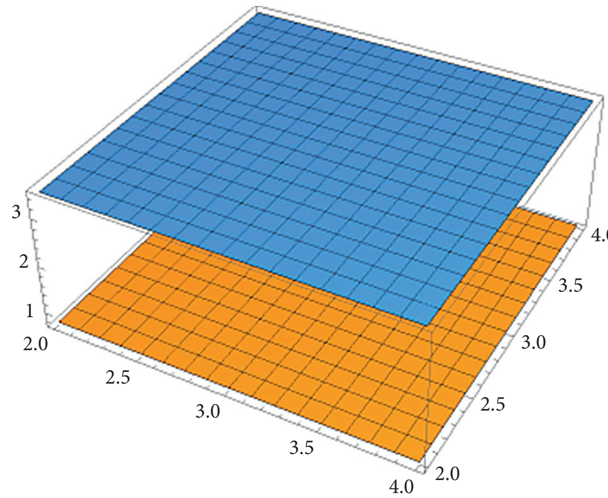

Figure 4: Dominance of right-hand side of equation (30) is visually checked for $\tau, \upsilon \in [2,4]$.

$$qp_b(R\tau, S\upsilon) \le \delta\{\max(qp_b(P\tau, R\tau), qp_b(P\tau, S\upsilon), qp_b(Q\upsilon, S\upsilon), qp_b(P\tau, S\upsilon), qp_b(R\tau, Q\upsilon))\} \\ + M\min\{qp_b(P\tau, R\tau), qp_b(Q\upsilon, S\upsilon), qp_b(P\tau, S\upsilon), qp_b(Q\upsilon, R\tau)\}, \tag{31}$$

with $\delta \in (0, 1)$, $M \ge 0$, and $p \ge 1$, for all τ, $\upsilon \in X$, and

(1) $SX \subset PX$ and $RX \subset QX$

(2) $((\delta + 2M)/\rho) < 1$

then the pairs (P, R) and (Q, S) have a coincidence point. Also P, Q, R, S have a common fixed point.

Proof. This can be done following the same steps as the proof of Theorem 1.

Example 2. Let $X = [0, \infty)$ equipped with quasi-partial b-metric $qp_b(\tau, \upsilon) = |\tau - \upsilon| + |\tau|$. Let P, Q, R, S be self-mappings on quasi-partial b-metric defined by

$$P\tau = \begin{cases} 2\tau, & \tau \in [0,2], \\ 4, & \tau > 2, \end{cases}$$
$$Q\tau = \begin{cases} \tau, & \tau \in [0,2], \\ 2, & \tau > 2, \end{cases}$$
$$R\tau = \begin{cases} \dfrac{\tau}{5}, & \tau \in [0,2], \\ 2, & \tau > 2, \end{cases} \tag{32}$$
$$S\tau = \begin{cases} \dfrac{2\tau}{5}, & \tau \in [0,2], \\ 1, & \tau > 2. \end{cases}$$

Here,

$$SX = \left[0, \frac{2}{5}\right] \cup \{1\} \cup [0,2] \cup \{4\} = PX, \\ RX = \left[0, \frac{1}{5}\right] \cup \{2\} \subset [0,1] \cup \{2\} = QX. \tag{33}$$

The point 0 is a coincidence point of these mapping. Furthermore, $PR0 = RP0 = 0$ and $SQ0 = QS0 = 0$, that is, the two pairs (P, R) and (Q, S), are weakly compatible.

Case 1. For $\tau, \upsilon \in [0,2]$, we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{\tau}{5} - \frac{2\upsilon}{5}\right| + \left|\frac{\tau}{5}\right|\right\} = \frac{1}{5}\{|4\tau - 2\upsilon| + |2\tau|\} \\ \le \frac{10}{9}\{|2\tau - \upsilon| + |2\tau|\}. \tag{34}$$

Dominance of the right-hand side of equation (34) is easily visually checked in Figure 5. Thus the inequality required in theorem holds for $\tau, \upsilon \in [0,2]$.

Case 2. For $\tau \in [0,2]$, $\upsilon > 2$ we have

$$Q(R\tau, S\upsilon) = \left\{\left|\frac{\tau}{5} - 1\right| + \left|\frac{\tau}{5}\right|\right\} \le \frac{10}{9}\{|2 - 1| + |2|\}. \tag{35}$$

Dominance of the right-hand side of equation (35) is easily visually checked in Figure 6. Thus the inequality required in theorem holds for $\tau \in [0,2], \upsilon > 2$.

Case 3. For $\tau > 2, \upsilon \in [0,2]$, we have

$$Q(R\tau, S\upsilon) = \left\{\left|2 - \frac{2\upsilon}{5}\right| + |2|\right\} \le \frac{10}{9}\left\{\left|4 - \frac{2\upsilon}{5}\right| + |4|\right\}. \tag{36}$$

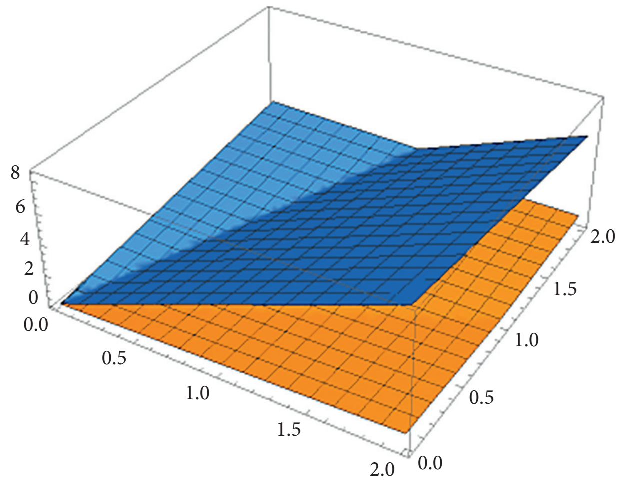

Figure 5: Dominance of the right-hand side of equation (34) is visually checked for $\tau, \upsilon \in [0, 2]$.

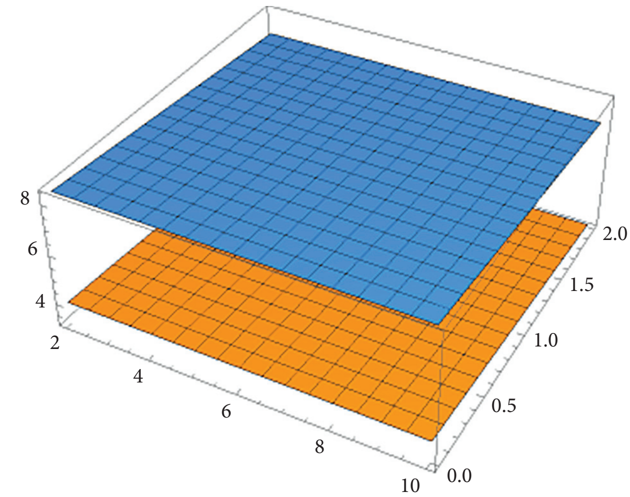

Figure 7: Dominance of the right-hand side of equation (36) is visually checked for $\tau > 2,\ \upsilon \in [0, 2]$.

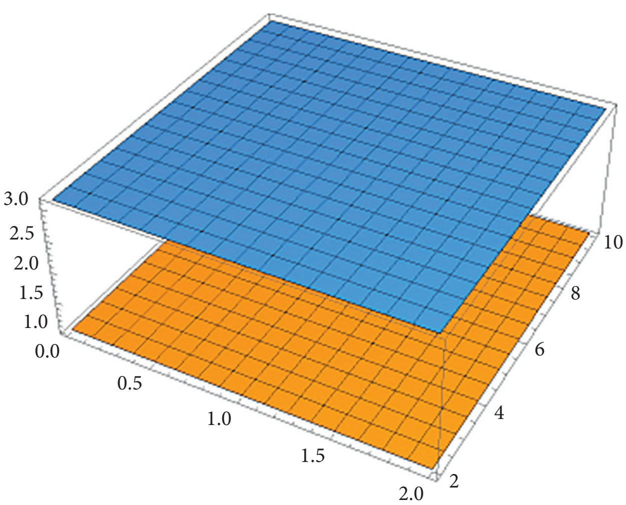

Figure 6: Dominance of the right-hand side of equation (35) is visually checked for $\tau \in [0, 2], \upsilon > 2$.

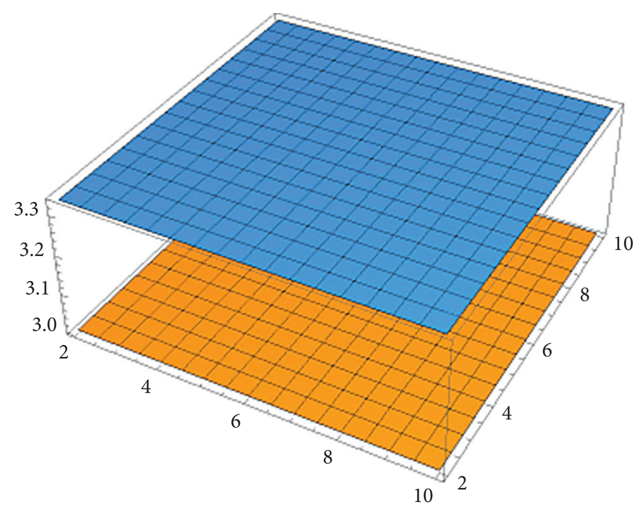

Figure 8: Dominance of the right-hand side of equation (37) is visually checked for $\tau, \upsilon > 2$.

Dominance of the right-hand side of equation (36) is easily visually checked in Figure 7. Thus the inequality required in theorem holds for $\tau > 2,\ \upsilon \in [0, 2]$.

Case 4. For $\tau, \upsilon > 2$, we have

$$Q(R\tau, S\upsilon) = \{|2 - 1| + |2|\} \leq \frac{30}{9}. \tag{37}$$

Dominance of the right-hand side of equation (37) is easily visually checked in Figure 8. Thus the inequality required in theorem holds for $\tau, \upsilon > 2$.

As a result, all postulates of Theorem 2 are satisfied ($\delta = (5/9), \rho = 2 \geq 1$, and $M = 0$), and 0 is a unique common fixed point of P, Q, R, S.

If $P = Q$ and $R = S$, we get a corollary.

Corollary 2. *Let P and S be self-mappings on quasi-partial b-metric space (X, qp_b). If for all $\tau, \upsilon \in X$, the pair of mapping (P, S) satisfies*

$$\begin{aligned} qp_b(R\tau, S\upsilon) \leq \delta\{&\max(qp_b(P\tau, R\tau), qp_b(P\tau, S\upsilon), qp_b(Q\upsilon, S\upsilon), qp_b(P\tau, S\upsilon), qp_b(R\tau, Q\upsilon))\} \\ &+ M\min\{qp_b(P\tau, R\tau), qp_b(Q\upsilon, S\upsilon), qp_b(P\tau, S\upsilon), qp_b(Q\upsilon, R\tau)\}, \end{aligned} \tag{38}$$

and P satisfies the following conditions:

(1) $SX \subset PX$

(2) $((\delta + 2M)/\rho) < 1$

then the pair (P, S) *has a coincidence point. Also P and S have a common fixed point if* (P, S) *are weakly compatible.*

Proof. Taking $P = Q$ and $R = S$ in Theorem 2, the above result can be obtained.

4. Conclusion

This paper expounds a new notion in quasi-partial b-metric space which is generalized condition (B) that helped to demonstrate coincidence and common fixed point for two weakly compatible pairs of self-mappings. The incentive behind using quasi-partial b-metric space is the fact that the distance from point x to point y may be different to that from y to x, and the self-distance of a point need not always be zero; also the distance between two points x and z is not equal to the sum of the two distances having a point y in between x and z. Furthermore, the results acquired are validated by explanatory examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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