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Recurrence relations for a family of iterations assuming Hölder continuous second order Fréchet derivative

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Abstract: The semilocal convergence using recurrence relations of a family of iterations for solving nonlinear equations in Banach spaces is established. It is done under the assumption that the second order Fréchet derivative satisfies the Hölder continuity condition. This condition is more general than the usual Lipschitz continuity condition used for this purpose. Examples can be given for which the Lipschitz continuity condition fails but the Hölder continuity condition works on the second order Fréchet derivative. Recurrence relations based on three parameters are derived. A theorem for existence and uniqueness along with the error bounds for the solution is provided. The R-order of convergence is shown to be equal to $3 + q$ when $\theta = \pm 1$; otherwise it is $2 + q$, where $q \in (0, 1]$. Numerical examples involving nonlinear integral equations and boundary value problems are solved and improved convergence balls are found for them. Finally, the dynamical study of the family of iterations is also carried out.

Keywords: dynamical systems; Hammerstein integral equation; Hölder condition; Lipschitz condition; semilocal convergence.

Mathematics Subject Classication (2000): 65G49; 47H99.

1 Introduction

Let X and Y be Banach spaces and consider solving

$$F(x) = 0 \tag{1.1}$$

where $F: \Omega \subseteq X \to Y$ be a nonlinear operator in an open convex domain $\Omega_0 \subseteq \Omega$ of X with values in Y. This problem is important in numerical analysis from the practical point of view. There exists a large number of applications leading to (1.1) depending on one or many parameters. The dynamical systems, elasticity and many other areas involve boundary value problems, partial differential equations, difference, differential and integral equations whose solutions are obtained by solving (1.1). Many optimization problems also require

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solving these equations. It is studied extensively in literature and several researchers [1–4] have significantly contributed to developing many diverse methods, both analytic and iterative, for them. The frequently used convergence analyses are the local [5–9] and the semilocal [10–13] using information around the solution and around the initial point, respectively. Other important problems also to be considered are the R-order of convergence and the domains of the convergence balls. In general, an attempt is made by considering additional hypothesis so that the convergence radii is enlarged as far as possible. The most widely used quadratically convergent Newton's method to solve (1.1) is defined for $k = 0, 1, 2, \ldots$, by

$$x_{k+1} = x_k - \Gamma_k F(x_k) \tag{1.2}$$

where $\Gamma_k = F'(x_k)^{-1}$ and x_0 are the initial points. The sufficient conditions for the semilocal convergence, the error estimates and existence-uniqueness regions for solutions for it are given by Kantorovich theorem [14]. Two types of proofs are considered. The first one uses majorizing functions and generates a sequence of iterates by the iteration in Banach space which is majorized by a sequence of iterates generated by the same iteration applied to a scaler function. The other is based on recurrence relations which has advantages, as the initial problem in Banach space can be reduced to a simpler problem with real sequences and vectors. Multi-points higher order iterative methods and their convergence analysis along with the error bounds are also used to solve (1.1). These methods require higher computational cost. Another drawback of them is the involvement of higher order derivatives which are either unbounded or difficult to compute at times. Also, their convergence domains are small. However, these methods are also required for some applications needing faster convergence, for example, those involving stiff systems of equations. Also, the second order Fréchet derivative used for the solutions of integral equations is diagonal by blocks and inexpensive. For quadratic equations, it is constant.

The semilocal convergence analysis of third order deformed Halley's method for solving (1.1) in Banach spaces is discussed in [15] under Hölder continuity conditions on the second order Fréchet derivative of the involved operator. The semilocal convergence of a Newton-like method of third order for solving (1.1) in Banach spaces is studied in [16, 17] under the Lipschitz and the Hölder continuity conditions on the second order Fréchet derivative of the involved operator. A family of the deformed Chebyshev iterative method for solving (1.1) and its semilocal convergence analysis under the Hölder continuity condition on the second order Fréchet derivative of the involved operator in Banach spaces is considered in [18]. Recently, the semilocal convergence analysis of a family of iterative methods for solving (1.1) in Banach space setting is described in [10] under the Lipschitz continuity condition on the second order Fréchet derivative of the involved operator. It is given for $k = 0, 1, 2, \ldots$, by

$$\begin{aligned} y_k &= x_k - \theta\, \Gamma_k F(x_k), \\ z_k &= y_k - \Gamma_k F(y_k), \\ x_{k+1} &= z_k - \Gamma_k F(z_k), \end{aligned} \tag{1.3}$$

where $\theta \in \mathbb{R} - \{0\}$ and x_0 are the starting points. For $\theta = \pm 1$, it leads to fourth order iterative methods. For other values of $\theta \in R$, it gives third order iterative methods. The novelty of this work is that it is applicable to many problems even if the Lipschitz condition on F'' fails. The following example illustrates this.

Example 1. Consider the integral equation

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+q}\, dt, \tag{1.4}$$

with $x, f \in C[0, 1]$, $s \in [0, 1]$, $q \in (0, 1)$ and $\lambda \in R$.

Under sup-norm, we get

$$\|F''(x) - F''(y)\| \leq |\lambda|(1+q)(2+q)\log\, 2\|x-y\|^q, \quad x, y \in \Omega.$$

Clearly, Lipschitz condition on F'' fails for $q \in (0,1)$. However, the Hölder condition works for it.

The aim of this paper is to establish the semilocal convergence using recurrence relations of a family of iterations for solving nonlinear equations in Banach spaces. It is done under the assumption that the second order Fréchet derivative satisfies the Hölder continuity condition. This condition is more general than the usual Lipschitz continuity condition used for this purpose. Examples can be given for which the Lipschitz continuity condition fails but the Hölder continuity condition works on the second order Fréchet derivative. Recurrence relations based on three parameters are derived. A theorem for existence and uniqueness along with the error bounds for the solution is provided. The R-order of convergence is shown to be equal to $3+q$ when $\theta = \pm 1$; otherwise it is $2+q$, where $q \in (0,1]$. Numerical examples involving nonlinear integral equations and boundary value problems are solved and improved convergence balls are found for them. Finally, the dynamical study of the family of iterations is also carried out.

The paper is organized as follows. Section 1 is the introduction. In Section 2, the semilocal convergence analysis of a family of iterative methods in Banach spaces is established under Hölder condition on second order Fréchet derivative. The theorems for existence and uniqueness are established for the solution along with error bounds. In Section 3, examples of different types are solved to demonstrate the applicability of our approach. The dynamical study is carried out in Section 4. Finally, conclusions are included in Section 5.

2 Semilocal convergence analysis

In this section, the semilocal convergence analysis of (1.3) is discussed. First, we shall show some preliminary results involving real sequences and their properties. Using these sequences, we shall establish the recurrence relations used in the convergence analysis. Next, an existence-uniqueness theorem and error bounds for the solution are provided.

2.1 Preliminary results

Let $\Gamma_0 = F'(x_0)^{-1} \in \mathrm{BL}(Y,X)$ exists at $x_0 \in \Omega$, where $\mathrm{BL}(Y,X)$ denotes the set of bounded linear operators from Y to X and the following conditions hold.

(1) $\|\Gamma_0\| \le \beta_0$

(2) $\|\Gamma_0 F(x_0)\| \le \eta_0$

(3) $\|F''(x)\| \le L_1$

(4) $\|F''(x) - F''(y)\| \le L_2 \|x-y\|^q, x, y \in \Omega, q \in (0,1]$.

Let $c_0 = L_1\beta_0\eta_0$, $d_0 = L_2\beta_0\eta_0^{1+q}$ and define the real sequences $\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ for $k = 0,1,2\dots$, by

$$c_{k+1} = c_k g_\theta(c_k)^2 h_\theta(c_k, d_k), \tag{2.1}$$

$$d_{k+1} = d_k g_\theta(c_k)^{2+q} h_\theta(c_k, d_k)^{1+q}, \tag{2.2}$$

$$\eta_{k+1} = \eta_k g_\theta(c_k) h_\theta(c_k, d_k), \tag{2.3}$$

where,

$$g_\theta(t) = \frac{1}{1 - t\varphi_\theta(t)}, \tag{2.4}$$

$$\varphi_\theta(t) = \left(1 + \frac{t}{2}\theta^2\right) + \phi_\theta(t), \quad \phi_\theta(t) = \frac{t}{2}\zeta_\theta^2(t) + |\theta| t \zeta_\theta(t), \quad \zeta_\theta(t) = \left(|1-\theta| + \frac{1}{2}\theta^2 t\right) \tag{2.5}$$

and

$$h_\theta(t,u) = \left(\left(t + \frac{t^2}{2}\theta^2\right)\phi_\theta(t) + \frac{t}{2}\phi_\theta(t)^2 + \frac{u}{(1+q)(2+q)}\phi_\theta(t)^{2+q}\right). \tag{2.6}$$

Let $v_\theta(t) = \varphi_\theta(t)t - 1$. Since $v_\theta(0) = -1$ and $\varphi_\theta(t)$ are increasing, $v_\theta(t)$ has a real root α. If $t \in (0, \alpha)$, we get $\varphi_\theta(t)t < 1$.

Lemma 1. *$g_\theta(t)$, $\varphi_\theta(t)$ and $h_\theta(t, u)$ are given by (2.4)–(2.6) respectively. If $0 < c_0 < \alpha$ and $g_\theta(c_0)^2 h_\theta(c_0, d_0) < 1$, then*

(i) *$g_\theta(t)$ and $\varphi_\theta(t)$ are the increasing functions and $g_\theta(t) > 1$, $\varphi_\theta(t) > 1$ for $t \in (0, \alpha)$.*
(ii) *For $u > 0$, $h_\theta(t, u)$ is an increasing function of t, for $t \in (0, \alpha)$.*
(iii) *$\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ are decreasing sequences and $c_k \varphi_\theta(c_k) < 1$ as well as $g_\theta(c_k)^2 h_\theta(c_k, d_k) < 1$ for $k \geq 0$.*

Proof. The proof of (i) and (ii) is obvious. The proof of (iii) can be given as follows. For $k = 0$, (2.1) gives $c_1 = c_0 g_\theta(c_0)^2 h_\theta(c_0, d_0) < c_0$. Similarly, using (2.2) and (2.3), we get $d_1 = d_0 g_\theta(c_0)^{2+q} h_\theta(c_0, d_0)^{1+q} < d_0(g_\theta(c_0)^2 h_\theta(c_0, d_0))^{1+q} < d_0$ and $\eta_1 = g_\theta(c_0) h_\theta(c_0, d_0)\eta_0 < \eta_0$. As, $g_\theta(c_0)^2 h_\theta(c_0, d_0) < 1$ and $g_\theta(c_0) > 1$, we get $g_\theta(c_0) h_\theta(c_0, d_0) < 1$. From $v_\theta(t)$, we get $c_0 \varphi_\theta(c_0) < 1$. Thus, (iii) holds for $k = 0$. Let it hold for some $k = n$. Since $g_\theta(t)$ and $\varphi_\theta(t)$ are the increasing functions and following in a similar manner, it is easy to show that it also holds for $k = m + 1$. Thus, by mathematical induction, (iii) holds $\forall k \geq 0$.

Lemma 2. *$\phi_\theta(t)$ and $h_\theta(t, u)$ are given by* (2.4) and (2.6), *respectively. If $\gamma \in (0, 1)$ then $\phi_\theta(\gamma t) < \gamma_\theta \phi(t)$ and $h_\theta(\gamma t, \gamma^{1+q} u) < \gamma^{1+q} h_\theta(t, u)$ $\forall\ \theta \in \mathbb{R}$. For $\theta = 1$, we have $\phi_1(\gamma t) < \gamma^2 \phi_1(t)$ and $h_1(\gamma t, \gamma^{1+q} u) < \gamma^{2+q} h_1(t, u)$.*

Proof. The proof is omitted here as it is trivial.

Lemma 3. *Let $\delta = g_\theta(c_0)^2 h_\theta(c_0, d_0)$, $0 < c_0 < \alpha$ and $\Delta = \frac{1}{g_\theta(c_0)}$. From Lemma 2, we have*

(i) $c_k \leq \delta^{(2+q)^{k-1}} c_{k-1} \leq \delta^{\frac{(2+q)^k - 1}{1+q}} c_0$ *and* $d_k \leq \left(\delta^{(2+q)^{k-1}}\right)^{1+q} d_{k-1} \leq \delta^{(2+q)^k - 1} d_0$.
(ii) $g_\theta(c_k) h_\theta(c_k, d_k) \leq \delta^{(2+q)^k} \Delta\ \forall\ k \in N$.
(iii) $\eta_k \leq \delta^{\frac{(2+q)^k - 1}{1+q}} \Delta^k \eta_0$ *and* $\sum_{n=k}^{k+m-1} \eta_n \leq \Delta^k \delta^{\frac{(2+q)^k - 1}{1+q}} \frac{1 - (\Delta \delta^{(2+q)^k})^m}{1 - \Delta \delta^{(2+q)^k}} \eta_0$.

For $\theta = 1$, the following results hold.

(i″) $c_k \leq \delta^{(3+q)^{k-1}} c_{k-1} \leq \delta^{\frac{(3+q)^k - 1}{2+q}} c_0$ *and* $d_k \leq \left(\delta^{(3+q)^{k-1}}\right)^{1+q} d_{k-1} \leq \left(\delta^{\frac{(3+q)^k - 1}{2+q}}\right)^{1+q} d_0$.
(ii″) $g(c_k) h(c_k, d_k) \leq \delta^{(3+q)^k} \Delta\ \forall\ k \in N$.
(iii″) $\eta_k \leq \delta^{\frac{(3+q)^k - 1}{2+q}} \Delta^k \eta_0$ *and* $\sum_{n=k}^{k+m-1} \eta_n \leq \Delta^k \delta^{\frac{(3+q)^k - 1}{2+q}} \frac{1 - (\Delta \delta^{(3+q)^k})^m}{1 - \Delta \delta^{(3+q)^k}} \eta_0$.

Proof. Taking $k = 0$ in (2.1) and (2.2), we get $c_1 = c_0 g_\theta(c_0)^2 h_\theta(c_0, d_0) \leq \delta c_0$ and $d_1 = d_0 g_\theta(c_0)^{2+q} h_\theta(c_0, d_0)^{1+q} \leq d_0 \left(g_\theta(c_0)^2 h_\theta(c_0, d_0)\right)^{1+q} = \delta^{1+q} d_0$. Thus, (i) holds for $k = 0$. Assume that (i) holds for $n < k$. If $\theta \neq 1$, then we have

$$\begin{aligned} c_k &= c_{k-1} g_\theta(c_{k-1})^2 h_\theta(c_{k-1}, d_{k-1}), \\ &\leq \delta^{(2+q)^{k-2}} c_{k-2} g_\theta(\delta^{(2+q)^{k-2}} c_{k-2})^2 h_\theta(\delta^{(2+q)^{k-2}} c_{k-2}, (\delta^{(2+q)^{k-2}})^{1+q} d_{k-2}), \\ &\leq \delta^{(2+q)^{k-2}} c_{k-2} g_\theta(c_{k-2})^2 \left(\delta^{(2+q)^{k-2}}\right)^{1+q} h_\theta(c_{k-2}, d_{k-2}), \\ &\leq \left(\delta^{(2+q)^{k-2}}\right)^{(2+q)} c_{k-2} g_\theta(c_{k-2})^2 h_\theta(c_{k-2}, d_{k-2}), \\ &\leq \delta^{(2+q)^{k-1}} c_{k-1}. \end{aligned} \tag{2.7}$$

Proceeding in this way, we get

$$\begin{aligned} c_k &\leq \delta^{(2+q)^{k-1}} c_{k-1} \leq \delta^{(2+q)^{k-1}} \delta^{(2+q)^{k-2}} c_{k-2} \\ &\leq \cdots \leq \delta^{(2+q)^{k-1}} \delta^{(2+q)^{k-2}} \ldots \delta^{2+q^0} c_0 = \delta^{\frac{(2+q)^k - 1}{1+q}} c_0. \end{aligned} \tag{2.8}$$

Now,

$$\begin{aligned}
d_k &= d_{k-1} g_\theta(c_{k-1})^{(2+q)} h_\theta(c_{k-1}, d_{k-1})^{1+q},\\
&\leq \left(\delta^{(2+q)^{k-2}}\right)^{1+q} d_{k-2} g_\theta(\delta^{(2+q)^{k-2}} c_{k-2})^{(2+q)} h_\theta\left(\delta^{(2+q)^{k-2}} c_{k-2}, \left(\delta^{(2+q)^{k-2}}\right)^{1+q} d_{k-2}\right)^{1+q},\\
&\leq \left(\delta^{(2+q)^{k-2}}\right)^{1+q} d_{k-2} g_\theta(c_{k-2})^{(2+q)} \left((\delta^{(2+q)^{k-2}})^{1+q}\right)^{1+q} h_\theta\,(c_{k-2}, d_{k-2})^{1+q},\\
&\leq \left((\delta^{(2+q)^{k-2}})^{2+q}\right)^{1+q} d_{k-2} g_\theta\,(c_{k-2})^{(2+q)} h_\theta\,(c_{k-2}, d_{k-2})^{1+q},\\
&\leq \left(\delta^{(2+q)^{k-1}}\right)^{1+q} d_{k-1}.
\end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned}
d_k &\leq \left(\delta^{(2+q)^{k-1}}\right)^{1+q} d_{k-1} \leq \left(\delta^{(2+q)^{k-1}}\right)^{1+q} \left(\delta^{(2+q)^{k-2}}\right)^{1+q} d_{k-2}\\
&\leq \dots \left(\delta^{(2+q)^{k-1}}\right)^{1+q} \left(\delta^{(2+q)^{k-2}}\right)^{1+q} \dots \left(\delta^{(2+q)^{0}}\right)^{1+q} d_0 = \left(\delta^{\frac{(2+q)^k-1}{1+q}}\right)^{1+q} d_0,\\
&= \delta^{(2+q)^k-1} d_0
\end{aligned} \tag{2.9}$$

Hence, (i) holds $\forall k \geq 0$ by using mathematical induction. Now, consider

$$\begin{aligned}
g_\theta(c_k) h_\theta(c_k, d_k) &\leq g_\theta(\delta^{\frac{(2+q)^k-1}{1+q}} c_0) h_\theta\left(\delta^{\frac{(2+q)^k-1}{1+q}} c_0, (\delta^{\frac{(2+q)^k-1}{1-q}})^{1+q} d_0\right),\\
&\leq \delta^{(2+q)^k-1} g_\theta(c_0) h_\theta\,(c_0, d_0) = \delta^{(2+q)^k} \Delta.
\end{aligned} \tag{2.10}$$

Thus (ii) is proved. From (2.3), we get

$$\begin{aligned}
\eta_k = g_\theta(c_{k-1}) h_\theta(c_{k-1}, d_{k-1}) \eta_{k-1} &\leq \prod_{n=0}^{k-1} g_\theta(c_n) h_\theta(c_n, d_n) \eta_0,\\
&\leq \prod_{n=0}^{k-1} \frac{\delta^{(2+q)^n}}{g_\theta(c_0)} \eta_0 \leq \delta^{\frac{(2+q)^k-1}{1+q}} \Delta^k \eta_0.
\end{aligned} \tag{2.11}$$

Therefore,

$$\begin{aligned}
\sum_{n=k}^{k+m-1} \eta_n &\leq \sum_{n=k}^{k+m-1} \delta^{\frac{(2+q)^n-1}{1+q}} \Delta^n \eta_0 \leq \delta^{\frac{(2+q)^k-1}{1+q}} \Delta^k \sum_{n=0}^{m-1} \Delta^n \left(\delta^{(2+q)^k}\right)^n \eta_0,\\
&\leq \delta^{\frac{(2+q)^k-1}{1+q}} \Delta^k \frac{1-\left(\Delta\delta^{(2+q)^k}\right)^m}{1-\Delta\delta^{(2+q)^k}} \eta_0.
\end{aligned} \tag{2.12}$$

Thus, (iii) is proved. On similar lines, we can easily prove (i′′), (ii′′) and (iii′′), respectively.

Since $\delta < 1$ and $\Delta < 1$, this gives $\sum_{n=0}^{\infty} \eta_n \leq \frac{1}{1-\Delta\delta}\eta_0$. Let $R = \frac{\varphi_\theta(c_0)}{1-\Delta\delta}$. Next, we shall establish the semilocal convergence of (1.3) in $B(x_0, R\eta_0)$.

2.2 Recurrence relations

Here, the recurrence relations for (1.3) will be established under the conditions discussed in the previous section. From (1.3) for $k = 0$, we get

$$z_0 - x_0 = -\Gamma_0(F(y_0) + \theta F(x_0)). \tag{2.13}$$

From Taylor expansion of $F(y_0)$ about x_0, we get

$$F(y_0) = F(x_0) + F'(x_0)(y_0 - x_0) + \int_0^1 (1-t)F''(x_0 + t(y_0 - x_0))(y_0 - x_0)^2\,dt \tag{2.14}$$

$$= (1-\theta)F(x_0) + \int_0^1 (1-t)F''(x_0 + t(y_0 - x_0))(y_0 - x_0)^2\,dt \tag{2.15}$$

Using (2.14) in (2.13) and taking norm, we get

$$\begin{aligned}\|z_0 - x_0\| &= \|\Gamma_0 F(x_0) + \Gamma_0 \int_0^1 (1-t)F''(x_0 + t(y_0 - x_0))(y_0 - x_0)^2\,dt\|\\ &= \|\Gamma_0 F(x_0)\| + \frac{1}{2}\|\Gamma_0\| L_1 \|y_0 - x_0\|^2\\ &= \left(1 + \frac{c_0}{2}\theta^2\right)\eta_0.\end{aligned} \tag{2.16}$$

Now,

$$\begin{aligned}\|z_0 - y_0\| &= \|\Gamma_0 F(y_0)\|,\\ &\le \|(1-\theta)\Gamma_0 F(x_0) + \Gamma_0 \int_0^1 (1-t)F''(x_0 + t(y_0 - x_0))(y_0 - x_0)^2\,dt\|,\\ &\le \left(|1-\theta| + \frac{1}{2}\theta^2 c_0\right)\eta_0 = \zeta_\theta(c_0)\eta_0.\end{aligned} \tag{2.17}$$

For $k = 0$, (1.3) gives

$$\|x_1 - z_0\| \le \|\Gamma_0\|\|F(z_0)\| \tag{2.18}$$

From Taylor expansion of $F(z_0)$ about y_0, we get

$$F(z_0) = F(y_0) + F'(y_0)(z_0 - y_0) + \int_0^1 (1-t)F''(y_0 + t(z_0 - y_0))(z_0 - y_0)^2\,dt$$

$$= (F'(y_0) - F'(x_0))(z_0 - y_0) + \int_0^1 (1-t)F''(y_0 + t(z_0 - y_0))(z_0 - y_0)^2\,dt \tag{2.19}$$

Using (2.17) and (2.19), we get

$$\|x_1 - z_0\| \le \left(|\theta| c_0 \zeta_\theta(c_0) + \frac{c_0}{2}\zeta_\theta^2(c_0)\right)\eta_0 = \phi_\theta(c_0)\eta_0 \tag{2.20}$$

where $\phi_\theta(t) = \left(|\theta| t\zeta_\theta(t) + \frac{c_0}{2}\zeta_\theta^2(t)\right)\eta_0 = \phi_\theta(c_0)$. Using (2.20) and (2.16), we get

$$\begin{aligned}\|x_1 - x_0\| &\le \|x_1 - z_0\| + \|z_0 - x_0\|,\\ &\le \left(\phi_\theta(c_0) + \left(1 + \frac{c_0}{2}\theta^2\right)\right)\eta_0 = \varphi_\theta(c_0)\eta_0.\end{aligned} \tag{2.21}$$

Now,

$$\|I - \Gamma_0 F'(x_1)\| \le \|\Gamma_0\|\|F'(x_1) - F'(x_0)\| \le \beta_0 L_1 \|x_1 - x_0\|$$

$$\le L_1 \beta_0 \eta_0 \varphi_\theta(c_0) = c_0 \varphi_\theta(c_0) < 1$$

Thus, by Banach lemma, we get

$$\|\Gamma_1\| \le \frac{\|\Gamma_0\|}{1 - c_0\varphi_\theta(c_0)} = \|\Gamma_0\| g_\theta(c_0). \tag{2.22}$$

From Taylor expansion of $F(x_1)$ about z_0, we get

$$F(x_1) = F(z_0) + F'(z_0)(x_1 - z_0) + \int_0^1 (1-t)F''(z_0 + t(x_1 - z_0))(x_1 - z_0)^2\,dt,$$

$$= (F'(z_0) - F'(x_0))(x_1 - z_0) + \frac{1}{2}F''(z_0)(x_1 - z_0)^2$$
$$+ \int_0^1 (1-t)(F''(z_0 + t(x_1 - z_0)) - F''(z_0))(x_1 - z_0)^2 \, dt \tag{2.23}$$

Taking norm and using (2.16) and (2.20), we get

$$\|F(x_1)\| \le L_1 \|z_0 - x_0\| \|x_1 - z_0\| + \frac{L_1}{2}\|x_1 - z_0\|^2 + \frac{L_2}{(1+q)(2+q)}\|x_1 - z_0\|^{2+q}. \tag{2.24}$$

Therefore,

$$\|\Gamma_1 F(x_1)\| \le \|\Gamma_1\| \|F(x_1)\| \le \|\Gamma_0\| g(c_0) \|F(x_1)\|,$$
$$\le g_\theta(c_0)\left(\left(c_0 + \frac{c_0^2}{2}\theta^2\right)\phi_\theta(c_0) + \frac{c_0}{2}\phi_\theta(c_0)^2 + \frac{d_0}{(1+q)(2+q)}\phi_\theta(c_0)^{2+q}\right)\eta_0,$$
$$= g_\theta(c_0) h_\theta(c_0, d_0)\eta_0 = \eta_1 \tag{2.25}$$

Using (2.25), we get

$$L_1 \|\Gamma_1\| \|\Gamma_1 F(x_1)\| \le L_1 g_\theta(c_0) \|\Gamma_0\| g_\theta(c_0) h_\theta(c_0, d_0)\eta_0,$$
$$\le L_1 \beta_0 \eta_0 g_\theta(c_0)^2 h_\theta(c_0, d_0) = c_1 \tag{2.26}$$

and

$$L_2 \|\Gamma_1\| \|\Gamma_1 F(x_1)\|^{1+q} \le L_2 g_\theta(c_0) \|\Gamma_0\| g_\theta(c_0)^{1+q} h_\theta(c_0, d_0)^{1+q} \eta_0^{1+q},$$
$$\le L_2 \beta_0 \eta_0^{1+q} g_\theta(c_0)^{2+q} h_\theta(c_0, d_0)^{1+q} = d_2. \tag{2.27}$$

Now, using mathematical induction, the following recurrence relations are proved for $k \ge 1$.

(I) $\|\Gamma_k\| \le g_\theta(c_{k-1})\|\Gamma_{k-1}\|$,
(II) $\|\Gamma_k F(x_k)\| \le g_\theta(c_{k-1}) h_\theta(c_{k-1}, d_{k-1})\eta_{k-1}$,
(III) $L_1 \|\Gamma_k\| \|\Gamma_k F(x_k)\| \le c_k$,
(IV) $L_2 \|\Gamma_k\| \|\Gamma_k F(x_k)\|^{1+q} \le d_k$,
(V) $\|x_k - x_{k-1}\| \le \varphi_\theta(c_{k-1})\eta_{k-1}, y_k,\ z_k,\ x_{k+1} \in \overline{B(x_0, R\eta_0)}$

Assume that $x_1, y_1, z_1 \in \Omega$ and $c_0 < \alpha$.
Now,

$$\|x_{k+1} - x_0\| \le \sum_{i=0}^{k} \|x_{i+1} - x_i\| \le \sum_{i=0}^{k} \varphi_\theta(c_i)\eta_i$$
$$\le \varphi_\theta(c_0)\sum_{i=0}^{k} \eta_i \le R\eta_0. \tag{2.28}$$

If $|\theta| \le 1$ then using (2.21), we get

$$\|y_k - x_0\| \le \|y_k - x_k\| + \|x_k - x_0\|,$$
$$\le |\theta|\eta_k + \sum_{i=0}^{k-1} \varphi_\theta(c_i)\eta_i \le \varphi_\theta(c_0)\sum_{i=0}^{k} \eta_i \le R\eta_0. \tag{2.29}$$

For $|\theta| > 1$, we get

$$\|y_k - x_0\| \le |\theta|\eta_k + \sum_{i=0}^{k-1} \varphi_\theta(c_i)\eta_i \le |\theta|\eta_0 + \varphi_\theta(c_0)\sum_{i=0}^{k-1} \eta_i,$$

$$\leq \left(|\theta| + \frac{\varphi_\theta(c_0)}{1-\Delta\delta}\right)\eta_0 \leq (|\theta| + R)\eta_0. \tag{2.30}$$

Using (2.5) and (2.16), we get

$$\begin{aligned}\|z_k - x_0\| &\leq \|z_k - x_k\| + \|x_k - x_0\|,\\ &\leq \left(1 + \frac{c_0}{2}\theta^2\right)\eta_k + \sum_{i=0}^{k-1}\varphi_\theta(c_i)\eta_i,\\ &\leq \varphi_\theta(c_0)\sum_{i=0}^{k}\eta_i \leq R\eta_0. \end{aligned} \tag{2.31}$$

This implies that $y_k, z_k, x_{k+1} \in \overline{B(x_0, R\eta_0)}$. Hence, for $k = 1$, the recurrence relations (I)–(IV) follow from (2.22) and (2.25)–(2.27), respectively. The recurrence relation (V) is already established for $k = 1$ in (2.21) followed by (2.28)–(2.31), respectively. Assume that they hold for some $k = n$. In a similar manner, it can be proved by mathematical induction that (I)–(V) hold for all $k \geq 1$.

2.3 Convergence theorem

Theorem 1. *Let $x_0 \in \Omega$ and assumptions (1–4) hold for $c_0 = L_1\beta_0\eta_0$, $d_0 = L_2\beta_0\eta_0^{1+q}$ with $c_0 \in (0, \alpha)$. If $\overline{B(x_0, (|\theta| + R)\eta_0)} \subseteq \Omega$, where $R = \frac{\varphi_\theta(c_0)}{1-\Delta\delta}$ with $\Delta = \frac{1}{g_\theta(c_0)}$ and $\delta = g_\theta(c_0)^2 h_\theta(c_0, d_0) < 1$, then $\{x_k\}$ given by* (1.3) *converges to the solution of* (1.1). *The R-order is three for any $\theta \in R$, and for $\theta = 1$ it becomes four. Moreover, $y_n \in \overline{B(x_0, (|\theta| + R)\eta_0)}$, $z_n, x_{n+1}, x^* \in \overline{B(x_0, R\eta_0)}$ and x^* are the unique solutions in $B\left(x_0, \frac{2}{L_1\beta_0} - R\eta_0\right) \cap \Omega$. The error bounds for $\theta \neq 1$ is given as follows.*

$$\|x_k - x^*\| \leq \varphi_\theta(c_0)\Delta^k \frac{\delta^{\frac{(2+q)^k-1}{1+q}}}{1 - \Delta\gamma^{(2+q)^k}}\eta_0.$$

For $\theta = 1$, it is given by

$$\|x_k - x^*\| \leq \varphi_\theta(c_0)\Delta^k \frac{\delta^{\frac{(3+q)^k-1}{2+q}}}{1 - \Delta\gamma^{(3+q)^k}}\eta_0.$$

Proof. To prove this theorem, we have to show that $\{x_k\}$ is a Cauchy sequence. Using (V), we get

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \sum_{i=k}^{k+m-1}\|x_{i+1} - x_i\|\\ &\leq \sum_{i=k}^{k+m-1}\varphi_\theta(c_i)\eta_i \leq \varphi_\theta(c_0)\sum_{i=k}^{k+m-1}\eta_i\\ &\leq \varphi_\theta(c_0)\delta^{\frac{(2+q)^k-1}{1+q}}\Delta^k\frac{1-(\Delta\gamma^{(2+q)^k})^m}{1-\Delta\gamma^{(2+q)^k}}\eta_0. \end{aligned} \tag{2.32}$$

Hence, $\{x_k\}$ is a Cauchy sequence and hence converges to x^* as $k \to$ s. Taking $m \to \infty$ in (2.32), we get

$$\|x_k - x^*\| \leq \varphi_\theta(c_0)\Delta^k \frac{\delta^{\frac{(2+q)^k-1}{1+q}}}{1 - \Delta\gamma^{(2+q)^k}}\eta_0. \tag{2.33}$$

Taking $k = 0$ in (2.33), we get $\|x^* - x_0\| \leq R\eta_0$. Hence, $x^* \in \overline{B(x_0, R\eta_0)}$. Now, we prove that x^* is the solution of $F(x) = 0$. Using Lemma 3 (iii), we get $\|\Gamma_k F(x_k)\| \to 0$, since $\|F'(x_k)\|$ is bounded. This gives $\|F(x_k)\| \leq \|F'(x_k)\|\|\Gamma_k F(x_k)\| \to 0$. Thus, by the continuity of F in Ω, we get $F(x^*) = 0$. To prove the uniqueness part, let $z^* \in B(x_0, \frac{2}{L_1\beta} - R\eta_0) \cap \Omega$ such that $F(z^*) = 0$, $z^* \neq x^*$. Then $0 = F(z^*) - F(x^*) = \int_0^1 F'(x^* + t(z^* - x^*))\mathrm{d}t(z^* - x^*) = T(z^* - x^*)$, where $T = \int_0^1 F'(x^* + t(z^* - x^*))\mathrm{d}t$. Now,

$$\begin{aligned}\|I - \Gamma_0 T\| &\le \|\Gamma_0\| \int_0^1 \left\| \left(F'(x^* + t(z^* - x^*)) - F'(x_0)\right) \right\| \mathrm{d}t \\ &\le L_1\beta \int_0^1 \|(1-t)(x^* - x_0) + t(z^* - x_0)\| \mathrm{d}t \\ &\le \frac{L_1\beta}{2} (\|x^* - x_0\| + \|z^* - x_0\|) \\ &\le \frac{L_1\beta}{2} \left(R\eta_0 + \frac{2}{L_1\beta} - R\eta_0 \right) \\ &= 1\end{aligned}$$

Therefore, $\|I - \Gamma_0 T\| < 1$. Thus, by Banach Lemma T^{-1} exists and hence $z^* = x^*$.

3 Numerical examples

In this subsection, different examples are worked out to demonstrate the efficiency of our approach.

Example 2. Consider nonlinear integral equation

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 K(s,t)x(t)^{2+q}\,\mathrm{d}t, \tag{3.34}$$

where $s \in [0,1]$, $x, f \in C[0,1]$, $\lambda \in \mathbb{R}$ and $K(s,t)$ denotes the Green's function in [0,1].

Clearly,

$$\|F''(x) - F''(y)\| \le \frac{1}{8}|\lambda|(1+q)(2+q)\|x-y\|^q.$$

Thus, Lipschitz condition does not hold for $q \in (0,1)$ but Hölder condition holds. Taking $\lambda = 1/7$, $f(s) = 1$, $x_0 = x_0(s) = 1$, $\theta = 1$ and $q = 0.7$, all the assumptions considered in the previous sections are satisfied. By applying Theorem 1, the existence and uniqueness balls are given by $\overline{B(x_0, 0.018777)}$ and $B(x_0, 23.206)$, respectively. However, for $q = 1$, the existence and uniqueness balls are given by $\overline{B(x_0, 0.018888)}$ and $B(x_0, 17.648)$, respectively. The values of $\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ are given in Table 1.

The error bounds for (1.3) are given in Table 2.

Table 1: The values of c_k, d_k and η_k.

k	c_k	d_k	η_k
0	1.6157×10^{-3}	9.9919×10^{-5}	1.8762×10^{-2}
1	3.4224×10^{-12}	1.7919×10^{-19}	3.9678×10^{-11}
2	6.8598×10^{-47}	1.8479×10^{-78}	7.9528×10^{-46}
3	1.1072×10^{-185}	2.0898×10^{-314}	1.2836×10^{-184}
4	7.5013×10^{-741}	3.418×10^{-1258}	8.7102×10^{-740}
5	1.5931×10^{-2961}	2.4476×10^{-5033}	1.8469×10^{-2960}

Table 2: Error bounds for (1.3).

k	$\|x_k - x^*\|$
0	0.018777
1	3.971×10^{-11}
2	3.1938×10^{-43}
3	5.7047×10^{-162}
4	2.456×10^{-601}
5	5.4693×10^{-2227}

Example 3. Consider nonlinear integral equation

$$F(x)(s) = x(s) - 1 - \frac{1}{4}\int_0^1 \frac{s}{s+t} x(t)^{11/5}\,dt. \tag{3.35}$$

where $s \in [0,1]$, $x \in C[0,1]$.

Clearly,

$$\|F''(x) - F''(y)\| \leq \frac{33}{50} \log 2\|x-y\|^{1/5}.$$

Thus, Lipschitz condition does not hold but Hölder condition holds. Taking $x_0 = x_0(s) = 1$, $\theta = 1$ and $q = 1/5$, all the assumptions considered in the previous sections are satisfied. By applying Theorem 1, the existence and uniqueness balls are given by $\overline{B(x_0, 0.31752)}$ and $B(x_0, 2.3876)$, respectively. The values of the sequences $\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ are given in Table 3.

The error bounds for (1.3) are given in Table 4.

In order to find the numerical solution of (3.35), we have approximated the integral by Gauss–Legendre formula

$$\int_0^1 f(t)dt \simeq \frac{1}{2}\sum_{j=1}^{m} \beta_j f(t_j)$$

Table 3: The values of c_k, d_k and η_k.

k	c_k	d_k	η_k
0	2.0705×10^{-1}	1.6052×10^{-1}	2.8005×10^{-1}
1	1.8373×10^{-3}	5.2504×10^{-4}	1.9057×10^{-3}
2	5.7267×10^{-12}	3.2546×10^{-14}	5.9289×10^{-12}
3	5.3777×10^{-46}	4.7834×10^{-55}	5.5676×10^{-46}
4	4.1818×10^{-182}	2.318×10^{-218}	4.3295×10^{-182}
5	1.5291×10^{-726}	1.0577×10^{-871}	1.5831×10^{-726}

Table 4: Error bounds for (1.3).

k	$\|x_k - x^*\|$
0	0.31752
1	2.1459×10^{-3}
2	4.4696×10^{-10}
3	3.3388×10^{-31}
4	1.4855×10^{-98}
5	7.9441×10^{-314}

where β_j and t_j denote the weights and nodes, respectively, and they are given in Table 5 for $m = 8$. If we denote the approximation of $x(t_i)$, $i = 1, 2, \dots, n$, by x_i, we obtain the following nonlinear system

$$x_i = 1 + \frac{t_i}{8}\sum_{j=1}^{8} \beta_j \frac{x_j^{11/5}}{t_i + t_j}, \quad i = 1, 2, \dots 8 \tag{3.36}$$

The above system can be written as

$$x_i = 1 + \sum_{j=1}^{8} a_{ij} x_j^{11/5}, \quad i = 1, 2, \dots 8 \tag{3.37}$$

where $a_{ij} = \frac{1}{8}\left(\frac{\beta_j t_i}{t_i + t_j}\right)$.

The numerical solution is displayed in Table 6 for $x_0 = (1, 1, \dots, 1)^T$ and $m = 8$. In Figure 1, interpolating function passing through (t_i, x_i) is an approximate solution of (3.37).

Example 4. Consider nonlinear integral equation

$$F(x)(s) = x(s) - 1 + \frac{1}{2}\int_0^1 s\cos(x(t))\mathrm{d}t, \tag{3.38}$$

where $s \in [0, 1]$ and $x \in \Omega = B(0, 2) \subset X$.

Clearly,

$$\|F''(x) - F''(y)\| \le \frac{1}{2}|\|x - y\|.$$

Thus, it satisfies the Lipschitz condition. Taking, $x_0 = x_0(s) = 1$, $\theta = 1$ and $q = 1$, all the assumptions considered in the previous sections are satisfied. By applying Theorem 1, the existence and uniqueness balls are given by $B(x_0, 0.66363)$ and $B(x_0, 1.6534)$, respectively. The values of the sequences $\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ are given in Table 7.

Table 5: Value of weights and nodes for $m = 8$.

j	1	2	3	4	5	6	7	8
β_j	0.10122854	0.22381034	0.31370665	0.36268378	0.36268378	0.31370665	0.22381034	0.10122854
t_j	0.01985507	0.10166676	0.237233795	0.40828268	0.59171732	0.762766205	0.89833324	0.98014493

Table 6: Solution of (3.37).

i	1	2	3	4	5	6	7	8
X_i	1.02415468657	1.07954327274	1.13201750646	1.17280585507	1.20186439201	1.22141388151	1.23360029059	1.23991266325

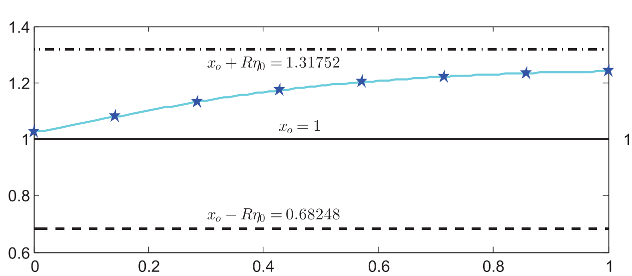

Figure 1: Approximate solution of (3.37).

Table 7: The values of c_k, d_k and η_k.

k	c_k	d_k	η_k
0	0.40255	0.18774	0.46637
1	0.078012	3.388×10^{-3}	0.04343
2	2.3277×10^{-5}	2.7711×10^{-10}	1.1905×10^{-5}
3	1.468×10^{-19}	1.1021×10^{-38}	7.5077×10^{-20}
4	2.3219×10^{-76}	2.7573×10^{-152}	1.1875×10^{-76}
5	1.4533×10^{303}	1.0802×10^{-606}	7.4327×10^{-304}

The error bounds for (1.3) are given in Table 8.

Example 5. Consider a typical example as [19]

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(S\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \upsilon(x^{11/4}), \quad t \in (0, 100), \tag{3.39}$$

for conductivity S and physical variable x. The boundary conditions are

$$x(0) = 1, \quad x(100) = 0.$$

Use cell-centered difference to discretize (3.39) by taking a mesh with 100 grid-cells. We take $S = 1$ and $\upsilon = 1/7413$. Solving (3.39) is equivalent to solve the non-linear system

$$F(x) = Ax - \upsilon H(x) - q, \tag{3.40}$$

where $x = (x_1, x_2, \dots, x_{100})^{\mathrm{T}}$, for $0 \le x_i \le 1$, $H(x) = (x_1^{11/4}, \dots, x_{100}^{11/4})$, $q = (2, 0, .., 0)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} 3 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ \dots & \dots & \dots & -1 & 3 \end{pmatrix}$$

Now, the successive derivatives of F are given by

$$F'(x) = A - 11/4\,\upsilon\ \mathrm{diag}(x_i^{7/4})$$
$$F''(x)y = -77/8\,\upsilon\ \mathrm{diag}(x_i^{3/4}y_i)$$

We take $x_0 = A^{-1}q$ as initial approximation to the solution. Clearly, $L_1 = L_2 = \frac{77}{8}\upsilon$ and $q = 3/4$, we get $c_0 = 0.47921 < 0.64132 = \alpha$, $d_0 = 0.20360$ and as a result, we have $g_\theta(c_0)^2 h_\theta(c_0, d_0) = 0.67883 < 1$.

Table 8: Error bounds for (1.3).

k	$\|x_k - x^*\|$
0	0.66363
1	0.056082
2	3.7985×10^{-5}
3	7.224×10^{-17}
4	8.5174×10^{-63}
5	1.4835×10^{-245}

Table 9: The values of c_k, d_k and η_k.

k	c_k	d_k	η_k
0	0.4792	0.2036	0.3194
1	0.3253	8.8026×10^{-3}	7.4663×10^{-3}
2	1.9826×10^{-3}	3.6905×10^{-6}	2.7446×10^{-3}
3	8.1646×10^{-9}	2.9074×10^{-17}	1.1076×10^{-9}
4	2.2218×10^{-29}	4.0509×10^{-67}	3.0142×10^{-30}
5	1.2184×10^{-115}	1.3464×10^{-266}	1.6529×10^{-116}
6	1.1018×10^{-460}	1.6431×10^{-1064}	1.4948×10^{-461}

Table 10: Error bounds for (1.3).

k	$\|x_k - x^*\|$
0	0.1111
1	8.2413×10^{-3}
2	1.2204×10^{-6}
3	5.6435×10^{-15}
4	1.0441×10^{-48}
5	6.2240×10^{-174}
6	2.9836×10^{-642}

By applying Theorem 1, the existence and uniqueness balls are given by $\overline{B(x_0, 0.57032)}$ and $B(x_0, 0.76272)$, respectively.

The values of the sequences $\{c_k\}$, $\{d_k\}$ and $\{\eta_k\}$ are given in Table 9.

The error bounds for (1.3) are given in Table 10.

4 Dynamics

Let us briefly analyze the dynamics of the family of iterative methods (1.3) in terms of parameter θ with the aim of choosing the values of this parameter more suitable for convergence with different starting points. Similar studies have been performed in [20–22] for other families of iterative methods. The dynamics of the relaxed Newton's method has been studied in [23].

Let us establish some notation. The iterates obtained starting from $z_0 \in \mathbb{C}$ can be denoted by $\{z_0, R(z_0), R^2(z_0), \dots, R^n(z_0), \dots\}$, where R is a rational function defined on the Riemann sphere $\hat{\mathbb{C}}$. This set is called the *orbit* of z_0.

Let $z \in \hat{\mathbb{C}}$ be a *fixed point* of the rational function R, that is to say $R(z) = z$. The *basin of attraction* of z consists of the points whose orbit tends to z. The behavior of the orbits near a fixed point z depends on the derivative $R'(z)$. If $|R'(z)| < 1$, the fixed point z is *attracting*; if $|R'(z)| > 1$, it is *repelling*; and if $|R'(z)| = 1$, it is *parabolic*. If $R'(z) = 0$, the fixed point is *superattracting*.

The set of points $z_0 \in \hat{\mathbb{C}}$ such that their families $R_n(z_0)$, $n \in \mathbb{N}$ are normal in some neighborhood $U(z_0)$ is the *Fatou set*, $\mathcal{F}(R)$ and its complement in $\hat{\mathbb{C}}$ is the *Julia set* $\mathcal{J}(R)$. Roughly speaking, the orbits of the points in $\mathcal{F}(R)$ present a stable behavior, whereas the orbits of the points in $\mathcal{J}(R)$ have chaotic behavior. In particular, the Fatou set contains the attraction basins of the attracting fixed points, whereas the Julia set contains the boundaries of the attraction basins.

Given an analytic function $f(z)$, consider the function associated to a step of the iterative method (1.3) $M_{\theta,f}: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that $M_{\theta,f}(x_k) = x_{k+1}$. The following scaling result holds for $M_{\theta,f}$:

Theorem 2. *Let f be an analytic function on $\hat{\mathbb{C}}$, and $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, an affine map. If $g(z) = \lambda(f \circ A)(z)$, $\lambda \neq 0$, then $M_{\theta,f}$ is analytically conjugated to $M_{\theta,g}$ by A, that is, $A \circ M_{\theta,g} \circ A^{-1} = M_{\theta,f}$.*

Any polynomial of second degree is conjugated by an affine transformation to a polynomial of the form $g(z) = z^2 + c,\ c \in \mathbb{C}$, so that in order to study the dynamics of $M_{\theta,f}$ on quadratic polynomials, it suffices to consider only polynomials of this form.

Then, if $g(z) = z^2 + c$, with $c \neq 0$, $M_{\theta,g}$ has the form

$$M_{\theta,f}(z) = -\frac{16z^4\left(c+z^2\right)\left(c+5z^2\right) + 8z^2\left(c+z^2\right)^3\theta^2 + \left(c+z^2\right)^4\theta^4}{128z^7}$$

The equation $M_{\theta,g}(z) = z$ can be written as

$$\left(c+z^2\right)\left(16z^4\left(c+5z^2\right) + 8z^2\left(c+z^2\right)^2\theta^2 + \left(c+z^2\right)^3\theta^4\right) = 0,$$

so that $M_{\theta,g}$ has eight fixed points, the roots of $f(z)$, $\sqrt{-c}$ and $-\sqrt{-c}$ and other six points called strange fixed points, which depend on θ. The two roots are superattracting, because

$$M'_{\theta,g}(z) = \frac{\left(c+z^2\right)^2\left(48z^4 - 8z^2\left(-5c+z^2\right)\theta^2 + \left(7c-z^2\right)\left(c+z^2\right)\theta^4\right)}{128z^8}$$

and then, $M'_{\theta,g}(\pm\sqrt{-c}) = 0$. The character of the remaining fixed points depends on θ.

The dynamical study can be simplified further by using the idea of analytical conjugation. If $B(z)$ is a Möbius map

$$B(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{4.41}$$

the rational maps M and N are analytically conjugated via B if $N = BMB^{-1}$. Then, $\mathcal{F}(N) = B(\mathcal{F}(M))$ and $\mathcal{J}(N) = B(\mathcal{J}(M))$.

Theorem 3. *Let $f(z)$ be a quadratic polynomial with simple roots. The fixed point operator $M_{\theta,g}(z)$ associated to the family of iterative methods* (1.3) *is analytically conjugated with*

$$N_\theta(z) = \frac{z^3\left(-(1+z)^4(2+z) + 2(1+z)^2\theta^2 + z\theta^4\right)}{-(1+z)^4(1+2z) + 2z^3(1+z)^2\theta^2 + z^4\theta^4}. \tag{4.42}$$

Proof. Considering the Möbius map $B(z) = \frac{z-\sqrt{(-c)}}{z+\sqrt{(-c)}}$, the conjugation $BM_{\theta,g}B^{-1}$ gives the desired result.

Function B sends the roots of g, $\sqrt{(-c)}$ and $-\sqrt{(-c)}$, to 0 and ∞, respectively. Furthermore, $B(\infty) = 1$. It is desirable from a numerical point of view that the basins of attraction of 0 and ∞ under N_θ cover almost all the plane. In order to analyze if this is the case for a given θ, we have to study the orbits of the free critical points of N_θ, namely, those that are not associated to the roots of the polynomial. The derivative of the conjugated function is

$$N'_\theta(z) = \frac{-2z^2(1+z)^6\left(3\left(-1+\theta^2\right)\left(1+z^4\right) + 2\left(-6+\theta^2+\theta^4\right)\left(z+z^3\right) - \left(18+2\theta^2+3\theta^4\right)z^2\right)}{\left(-1-6z-14z^2+2\left(-8+\theta^2\right)z^3+\left(-9+4\theta^2+\theta^4\right)z^4 2+\left(-1+\theta^2\right)z^5\right)^2}. \tag{4.43}$$

There are twelve critical points, counting multiplicities: 0 and ∞ associated to the roots of the polynomial, a root with multiplicity six, -1, associated to 0, and the four roots of a self-reciprocal polynomial depending on θ. Without loss of generality, we will study the orbit of one of these roots, called $z_{0,\theta}$, under the iteration function N_θ. We are interested in knowing if the orbit either converges to 0 or ∞, or has another behavior, converging to a strange fixed point, to a periodic cycle or exhibiting a more complex behavior.

The results are depicted in the so-called parameter plane, in which point $\theta \in \mathbb{C}$ is colored according to the behavior of the orbit of $z_{0,\theta}$. Following the convention established in [9], point θ is colored in cyan if the orbit converges to the fixed point 0, in magenta if it converges to ∞ and in yellow if converges to 1. If the orbit converges to another fixed point, point θ is colored in red. The convergence to 2-cycles is depicted in orange, to 3-cycles in light green, to 4-cycles in dark red, to 5-cycles in dark blue, to 6-cycles in dark green, to 7-cycles in dark yellow, 8-cycles in white and to higher order cycles in dark magenta. The points that neither converge nor tend to a cycle after 1000 iterations are colored in black.

Figure 2 shows the parameter plane of the iteration function N_θ. The image is symmetric with respect to the real and complex axis but only the positive imaginary semiplane is shown. The dynamics is richer near the imaginary axis.

Figure 3 shows the basins of attraction in the case $\theta = 1.5$. The basin of 0 is painted in cyan and that of ∞ in magenta. In this case, the numerical behavior is straightforward, as the orbits tend to one of the roots. Let us now show the dynamical planes for some values of θ that present involved behaviors.

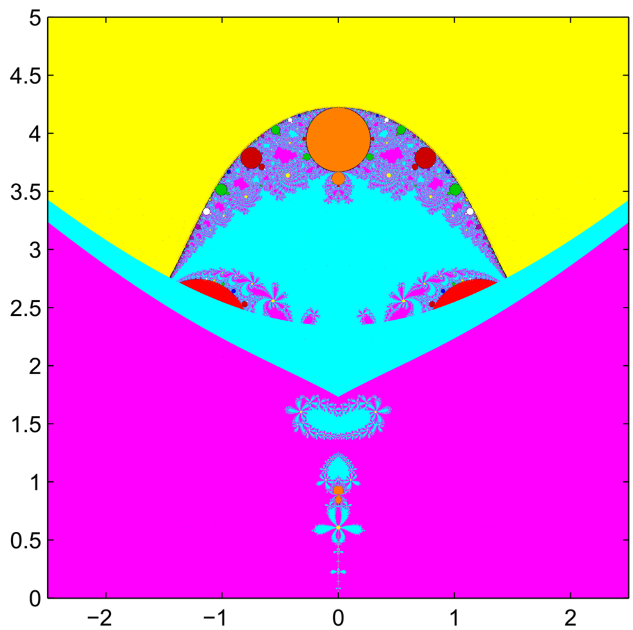

Figure 2: Parameter plane of N_θ.

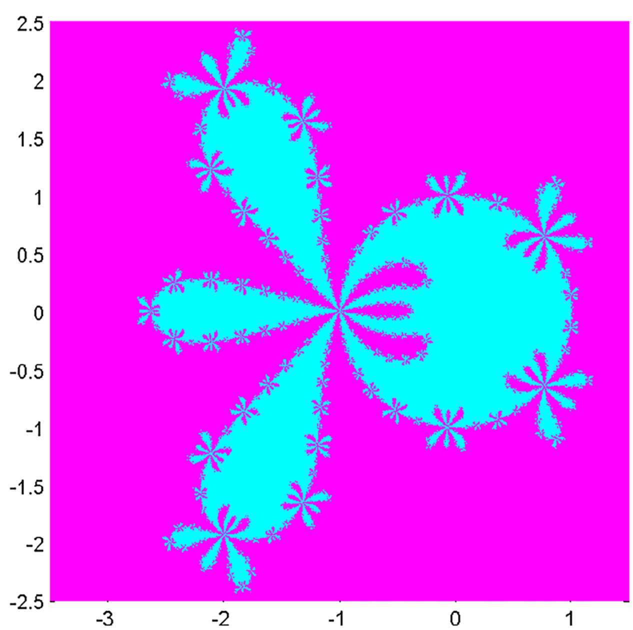

Figure 3: Attraction basins for $\theta = 1.5$.

Figure 4, for $\theta = 0.45 + 3.64i$, shows the cyan and magenta regions deeply intertwined in a fractal shape. Although no other regions are visible, this indicates instability in the iteration function.

For $\theta = 0.9i$ (see Figure 5), besides the cyan and magenta zones corresponding to the roots, there appear black regions where the iterations present a periodical behavior.

Figure 6 shows the case of $\theta = 1.1 + 2.6i$. The black zones occupy more space alternating with the basins of the proper fixed points.

In the case of $\theta = 4.25i$ shown in Figure 7 the basin of 0, in cyan, is very small. Instead, there are big yellow zones that are the basin of 1, corresponding to points whose orbit for the original iteration function $M_{\theta,g}$ would tend to infinity.

The dynamical study shows the good behavior of the method, provided the parameter θ is suitably chosen.

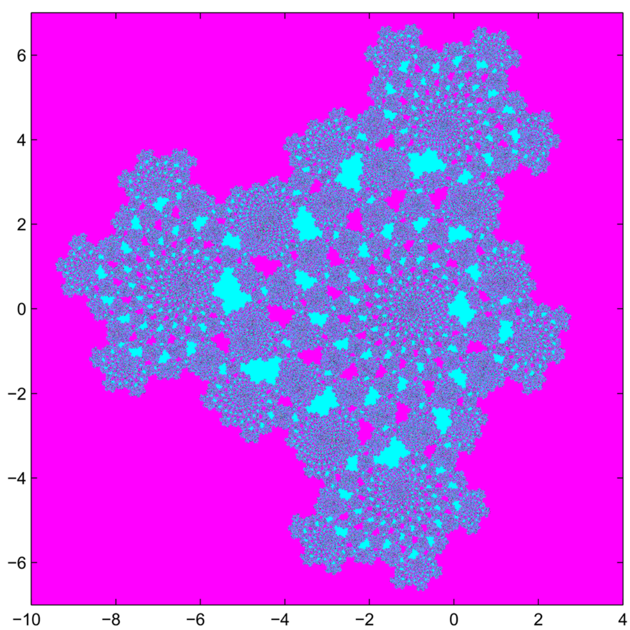

Figure 4: Attraction basins for $\theta = 0.45 + 3.64i$.

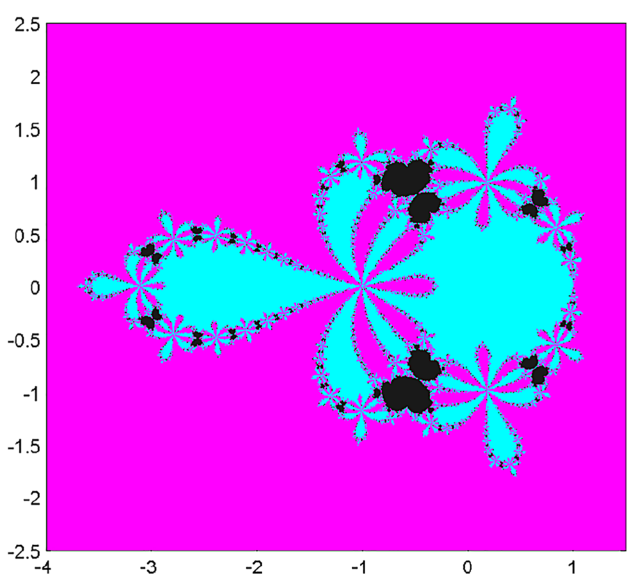

Figure 5: Attraction basins for $\theta = 0.9i$.

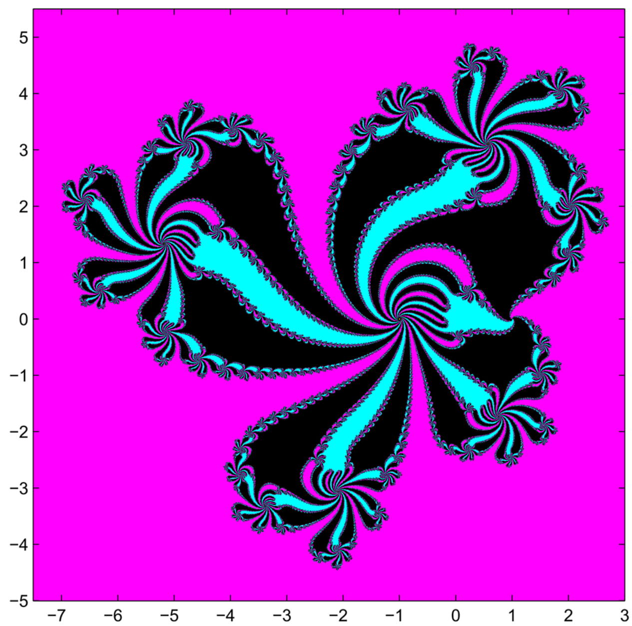

Figure 6: Attraction basins for $\theta = 1.1 + 2.6i$.

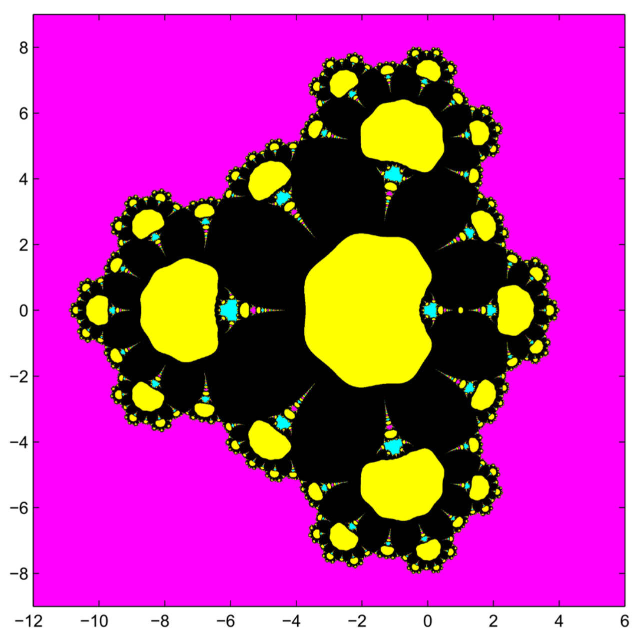

Figure 7: Attraction basins for $\theta = 4.25i$.

4.1 Critical points of operator $N_\theta(z)$

Recall that the critical points of $N_\theta(z)$ are the roots of $N'_\theta(z) = 0$. That is $z = 0$, $z = \infty$ are associated to the roots of the polynomials and remaining are the roots depending upon θ is given by the polynomial

$$3\left(-1+\theta^2\right)\left(1+z^4\right)+2\left(-6+\theta^2+\theta^4\right)\left(z+z^3\right)-\left(18+2\theta^2+3\theta^4\right)z^2=0$$

whose solution are given by

$$z = \frac{-(2\theta^4 + 2\theta^2 - 12 + 2\theta^2\sqrt{4 + 11\theta^2 + \theta^4} \pm E)}{12(\theta^2 - 1)} \tag{4.44}$$

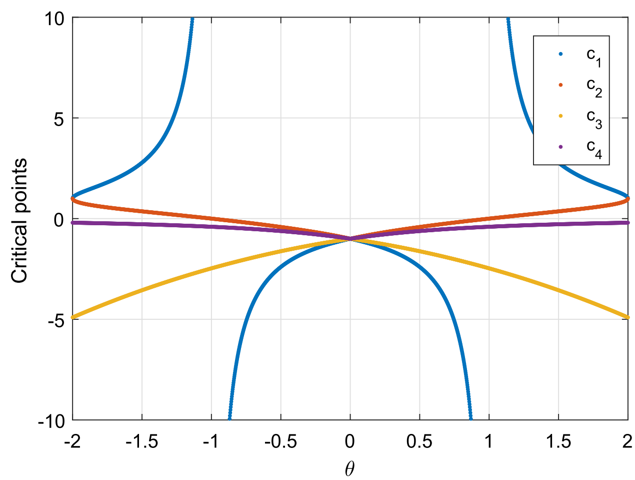

Figure 8: Dynamical behavior of critical points for $-2 \le \theta \le 2$.

and

$$z = \frac{-(2\theta^4 + 2\theta^2 - 12 - 2\theta^2\sqrt{4 + 11\theta^2 + \theta^4} \pm F)}{12(\theta^2 - 1)} \tag{4.45}$$

We denote the critical points obtained by (4.44) with c_1, c_2 and by (4.45) with c_3, c_4 where

$$E = 2\theta\sqrt{2\theta^6 + 13\theta^4 - 43\theta^2 + 60 + (2\theta^4 + 2\theta^2 - 12)\sqrt{\theta^4 + 11\theta^2 + 4}}$$

and

$$F = 2\theta\sqrt{2\theta^6 + 13\theta^4 - 43\theta^2 + 60 - (2\theta^4 + 2\theta^2 - 12)\sqrt{\theta^4 + 11\theta^2 + 4}}$$

Clearly $c_1 = \frac{1}{c_2}$ and $c_3 = \frac{1}{c_4}$.

In Figure 8, we represent the behavior of critical points for real values of θ between −2 and 2. We observe that the critical points are symmetric along the line $\theta = 0$. All the critical points meet at $z = -1$ for $\theta = 0$, and $c_1 = c_2 = 1$ for $\theta = \pm 2$.

Moreover, we can see that when $\theta \to \pm 1$, $c_1 \to \pm\infty$, and $c_2 = 0$. Outside $[-2, 2]$ the critical points can take complex values. It is observed that when $\theta \to \pm\infty$, the real parts of c_2 and c_3 tend to $\pm\infty$, respectively, whereas the real parts of c_1 and c_4 tend to 0.

5 Conclusions

A family of iterative methods and its semilocal convergence using recurrence relations for solving nonlinear equations in Banach spaces are presented. The convergence is established under the Hölder continuity condition on second derivatives which generalize the results obtained in [10]. Recurrence relations based on three parameters are derived. A theorem for existence and uniqueness along with the error bounds for the solution is provided. The R-order of convergence is shown to be equal to $3 + q$ when $\theta = \pm 1$; otherwise it is $2 + q$, where, $q \in (0, 1]$. Numerical examples involving nonlinear integral equations and boundary value problems are solved and improved convergence balls are found for them. Finally, the dynamical study of the family of iterations is also carried out.

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