SOLITARY WAVES IN NONLINEAR PHONONIC CRYSTALS

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Abstract

We discuss two possible regimes of solitary wave formation in acoustic layered media. In the weakly dispersive limit, KdV-type solitons are formed, consisting of broad pulses with a width much larger than the lattice periodicity. Such KdV solitons are shown to exist even far from the weakly dispersive conditions. On the other hand, in the strongly dispersive regime, gap acoustic solitons are demonstrated. They are formed by a fast carrier wave inside the band-gap of the structure, near the Bragg frequency (whose propagation is not allowed in the case of linear waves), modulated by a wide envelope, whose width lies inside the gap. Gap solitons propagate slower than linear waves, or can be even reach a stationary non-propagating state within the medium. The parameters for a realistic acoustic medium supporting both types of solitary waves are discussed.

Keywords: solitons, nonlinear, dispersion, multilayer, band-gap.

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1 Introduction

Sound wave propagation in periodic media has become increasingly popular in the last years, after the introduction of the ideas of sonic and phononic crystals. Exploiting the analogies with other type of waves (mainly electrons and light) in the corresponding media, many interesting effects as forbidden propagation bands (band-gaps), focalization, self-collimation, negative refraction, and many others have been proposed. The most of the studies have been done assuming low-amplitude waves, or linear regime, where the mentioned analogies apply. In linear systems, frequency is conserved. Intense wave propagation in nonlinear periodic media has been much less explored, particularly in acoustics [1,2]. Here we discuss localization phenomena related to sound wave propagation in a nonlinear medium with periodically modulated properties. Space dependent linear properties (as density, or sound velocity) introduce dispersion, which may be very strong at some frequency ranges. On the other hand, propagation in a nonlinear medium leads to waveform distortion. This work explores several effects of strong dispersion in acoustic waves propagating in nonlinear media. It is known that the interplay between nonlinearity and dispersion allows for the propagation of solitary waves. Two type of solitary waves are discussed, KdV solitons occurring for weak dispersion, and gap solitons, envelope waves with carrier frequency within a stopband (gap) such that any linear wave turns out to be exponentially decaying. Their existence is based on different physical arguments.

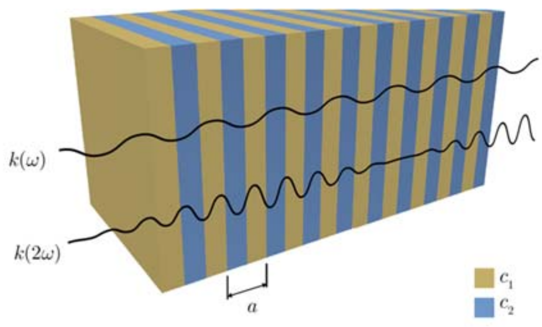

Figure 1 – Schematic model of a multilayered structure and waves propagating near the bragg regime.

2 Model

The propagation of sound can describe by the Navier-Stokes equations, namely the continuity and Euler equations, completed by the equation of state, which in 1D read

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}, \\ &\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial P}{\partial x}, \\ &P = P(\rho) \end{aligned} \tag{1}$$

where $P = p_0 + p$ is the pressure, v is the particle velocity (in general a vector, here a scalar), $\rho = \rho_0 + \rho'$ is the density. Note that p is called the acoustic pressure, and subscript 0 refers to equilibrium values.

Apart from the convective nonlinearity in the Euler equation, also an acoustical nonlinearity appears in the state equation. Expanding up to second order, we deal with a quadratic nonlinearity which is dominant for most of the fluids

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0}\frac{B}{2A}\rho'^2 \tag{2}$$

where, c_0 the (linear) sound speed, and B/A is the nonlinearity parameter. Again, the medium parameters ρ_0, c_0 and B/A can be considered as space-dependent (periodic in the case of a crystal). Here we will consider only c_0 as a space dependent parameter, the analysis is easily extendable to other cases.

Under some assumptions [3] the acoustic equations Eqs. (1) can be written in the form of a nonlinear wave equation, the so called Westervelt equation, which is more convenient for the analysis,

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}, \tag{3}$$

The Westervelt equation is still valid for inhomogeneous media, whose properties are space dependent. Here we consider that the medium is a one-dimensional periodic structure, consisting of layers of elastic isotropic materials of two types, periodically distributed (along the propagation axis), with thickness d_i (i = 1, 2), being c_i the velocity of the sound wave in the *i*-th layer. The structure is periodic with period (lattice constant) $d = d_1 + d_2$. Such a structure is often called an elastic superlattice (SL).

The dispersion relation, or band structure, of the multilayer can be expressed analytically. It is given by the so called Rytov formula [4]

$$\cos(kd) = \cos(k_1 d_1)\cos(k_2 d_2) - \frac{1}{2}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\sin(k_1 d_1)\sin(k_2 d_2) \tag{4}$$

Figure 2 shows the dispersion relation given by Eq. (4) with dashed lines, where the appearance of different propagation bands and bandgaps between them is observed. Four propagation bands are shown.

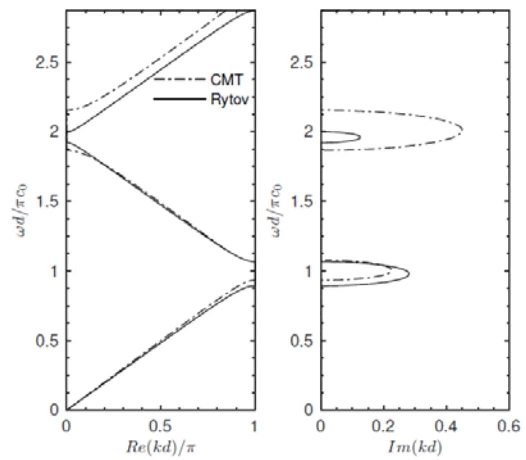

Figure 2 – Band structure obtained for a water-glycerol multilayer with the following parameters: d_1=0.02 m, d_2=0.02 m, c_1=1483 m/s, ρ_1=998 kg/m^3, c_2=1920 m/s, ρ_2=1260 kg/m^3.

A wavepacket propagating in the layered medium will experience the effects of nonlinearity and dispersion. Depending of the frequency, is possible to obtain solitary wave propagation regimes under two different regimes, as discussed in the following sections.

3 KdV solitons

Such propagative solutions correspond to broad localized perturbations (with a typical width much larger that the lattice constant) that propagate keeping its shape in the small wavenumber region of the dispersion relation [4]. Using travelling coordinates, Westervelt equation can be converted into the Burgers equation, as is well known in the literature [3]. To describe a layered medium, this equation includes an additional cubic dispersion term, which result from the asymptotic expansion of the Rytov dispersion relation at low wavenumbers. In particular, Eq. (4) takes the form $\omega = c_0 k + bk^3$, where

$$c_0 = \sqrt{\frac{d c_1^2 c_2^2}{d_2 c_1^2 + d_1 c_2^2}}, \qquad b = \frac{d^2}{24} c_0 \tag{5}$$

are the linear sound velocity in the layered media and the dispersion coefficient. The corresponding evolution equation can be cast then in the form of the well-known KdV equation

$$\frac{\partial v}{\partial t} - \left(c_0 + \beta v\right)\frac{\partial v}{\partial x} + b\frac{\partial^3 v}{\partial x^3} = 0 \ . \tag{6}$$

Equation (6) is known to possess exact solitary wave solutions in the form

$$p(x,t) = A\operatorname{sec}h^2\left(\gamma(x - Vt)\right) \ . \tag{7}$$

An example of a typical KdV solitary wave, is shown in Fig. 3, where the numerical solution of the full nonlinear acoustic equation system, Eqs. (1) is compared with the numerical solutions of the derived KdV equation, Eq. (6).

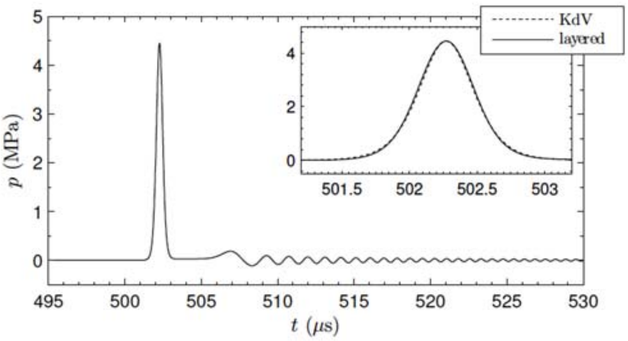

Figure 3 – KdV solitary wave in the layered medium with c_2/c_1=0.1.

The presented KdV-like soliton exist only for a particular set of conditions. In general, the dispersion near the band gap cannot be accurately described by the KdV dispersion. Furthermore, in case of high-amplitude/low-width solitary waves, the soliton spectrum can reach higher bands, that are inexistent in the KdV model. Here, we have briefly presented the existence of solitons when its width is comparable to the lattice constant. Figure 4 shows four examples of propagating localized waves with different amplitudes. As can observed, as the soliton amplitude increases its speed increases and its width reduces, as is commonly observed in other solitary waves. However, for high amplitude solitons, in which the width of the localized wave is of the order of the lattice constant, there exist remarkable differences with the continuum KdV approximation. In these simulations, the initial excitation is the KdV solution. Thus, for the low amplitude regime (Fig. 4(a)) the soliton solution matches the excitation and propagates without changing its shape, amplitude and velocity. However, for higher amplitudes (Fig. 4(b, c)), it can be observed that the KdV soliton is not the exact solution: the solitary wave breaks into other wave-packets plus an oscillatory tail that its left behind. Finally, in the regime where the width of the soliton spectrum extends over the band-gap and up to the second band, i.e. for very localized waves (Fig. 4 (d)), the solitary wave suffers from a process of radiation

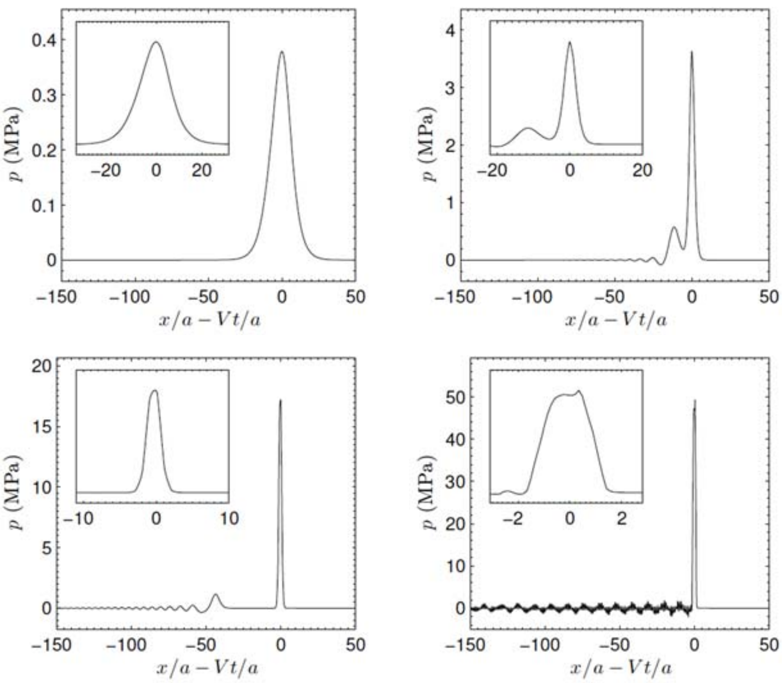

Figure 4 – Examples of KdV solitons for different amplitudes.

4 Gap solitons

Gap solitons are wavepackets with a carrier frequency inside the bandgap (where linear waves can not propagate) modulated by a broad envelope [6], and therefore different from KdV solitons. In order to demonstrate the existence of gap solitons in acoustic layered media, we follow a usual procedure, by deriving a simplified system of coupled-mode equations which describe the evolution of the envelopes of the first and second harmonics (higher harmonics are ignored in this approximation). Near the band-gaps, waves are strongly scattered backwards, therefore the acoustic field can be expanded as a sum of forward (A_n) and backward (B_n) propagating components for each harmonic

$$p(x,t)=\sum_{n=1,2} A_n(x,t)e^{ik_nz-in\omega t}+B_n(x,t)e^{-ik_nz-in\omega t}$$

and assume that the velocity in the medium is modulated as $c(x)=c_0+\Delta c\sin(2k_Bx)$, where $k_B=\pi/d$ denotes the wavenumber at the bragg frequency (at the center of the gap). The slow envelopes obey

$$\begin{aligned}
i\left(\frac{1}{v_1}\frac{\partial A_1}{\partial t}-\frac{\partial A_1}{\partial z}\right)&=\Delta k_1A_1+m_1B_1+\gamma_1A_2A_1^*\\
i\left(\frac{1}{v_1}\frac{\partial B_1}{\partial t}-\frac{\partial B_1}{\partial z}\right)&=\Delta k_1B_1+m_1A_1+\gamma_1B_2B_1^*\\
i\left(\frac{1}{v_2}\frac{\partial A_2}{\partial t}-\frac{\partial A_2}{\partial z}\right)&=\Delta k_2A_2+m_2B_2+\gamma_2A_1^2\\
i\left(\frac{1}{v_2}\frac{\partial B_2}{\partial t}-\frac{\partial B_2}{\partial z}\right)&=\Delta k_2B_2+m_2A_2+\gamma_2B_1^2
\end{aligned} \tag{8}$$

where v_j are group velocities, $m_n = (\Delta c / c_0) k_n / 2$ are the modulation parameters (for n=1,2), and $\Delta k_n = k_n - n k_B$ are the detuning with respect to the bragg resonances and $\gamma_n = (\beta / \rho_0 c_0^2) 2k_1 / k_n$ is the nonlinearity parameter. Linearizing, it is straightforward to obtain the dispersion relation in a very simple form. For the first band gap it reads

$$\omega = \omega_B \pm \sqrt{m^2 + c_0^2 (k_B - k)^2} \tag{9}$$

and analogously for the second bandgap. In Fig. 2 the dispersion relations given by Eq. (9) are shown together with the exact solution for a multilayer. Note the validity of Eq. (9) to describe the dispersion of the system around the bragg frequencies.

Equations similar to Eqs. (8) have been derived by Conti et al. [7] for light propagation in periodic media, where gap solitons have been demonstrated. We show here the analogous solutions for the acoustic field. We have solved Eqs. (8) for an incident pulse with frequency inside the first bandgap (but near the lower band). The result is shown in Fig. 5, which show the wave amplitude in a space-time diagram. The wave reaches the periodic medium from the bottom, and is partially reflected and partially transmitted. The medium of incidence is linear, and therefore the second harmonic is not present in the wave incoming in the nonlinear periodic medium. In the medium, a localized wavepacket is excited, with first and second harmonic components, both propagating at the same group velocity. The corresponding solution is a gap soliton or simulton.

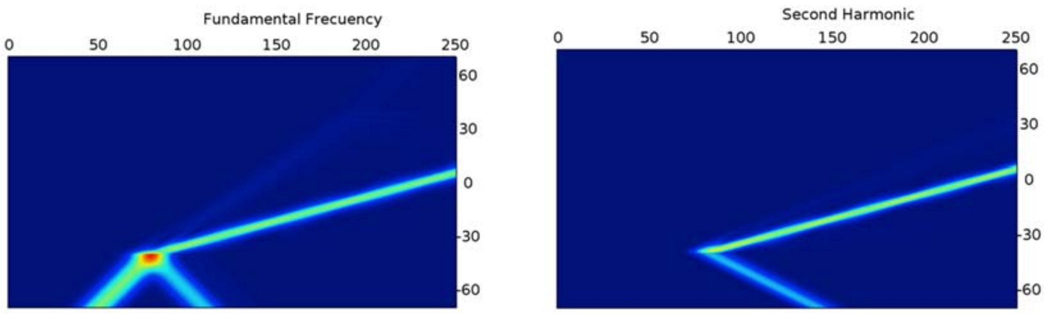

Figure 5 – Gap solitons obtained from numerical solution of the acoustic coupled mode equations for m=1 and $\Delta k = -0.9$.

5 Conclusions

Two type of solitary waves have been predicted for acoustic multilayers composed by alternating fluid layers of different properties. These solutions have been studied by deriving a Korteweg-de Vries (KdV) and coupled-mode equations respectively.

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