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Topological transitivity of the normalized maps induced by linear operators

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Abstract

In this article, we provide a simple geometric proof of the following fact: The existence of topologically transitive normalized maps induced by linear operators is possible only when the underlying space's real dimension is either 1 or 2 or infinity. A similar result holds for projective transformation as well.

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1. Introduction

In general, a topological dynamical system is a pair (X, f), where X is a topological space and f is a self map on X. The study of dynamics is mainly about the eventual behavior of orbits. In our setting, we take X as a metric space and f as a continuous self map on X. For $x \in X$, the f-orbit of x is denoted by $O(f, x)$ and is defined by $\{f^n(x) : n \in \mathbb{N}_0\}$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, 3, ...\}$. Here $f^n(x) = f \circ f \circ ... \circ f(x)$ (n-times) and $f^0(x) = I(x) = x$ (I denotes the identity map). Now f is said to have dense orbit if there exists $x \in X$ such that $O(f, x)$ is dense in X and f is said to be topologically transitive if for any two non-empty open sets U, V in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$. For several equivalent formulations of topological transitivity, see [3], [9]. Having dense orbit and

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topological transitivity are not equivalent notions in general. However, in some 'nice spaces' both the notions are equivalent. For example, if X is a separable complete metric space without isolated points. Then the following assertions are equivalent:

(i) f is topologically transitive;

(ii) there exists $x \in X$ such that $O(f, x)$ is dense in X.

This is known as Birkhoff transitivity theorem (see [6]).

Let $T : X \to X$ be an invertible continuous linear operator, where X is a real separable Hilbert space. Hereafter, in this article, X denotes a real separable Hilbert space, unless otherwise mentioned. Let $L(X)$ be the set of all continuous linear operators on X and $GL(X)$ be the set of all invertible continuous linear operators on X. The map $T : X \to X$ induces a map $\overline{T} : S_X \to S_X$, where $S_X := \{x \in X : ||x|| = 1\}$ and is defined by $\overline{T}x = \frac{Tx}{||Tx||}$, whenever $T \in GL(X)$. We call $\overline{T}$ as the normalized map induced by T. There are some excellent monographs and expository articles which dealt with dynamics of linear operators in great detail (see [2], [4], [6]). Here we are interested to study a particular notion namely, the topological transitivity of the map $\overline{T}$ on S_X. More precisely, we ask the following questions:

Question 1.1. *For a given X, does there exist $T \in GL(X)$ such that $\overline{T}$ is topologically transitive on S_X?*

If the answer to the above is positive, we would like to investigate the following also.

Question 1.2. *What are all invertible continuous linear operators whose normalized maps are topologically transitive?*

Let $\dim(X)$ be the Hilbert dimension of X i.e., the cardinality of an orthonormal basis of X. In this article, as an answer to the Question 1.1, we show the following:

Theorem 1.3. *There exists $T \in GL(X)$ such that $\overline{T}$ is topologically transitive on S_X if and only if $\dim(X) \in \{1, 2, \infty\}$.*

In the context of linear dynamics, topological transitivity and having dense orbit are equivalent (due to Birkhoff transitivity theorem) and it is known as hypercyclicity. Supercyclicity and Positive supercyclicity are two weaker notions than the notion of hypercyclicity. A linear operator T is said to be supercyclic (resp. positive supercyclic) if there exists a vector $x \in X$ whose projective orbit (resp. positive projective orbit) $\mathbb{R}.O(T, x)$ (resp. $\mathbb{R}^+.O(T, x)$) defined by $\{\lambda T^n(x) : n \in \mathbb{N}_0 \text{ and } \lambda \in \mathbb{R}\}$ (resp. $\{\lambda T^n(x) : n \in \mathbb{N}_0 \text{ and } \lambda \in \mathbb{R}^+\}$) is dense in X, where $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all positive real numbers respectively. For equivalent formulations on these notions, one can refer [2], [6]. In [7], G. Herzog proved that supercyclic vector exists if and only if the dimension of the space is 1, 2, ∞ (over real) and 1, ∞ (over complex). Proof of Herzog's theorem is based on the techniques from functional analysis and operator theory. Here we essentially reprove the Herzog's theorem in the light of dynamics on unit sphere. Although our proof is relatively longer, it is still simple. It is intuitive according to our expositions and based on basic and well-known techniques from linear algebra and theory of dynamical systems. In the end, we also list all such invertible continuous linear operators whose normalized maps are topologically transitive as a possible answer to the Question 1.2.

Similarly, the map $T \in GL(X)$ induces a map $\widetilde{T} : S_X/\sim \to S_X/\sim$, where $S_X/\sim$ is the quotient space of S_X under the relation $\sim$. Here '$\sim$' identifies the antipodal points. The map $\widetilde{T}$ is known as real projective transformation. For a detailed study on dynamics of projective transformation, see [4], [8]. One can ask similar questions as above and expect that similar results also hold for real projective transformation.

2. Cone Transitivity and Basic Properties

In this section, we introduce a weak notion of topological transitivity, which we call cone transitivity. This notion helps us to study the dynamics of the normalized maps induced by linear operators. First, we start with a definition.

Definition 2.1 (Open cone)**.** An open set $V \subset X$ is said to be an open cone if $\lambda V \subset V$ for any $\lambda > 0$.

Remark 2.2. For any non-empty open cone V, there exists a non-empty open subset $\mathcal{S}$ (namely $S_X \cap V$) of S_X (in the subspace topology) such that $V = \cup_{x \in \mathcal{S}} L_x$, where $L_x := \{\lambda x : \lambda > 0\}$.

Definition 2.3 (Cone transitivity)**.** A $T \in L(X)$ is said to be cone transitive, if for any two non-empty open cones U, V in X, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

We now provide an alternative definition for the cone transitivity in the following theorem, which is analogous to Birkhoff transitivity theorem.

Theorem 2.4. *Let $T \in L(X)$. Then the following are equivalent:*

(i) There exists $x \in X$ such that for any non-empty open cone V, $T^n x \in V$ for some $n \in \mathbb{N}$.

(ii) T is cone transitive.

Proof. Let U and V be two non-empty open cones. By (i), there exists $x \in X$ such that $T^n(x) \in V$ and $T^m(x) \in U$ for some $m, n \in \mathbb{N}$. Since X does not contain any isolated point, without loss of generality we can choose $n > m$. Therefore $T^{n-m}(U) \cap V \neq \varnothing$. Hence, (i) implies (ii).

Since X is separable, it has a countable dense set say $\{v_1, v_2, v_3, ...\}$. Consider the open balls V'_k of radius $\epsilon > 0$ around v_k for each $k \geq 1$, form a countable base of the topology of X. Let V_k be the smallest open cone containing V'_k. If $T^{-n}(V_k)$ is the n-th pre-images of V_k, then $\cup_{n=0}^{\infty} T^{-n}(V_k)$ is the set of points which visit V_k at least once. We claim that $\cup_{n=0}^{\infty} T^{-n}(V_k)$ is dense in X. If not, then there exists an non-empty open set U' in X such that $U' \cap \cup_{n=0}^{\infty} T^{-n}(V_k) = \varnothing$. Therefore for each $n \in \mathbb{N}$, $U' \cap T^{-n}(V_k) = \varnothing$. Since

V_k is a cone and T is linear, we have for any $\lambda > 0$, $\lambda U' \cap T^{-n}(V_k) = \varnothing$. Therefore if U is the smallest open cone containing U', then $U \cap T^{-n}(V_k) = \varnothing$ i.e., $T^n(U) \cap V_k = \varnothing$. This is a contradiction to the fact that T is a cone transitive operator. Hence $\cup_{n=0}^{\infty} T^{-n}(V_k)$ is dense in X. Therefore $\cap_{k=1}^{\infty} \cup_{n=0}^{\infty} T^{-n}(V_k)$ is non-empty, by Baire's category theorem. Hence, (ii) implies (i). □

Remark 2.5. If T is a cone transitive operator, then the x defined in Theorem 2.4 is called a cone transitive vector of T. We denote by $CV(T)$ the set of all cone transitive vectors of T. Here X does not contain any isolated point. Therefore there exists a sequence of natural numbers (n_k) such that $T^{n_k}x \in V$ where $n_k \to \infty$ and $O(x, T) \subset CV(T)$.

One can show that the notions of cone transitivity and positive supercyclicity are equivalent. To keep our exposition self-contained and geometrically intuitive, we prefer to use cone transitivity instead of positive supercyclicity. Let us see some examples of cone transitivity:

Example 2.6. Any topologically transitive continuous linear operator in infinite dimensional space (known as Hypercyclic operator) is cone transitive.

To see this: take any two non-empty open sets as any two non-empty open cones.

In the next example, we see that there exists a cone transitive linear operator which is not topologically transitive. Furthermore, cone transitivity is not an infinite dimensional property like the topological transitivity for linear operators.

Example 2.7. Let $T := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\frac{\theta}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$.

Here $\overline{T} = T \mid_{S_{\mathbb{R}^2}}$. Let U and V be two open cones in $\mathbb{R}^2$. Then $U' := U \cap S_{\mathbb{R}^2}$ and $V' = V \cap S_{\mathbb{R}^2}$ are two non-empty open sets in $S_{\mathbb{R}^2}$. It is well-known that any irrational circular rotation is topologically transitive on $S_{\mathbb{R}^2}$. Therefore there exists $n \in \mathbb{N}$ such that $\overline{T}^n(U') \cap V' \neq \varnothing$. Hence $T^n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$. Therefore T is a cone transitive operator on $\mathbb{R}^2$.

We ask the following question:

Question 2.8. *Does there exist any relation among the cone transitivity of T on X and the topological transitivity of $\overline{T}$ on S_X?*

First, observe that a non-invertible linear operator can not be cone transitive as $T(X)$ is a proper subspace of X, when $\dim(X) < \infty$. However, we find an affirmative answer to Question 2.8. In fact, both the notions are equivalent. In this section, we prove the equivalence together with some basic properties of cone transitivity, which are useful throughout our discussions.

Theorem 2.9. *Let $T \in GL(X)$. Then $\overline{T} : S_X \to S_X$ is topologically transitive if and only if $T : X \to X$ is cone transitive.*

Proof. ($\Rightarrow$) Let U and V be two non-empty open cones in X. In view of Remark 2.2, $U' := U \cap S_X$ and $V' := V \cap S_X$ are two non-empty open sets in S_X. Since $\overline{T} : S_X \to S_X$ is topologically transitive, there exists $n \in \mathbb{N}$ such that $\overline{T}^n(U') \cap V' \neq \varnothing$, i.e., there exists $x \in U'$ such that $\overline{T}^n x \in V'$. Since V is an open cone containing V', we have $T^n x \in V$. This shows that $T^n(U) \cap V \neq \varnothing$. Hence T is cone transitive.

($\Leftarrow$) Let U' and V' be two non-empty open sets in S_X. Take $U := \cup_{\lambda>0}\lambda U'$, the smallest open cone containing U' and $V := \cup_{\lambda>0}\lambda V'$, the smallest open cone containing V'. Since T is cone transitive, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$ i.e., there exists $x \in U$ such that $T^n x \in V$. Consider $x' := \frac{x}{||x||}$. In view of Remark 2.2, $x' \in U'$. Since V is an open cone, we have $T^n x' \in V$. Therefore $\overline{T}^n(x') = \frac{T^n x'}{||T^n x'||} \in V'$. This shows that $\overline{T}^n(U') \cap V' \neq \varnothing$. Hence $\overline{T}$ is topologically transitive. $\square$

Proposition 2.10. *Let $T \in L(X)$. If x is a cone transitive vector of T, then λx is also a cone transitive vector of T for any $\lambda > 0$.*

Proof. Let V be any non-empty open cone. Since x is cone transitive vector of T, there exists $n \in \mathbb{N}$ such that $T^n x \in V$. Therefore $T^n(\lambda x) = \lambda T^n x \in \lambda V \subset V$ for $\lambda > 0$. Hence λx is also a cone transitive vector of T for any $\lambda > 0$. $\square$

Remark 2.11. For any $T \in L(X)$, $TV(\overline{T}) = \{\frac{x}{||x||} : x \in CV(T)\}$ and $CV(T) := \{\lambda x : x \in TV(\overline{T}) \text{ and } \lambda > 0\}$, where $TV(\overline{T})$ is the set of all transitive vectors for $\overline{T}$. Moreover $CV(T)$ is either empty or dense in X.

Definition 2.12 (Linear conjugacy)**.** Let $T, S \in L(X)$. Now T and S are said to be linearly conjugate if there exists $P \in GL(X)$ such that $S = P^{-1}TP$.

Proposition 2.13. *Cone transitivity is preserved by linear conjugacy.*

Proof. Let $T \in L(X)$ be cone transitive. We claim that for any $P \in GL(X)$, $S = P^{-1}TP$ is also cone transitive. Let U be a non-empty open set and V be a non-empty open cone. Then observe that $P(U)$ is a non-empty open set and $P(V)$ is a non-empty open cone. Since T is cone transitive, we have $T^n(P(U)) \cap P(V) \neq \varnothing$ for some $n \in \mathbb{N}$. On the other hand, $PS^n = T^n P$. We conclude that $PS^n(U) \cap P(V) \neq \varnothing$, which readily gives $S^n(U) \cap V \neq \varnothing$. Hence S is cone transitive. $\square$

Proposition 2.14. *If $T, S \in GL(X)$ are linearly conjugate, then $\overline{T}$, $\overline{S}$ are topologically conjugate.*

Proof. Let S and T be two invertible linear operators which are linearly conjugate. We claim that $\overline{S}$ and $\overline{T}$ are topologically conjugate. Since S and T are linearly conjugate, there exists a invertible linear operator P on X such that $S = P^{-1}TP$. For $x \in X$, $\overline{P^{-1}}\ \overline{T}\ \overline{P}x = \overline{P^{-1}}\ \overline{T}(\frac{Px}{||Px||}) = \overline{P^{-1}}(\frac{T(Px)/||Px||}{||TPx||/||Px||}) = \frac{P^{-1}TPx/||TPx||}{||P^{-1}TPx||/||TPx||} = \frac{P^{-1}TPx}{||P^{-1}TPx||} = \overline{S}x$. Take $h = \overline{P}$, then $\overline{S} = h^{-1}\overline{T}h$. $\square$

Theorem 2.15. *If $T \in GL(X)$ is cone transitive, then T^{-1} is also cone transitive.*

Proof. Observe that $\overline{T}^{-1} = \overline{T^{-1}}$. Again the proof goes via topological transitivity of $\overline{T}$ on S_X, by Theorem 2.9 and Proposition 1.14 in [6]. $\square$

Remark 2.16. If $\dim(X) \geq 2$, we may use Birkhoff transitivity theorem to show that topological transitivity of $\overline{T}$ and having dense orbit of $\overline{T}$ are equivalent. We claim that both the notions are also equivalent for $\dim(X) = 1$. In this case, $X \cong \mathbb{R}$ and $S_X \cong \{-1, +1\}$. If $\overline{T}$ is not topologically transitive but it contains a dense orbit, then we get $||T(1)|| = -||T(-1)||$, which is absurd. It is obvious that S_X is a Baire space with a countable basis of open sets. Therefore topological transitivity of $\overline{T}$ ensures a dense orbit of $\overline{T}$. Hence in our setting both the notions are equivalent.

Theorem 2.17 (A necessary condition)**.** *Let $T_i \in L(X_i)$ for $i = 1, 2$ and $T := T_1 \bigoplus T_2 : X_1 \bigoplus X_2 \to X_1 \bigoplus X_2$, where each X_i is real separable Hilbert space. If $\overline{T}$ is topologically transitive on $S_{X_1 \bigoplus X_2}$, then $\overline{T_i}$ is also topologically transitive on S_{X_i} for $i = 1, 2$.*

Proof. Since $\overline{T}$ is topologically transitive on $S_{X_1 \bigoplus X_2}$, there exists $(x_1, x_2) \in S_{X_1 \bigoplus X_2}$ such that $\{\overline{T}^n(x_1, x_2) : n \in \mathbb{N}\}$ is dense in $S_{X_1 \bigoplus X_2}$. It is enough to prove that T_i is cone transitive on X_i for $i = 1, 2$. First, we claim that x_1 is cone transitive vector of T_1. If not, then there exists an open cone V_1 in X_1 such that $V_1 \cap \{T_1^n x_1 : n \in \mathbb{N}\} = \varnothing$. This implies $\lambda T_1^n x_1 \notin V_1$ for any $\lambda > 0$. Let V_2 be any open cone in X_2. Then $V_1 \times V_2$ is an open cone in $X_1 \bigoplus X_2$. Since $\lambda T_1^n x_1 \notin V_1$ for any $\lambda > 0$, we have $\frac{(T_1^n x_1, T_2^n x_2)}{||(T_1^n x_1, T_2^n x_2)||} \notin V_1 \times V_2$. Therefore $\overline{T}$ is not topologically transitive on $S_{X_1 \bigoplus X_2}$, which is a contradiction. $\square$

The converse of the above theorem is not true in general. We encounter this on several occasions in the proof of the main result.

3. Main result

Theorem 3.1 (Main theorem)**.** *If $\dim(X) \in \{1, 2, \infty\}$, then there exists a $T \in GL(X)$ such that $\overline{T}$ is topologically transitive on S_X. If $\dim(X) \notin \{1, 2, \infty\}$, then there exists no $T \in GL(X)$ such that $\overline{T}$ is topologically transitive on S_X.*

Proof. The proof depends on the dimension of the space X. Thus we prove the result by considering various cases.

Case A: Let $\dim(X) = 1$. Then $X \cong \mathbb{R}$ and $S_X \cong \{+1, -1\}$. Take $Tx = -x$, where $x \in X$. Then $\overline{T}$ is topologically transitive.

Case B: Let $\dim(X) = 2$. Then $X \cong \mathbb{R}^2$ and $S_X \cong S_{\mathbb{R}^2} := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 = 1\} = \mathbb{S}^1$. Take $T := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\frac{\theta}{\pi}$ is an irrational number. Then $\overline{T} : \mathbb{S}^1 \to \mathbb{S}^1$ is an irrational rotation and it is well-known that irrational rotation is topologically transitive on $S_{\mathbb{R}^2}$ (for a proof, see Example 1.12 in [6]).

Case C: Let $\dim(X) = \infty$. Since X is an infinite dimensional separable Hilbert space, we have $X \cong l^2(\mathbb{Z})$ (see [5]). Let T be an invertible hypercyclic operator (see [6]). Then $\overline{T} : S_X \to S_X$ is topologically transitive.

Case D: Let $2 < \dim(X) < \infty$. If S is a linear operator on X, then using real Jordan canonical form, S has at least one block-diagonal which is linearly conjugate to one of the following:

(1) $\begin{pmatrix} r\cos\theta & r\sin\theta & 0 & 0 \\ -r\sin\theta & r\cos\theta & 0 & 0 \\ 0 & 0 & s\cos\phi & s\sin\phi \\ 0 & 0 & -s\sin\phi & s\cos\phi \end{pmatrix}$ on $\mathbb{R}^4$, where $r, s \in \mathbb{R}\backslash\{0\}$;

(2) $\begin{pmatrix} \lambda & 1 & 0 & ... & 0 \\ 0 & \lambda & 1 & ... & 0 \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & \lambda & 1 \\ 0 & 0 & 0 & ... & \lambda \end{pmatrix}$ on $\mathbb{R}^n$, where $\lambda \in \mathbb{R}\backslash\{0\}$;

(3) $\begin{pmatrix} J_1 & I_2 & 0 & ... & 0 \\ 0 & J_2 & I_2 & ... & 0 \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & J_{n-1} & I_2 \\ 0 & 0 & 0 & ... & J_n \end{pmatrix}$ on $\mathbb{R}^{2n}$, where $J_i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(4) $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & r\cos\phi & r\sin\phi \\ 0 & -r\sin\phi & r\cos\phi \end{pmatrix}$ on $\mathbb{R}^3$, where $\alpha, r \in \mathbb{R}\backslash\{0\}$;

(5) $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ on $\mathbb{R}^2$, where $\alpha, \beta \in \mathbb{R}\backslash\{0\}$.

We now show that each of the above representation can not be cone transitive on their respective invariant subspaces.

For (1): Let $S\mid_{\mathbb{R}^4} = T$. By Theorem 2.15, we may assume either $|s| \geq |r|$, $|s| > 1$ or $|s| = |r| = 1$. If possible, let us assume (x, y, u, v) be a cone transitive vector of T such that $||(x, y, 0, 0)|| \leq ||(0, 0, u, v)||$. Identify $\mathbb{R}^4$ as $xyuv$-space and $T^n(x, y, u, v) = (xr^n \cos n\theta + yr^n \sin n\theta, -xr^n \sin n\theta + yr^n \cos n\theta, us^n \cos n\phi + vs^n \sin n\phi, -us^n \sin n\phi + vs^n \cos n\phi)$. It is clear from the definition that the orbit of $(x_0, y_0, 0, 0)$ lies on xy-plane and the orbit of $(0, 0, u_0, v_0)$ lies on uv-plane. Since (x, y, u, v) is a cone transitive vector of T, we have $(x, y) \neq (0, 0)$ and $(u, v) \neq (0, 0)$. Let $(x', y', 0, 0)$ be any point in the xy-plane. Then $||T^n(x, y, u, v) - (x', y', 0, 0)|| \geq ||(0, 0, us^n \cos n\phi + vs^n \sin n\phi, -us^n \sin n\phi + vs^n \cos n\phi)|| = |s|^n ||(0, 0, u, v)||$ and $||T^n(x, y, u, v)|| \leq 2 \times \max\{|r|^n, |s|^n\} \leq M^n$, for some M. If $|s| > 1$, then we take an open 4-ball centered at $(x, y, 0, 0)$ with radius r', where $r'M < \frac{||(x,y,0,0)||}{2}$. Let V be the smallest cone containing the 4-ball. Then $T^n(x, y, u, v)) \notin V$, for any $n \in \mathbb{N}$.

If $|s| = |r| = 1$, then observe that $T = \overline{T}$ on $S_{\mathbb{R}^4}$. If $(x', y', 0, 0)$ is any point in $S_{\mathbb{R}^4}$, similarly, we have $||\overline{T}^n(x, y, u, v) - (x', y', 0, 0)|| \geq ||(0, 0, u, v)||$. This means $\overline{T}$ is not topologically transitive on $S_{\mathbb{R}^4}$.

For (2): Let $S\mid_{\mathbb{R}^n}= T$. By Theorem 2.15, we may assume that $|\lambda| \geq 1$. Let $(x_1, x_2, ..., x_n)$ be any point in $\mathbb{R}^n$. Let B be an open ball centered at $(0, 0, ..., x_n)$ with radius $r < ||(0, 0, ..., x_n)||$ and V be the smallest open cone B. It is straightforward to verify that there exists $C > 0$ such that for every $(\alpha_1, \alpha_2, ..., \alpha_n) \in V$, we have $\frac{|\alpha_1|}{|\alpha_n|} < C$. In particular, if $r < \frac{||(0,0,...,x_n)||}{2}$, then $C < 1$. If for some $N \geq n_0$, we have $T^N(x) \in V$, then

$$|\lambda^N x_1 + N\lambda^{N-1}x_2 + ... + \binom{N}{n-1}\lambda^{N-n+1}x_n| < C|\lambda^N x_n|$$

or,

$$|x_1 + \frac{N}{\lambda}x_2 + ... + \binom{N}{n-1}\frac{1}{\lambda^{n-1}}x_n| < C|x_n|.$$

Observe that it is not true, if we choose n_0 large enough. Therefore there exists a $n_0 \in \mathbb{N}$, such that $T^m(x) \notin V$ for $m \geq n_0$.

For (3): The proof is in similar lines as that of the previous case (i.e., **(2)**). Thus we omit the details.

For (4): Let $S\mid_{\mathbb{R}^3}= T$. Here $T^n(x_1, x_2, x_3) = (\alpha^n x_1, r^n x_2 \cos n\phi + r^n x_3 \sin n\phi, -r^n x_2 \sin n\phi + r^n x_3 \cos n\phi)$ for any $(x_1, x_2, x_3) \in \mathbb{R}^3$. If $|\alpha| > |r|$, then take a sphere centered at $(0, x_2, x_3)$ with radius r', where $r' < \frac{||(0,x_2,x_3)||}{2}$. If V be the smallest cone containing the sphere, then we claim that there exists a $n_0 \in \mathbb{N}$ such that $T^n(x_1, x_2, x_3) \notin V$ for $n \geq n_0$. If possible, for some $N \geq n_0$, we have $T^N(x_1, x_2, x_3) \in V$. Then $\frac{|\alpha|^N ||(x_1,0,0)||}{|r|^N ||(0,x_2,x_3)||} < 1$, which is not true if we take n_0 large enough. If $|r| > |\alpha|$, then we take the sphere centered at $(x_1, 0, 0)$ with radius $r'' < \frac{||(x_1,0,0)||}{2}$.

If $|\alpha| = |r|$, then $\frac{||T^n(x_1,0,0)||}{||T^n(0,x_2,x_2)||} = \frac{||(x_1,0,0)||}{||(0,x_2,x_3)||}$. Hence T is not cone transitive.

For (5): Let $S\mid_{\mathbb{R}^2}= T$. Observe that for any point (x, y), there exists an arbitrarily small $\epsilon > 0$ such that the smallest open cone containing $B((x, y), \epsilon)$ contains only finitely many $T^n(x, y)$ when $|\alpha| \neq |\beta|$. If $|\alpha| = |\beta|$, then all $T^n(x, y)$ is in the smallest cone containing $B((x, y), \epsilon)$, $B((-x, y), \epsilon)$, $B((x, -y), \epsilon)$ and $B((-x, -y), \epsilon)$. Hence in both the cases, T is not cone transitive.

By Theorem 2.15, S is not cone transitive, whenever $2 < \dim(X) < \infty$.

Hence the result.

□

Remark 3.2. The proof of Theorem 3.1 suggests that a similar argument can also ensure the non-existence of topologically transitive projective transformation on $S_X/{\sim}$, when $2 < dim(X) < \infty$. In this case instead of taking a positive open cone, we may need to take a full open cone i.e., instead of $\lambda > 0$, we take $\lambda \neq 0$.

Remark 3.3. In contrast with the above result, for $n \geq 2$, every n-dimensional compact manifold admits a chaotic homeomorphism. For a proof, see [1]. In particular, for each $n \in \mathbb{N}$, there exists a homeomorphism h_n on $S_{\mathbb{R}^n}$ such that h_n is topologically transitive on $S_{\mathbb{R}^n}$.

We conclude by providing a complete list of linear operators whose normalized maps are topologically transitive (similarly, for real projective transformation), which is apparent from the above discussion. Since linear conjugacy preserves cone transitivity, we make a list as follows:

$\dim(X)$	$T \in GL(X)$ such that $(\overline{T}, S_X)$ is topologically transitive	$T \in GL(X)$ such that $(\widetilde{T}, S_X/\sim)$ is topologically transitive
1	rI, for $r < 0$.	rI, for $r \neq 0$.
2	$\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$, where $\frac{\theta}{\pi} \in \mathbb{R}\backslash\mathbb{Q}$ and $r \neq 0$.	$\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$, where $\frac{\theta}{\pi} \in \mathbb{R}\backslash\mathbb{Q}$ and $r \neq 0$.
∞	Invertible positive supercyclic operators.	Invertible supercyclic operator.

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