# MAXIMAL FACTORIZATION OF OPERATORS ACTING IN KÖTHE-BOCHNER SPACES

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Abstract. Using some representation results for Köthe-Bochner spaces of vector valued functions by means of vector measures, we analyze the maximal extension for some classes of linear operators acting in these spaces. A factorization result is provided, and a specific representation of the biggest vector valued function space to which the operator can be extended is given. Thus, we present a generalization of the optimal domain theorem for some types of operators on Banach function spaces involving domination inequalities and compactness. In particular, we show that an operator acting in Bochner spaces of $p$-integrable functions for any $1 < p < \infty$ having a specific compactness property can always be factored through the corresponding Bochner space of 1-integrable functions. Some applications in the context of the Fourier type of Banach spaces are also given.

## 1. Introduction

The effective computation of the maximal domain of operators that are defined on a certain class of Banach spaces is an important topic in functional analysis. For instance, some classical problems on almost everywhere convergence of trigonometric series in spaces of integrable functions can be understood as an attempt of finding the bigger function space preserving this property. This problem has sometimes a solution —that is, there is a maximal space satisfying the given property, see for example Ch.4 and Ch.7 in [22]—, but in other cases such a space does not

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exist. For instance, compactness of the operator has been one of the properties studied from this point of view, for which there is no such maximal factorization [21] (see also [5]).

In recent years, some effort has been made for the determination of the maximal domains —also called optimal domains— of some classical operators in the class of the order continuous Banach function spaces. The most widespread technique for giving a representation of such spaces is based on vector measures and integration. Generalizing this method, in this paper we are interested in the analysis of the existence of the optimal domain of operators acting in Köthe-Bochner spaces of vector valued integrable functions $X(\mu, Y)$, for the class of all such spaces of function having values in a fixed Banach space $Y$, where $X(\mu)$ is an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$.

In this paper we prove the following maximality result: under some additional requirements, given a Banach space valued operator $T : X(\mu, Y) \to E$, we can find a maximal space $Z(\eta, Y)$ such $X(\mu, Y) \subseteq Z(\eta, Y)$ and $T$ can be extended to it preserving continuity. A complete description of such space is given, as a Köthe-Bochner space in which the functional part is a space $L^1$ of a vector measure defined by the operator (Theorem 6). If some additional compactness property is required for the operator, the optimal factorization space can be characterized as a specific $L^1$-space, of a scalar measure in this case (Theorem 12).

Actually, our analysis involves factorization of bilinear maps through the pointwise product of functions in $X(\mu)$ and vectors in $Y$, and a linear map acting in a Köthe-Bochner function space. Therefore, our method combines factorization of bilinear operators and integral representation of linear maps. As a consequence, we prove that under a compactness requirement for the operator —to be right-uniformly-compact—, a continuous linear map acting in a Bochner space $L^p(\mu, Y)$ can be factored through $L^1(\mu, Y)$ (Corollary 14).

## 2. Preliminaries and notation

Since the original work of Bochner [3] on integration of vector valued functions was published in the first part of the 20th century, Köthe-Bochner spaces have been fixed as a relevant tool in functional analysis. Let us recall now the definition of Köthe-Bochner space (sometimes also called mixed norm spaces, see Ch.7 in [16]). Fix a finite measure space $(\Omega, \Sigma, \mu)$. If $X(\mu)$ is a Banach space consisting of (equivalence classes of) measurable functions, then we say that $X(\mu)$ is a *Banach function space* if it is an *ideal* of measurable functions and has a *Riesz norm*. That is, if $g$ is a measurable function, $f \in X(\mu)$ and $|g| \leq |f|$, then we always have $g \in X(\mu)$ and $\|g\| \leq \|f\|$. Thus, we follow the standard definition given in [19, p.28] under the name "Köthe function space". A Banach function space is order continuous if each decreasing sequence $f_n \downarrow$ in it converging almost everywhere to 0, converges to 0 also in the norm.

If $X(\mu)$ is a Banach function space and $Y$ a Banach space, the Köthe-Bochner space $X(\mu, Y)$ is defined to be the space of ($\mu$-a.e. classes of) strongly measurable functions $f : \Omega \to Y$ satisfying that $w \rightsquigarrow \|f(w)\|_Y \in X(\mu)$, endowed with the norm

$$\|f\|_{X(\mu,Y)} := \big\|\|f\|_Y(w)\big\|_{X(\mu)}.$$

The reader can find more information in [6, 7, 18]. A particular relevant case of this class of spaces are the Bochner spaces of $p$-integrable functions, in which $X(\mu) = L^p(\mu)$ for a finite measure $\mu$. These spaces allow a tensor product representation that will be relevant for this paper. It is well-known that

$$L^1(\mu, Y) = L^1(\mu)\hat{\otimes}_\pi Y,$$

the completion of the projective tensor product (see for example Proposition 1.8.6 in [18]; see [9] for more information about topological tensor products). For the case of $1 < p \leq \infty$, a similar representation can be obtained, but the projective norm $\pi$ has to be substituted by the so called natural norm $\Delta_p$, defined in the obvious way by using the $L^p$-norm.

The study of the relation of vector measure integration and representation of operators was started by Bartle, Dunford and Schwartz in the foundational paper [2]. The corresponding theory of integration was fixed by [17] and other authors (see [22] and the references therein), and has found a lot of applications, mainly by using the integral representation of operators that if provides.

Our notation is standard. For a Banach space $X$ we write as usual $B_X$ for its closed unit ball and $X^*$ for its dual space. We will consider finite measure spaces $(\Omega, \Sigma, \mu)$, and Banach space valued measures $\nu : \Sigma \to X$ on the measurable space $(\Omega, \Sigma)$. Once the vector measure $\nu$ is fixed, we can define the family of scalar measures given by the variations $|\langle \nu, x^* \rangle|$ of all the scalarizations $\langle \nu, x^* \rangle(\cdot) := x^*(\nu(\cdot))$ of the measure $\nu$, $x^* \in B_{X^*}$.

If $\nu$ is a vector measure, the reader can find in [11, Chapter IX]) the definition of Rybakov measure. Such a positive finite measure can be written as the variation $|\langle \nu, x^* \rangle|$ of a scalar measure as $\langle \nu, x^* \rangle$ for a certain norm one element $x^* \in X^*$ such that $\nu$ and $|\langle \nu, x^* \rangle|$ are equivalent, that is, they have the same null sets.

A function $f : \Omega \to \mathbb{R}$ is *integrable with respect to* $\nu$ if it is measurable and integrable with respect to all the scalar measures $|\langle \nu, x^* \rangle|$, $x^* \in X^*$, and satisfies that for each $A \in \Sigma$ there exists a vector $\int_A f \, d\nu \in X$ such that

$$\int_A f \, d\langle \nu, x^* \rangle = \langle \int_A f \, d\nu, x^* \rangle, \quad x^* \in X^*.$$

The space $L^1(\nu)$, whose elements are the (classes of) integrable functions with respect to $\nu$, is an order continuous Banach function space over any Rybakov measure $\mu$. Its norm —that is a lattice norm— is given by the formula

$$\|f\|_{L^1(\nu)} = \sup\{\int_\Omega |f| d|\langle \nu, x^* \rangle| : x^* \in B_{X^*}\}.$$

The expression

$$|||f|||_{L^1(\nu)} = \sup_A \big\| \int_A f \, d\nu \big\|_E$$

gives an equivalent norm for $L^1(\nu)$, that is not a lattice norm. Moreover, if $E$ is a Banach lattice, we can compute the norm by the formula

$$\|f\|_{L^1(\nu)} = \Big\| \int_\Omega |f| d\nu \Big\|_E.$$

The space $L^1(\nu)$ has also a weak unit, the function $\chi_\Omega$. The reader can find this and a complete explanation of related concepts in [22, Ch.2 and Ch.3].

Let us mix now both classes of function spaces to get a representation that will be relevant for the paper. It must be said that, by a well-known representation theorem for Banach function lattices, every order continuous Banach function space with a weak order unit can be written isometrically and in the order as a space $L^1(\nu)$ of a vector measure $\nu$ (Proposition 3.9 in [22]). Recall that we are considering a measurable space $(\Omega, \Sigma)$ and a vector measure $\nu : \Sigma \to X$ over it. If $Y$ is a Banach space and $\mu$ is a scalar Rybakov measure for $\nu$, we will consider the Köthe-Bochner space $L^1(\nu, Y) := X(\mu, Y)$, where $X(\mu) = L^1(\nu)$. Note also that the order continuity of $L^1(\nu)$ allows to deduce that simple functions are dense in the corresponding Köthe-Bochner space with the norm defined by the expression above (see for example [15, Lema 1.51]). Operators acting in Köthe-Bochner spaces is a classic topic of current interest. The interested reader can find recent papers on this subject, for example on special classes of operators on these spaces [1, 12, 13], and on the integral representation of such operators ([14, 20]).

## 3. Linear operators acting in Köthe-Bochner spaces, bilinear maps and representation of maximal Köthe-Bochner spaces

We will show two representations of the space of integrable functions that will become the optimal domain of an operator $T$ acting in a Köthe-Bochner space. In fact, we will consider maximality for the bilinear map given by any linear operator like this, following some ideas presented in [4] and [23]. In order to do that, we will consider vector measures on $\ell^\infty$-spaces as well as operator-valued measures. One of the spaces is the one composed by integrable functions with respect to an operator valued vector measure, that we will denote by $L^1(\nu_{T_B})$. Since the description is not giving a direct information about its structure, we describe another space of integrable functions —$L^1(m_B)$—, in this case with respect to a *positive* vector measure with values in an $\ell^\infty$-space. The definition of the norm in this space gives a clear description of how the space is constructed, starting from the operator $T$. We begin by proving that both spaces $L^1(\nu_{T_B})$ and $L^1(m_B)$ actually coincide.

Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$ and let $Y$ and $E$ be Banach spaces. Consider a bilinear map $B : X(\mu) \times Y \to E$. It defines a linear operator $T_B : X(\mu) \to \mathcal{L}(Y, E)$ in the usual manner—$T_B(f)(\cdot) := B(f, \cdot)$—, that has an associated vector measure $\nu_{T_B} : \Sigma \to \mathcal{L}(Y, E)$.

On the other hand, let us show that a related $\ell^\infty$-space-valued measure can be defined in the following simple way. Note first that for each $y \in Y$, the map $B(\chi_\cdot, y)$ given by $A \mapsto B(\chi_A, y)$ is a countably additive vector measure, by the continuity of $B$ and the order continuity of $X(\mu)$. Moreover, for each $y \in Y$ and $x' \in B_{E^*}$, we can also consider the scalar measure

$$m_{B,y,x'}(A) := \langle B(\chi_A, y), x' \rangle, \quad A \in \Sigma.$$

This allows to define the *positive* finitely additive measure by means of the semivariations of these measures as

$$m_B(A) := \Big( |m_{B,y,x'}|(A) \Big)_{y \in B_Y,\, x' \in B_{E^*}} \in \ell^\infty(B_Y \times B_{E^*}).$$

Let us show first that it is well-defined. Clearly, each measure $A \mapsto \langle B(\chi_A, y), x' \rangle$ is countably additive by the order continuity of $X(\mu)$. On the other hand, if we fix $A \in \Sigma$ we get

$$\sup_{y \in B_Y,\, x' \in B_{E^*}} \big| \langle B(\chi_A, y), x' \rangle \big| \le \|B\| \, \|\chi_A\|_{X(\mu)} \, \|y\| \le \|B\| \, \|\chi_\Omega\|_{X(\mu)} < \infty.$$

Therefore, the set function $m_B : \Sigma \to \ell^\infty(B_Y \times B_{E^*})$ is well-defined. Let us show now that in fact the vector measure $m_B$ is countably additive.

**Lemma 1.** *For a bilinear operator $B : X(\mu) \times Y \to E$ and an order continuous Banach function space $X(\mu)$ over a finite measure $\mu$, the function $m_B$ defined as above is a countably additive vector measure.*

*Proof.* Take a disjoint sequence of measurable sets $(A_i)_{i=1}^{\infty}$. Let us show that

$$\lim_n \|m_B(\cup_{i=n}^{\infty} A_i)\|_E = 0.$$

Indeed, for $y \in B_Y$ and $x' \in B_{E^*}$ we have that there is a measurable set $C \subseteq \cup_{i=n}^{\infty} A_i$ such that

$$|m_{B,y,x'}|(\cup_{i=n}^{\infty} A_i) = m_{B,y,x'}(C) - m_{B,y,x'}(\cup_{i=n}^{\infty} A_i \setminus C)$$

$$\leq \|B(\chi_C, y)\|_E + \|B(\chi_{\cup_{i=n}^{\infty} A_i \setminus C}, y)\|_E \leq 2\,\|B\|\,\|\chi_{\cup_{i=n}^{\infty} A_i}\|_{X(\mu)}\,\|y\|_Y.$$

Since $X(\mu)$ is order continuous, this gives a uniform bound that converges to 0. Thus, we get the result. □

**Lemma 2.** *Let $B : X(\mu) \times Y \to E$ be a Banach space valued bilinear map. Suppose that $X(\mu)$ is an order continuous Banach function space over a finite measure $\mu$. Consider the vector measures $m_B$ and $\nu_{T_B}$ associated to $B$ and defined as above. Then the corresponding spaces of integrable functions coincide, that is*

$$L^1(\nu_{T_B}) = L^1(m_B).$$

*Moreover,*

$$|||\cdot|||_{L^1(m_B)} \leq 2\,|||\cdot|||_{L^1(\nu_{T_B})} \leq 2\,\|\cdot\|_{L^1(m_B)} \leq 4\,|||\cdot|||_{L^1(m_B)}.$$

*Proof.* First notice that, by Lemma 1, $m_B$ is a countably additive vector measure and so $L^1(m_B)$ is well-defined. Consider a simple function $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$, where $A_1, ..., A_n$ are disjoint sets, and note that

$$|||f|||_{L^1(\nu_{T_B})} = \sup_{A, y \in B_Y} \|T_B(f\chi_A)(y)\|_E = \sup_{A, y \in B_Y} \|B(f\chi_A, y)\|_E.$$

On the other hand, for each couple $(y,x')\in Y\times E^*$ the Hahn decomposition of a measure gives a set $C_{y,x'}\in\Sigma$ such that the following inequalities hold.

$$|\|f\||_{L^1(m_B)}=\sup_{A\in\Sigma, y\in B_Y,\, x'\in B_{E^*}}\Big(\int_A f\, d|m_{B,y,x'}|\Big)$$

$$=\sup_{A\in\Sigma, y\in B_Y,\, x'\in B_{E^*}}\Big(\Big\langle\int_{A\cap C_{y,x'}} f\, dB(\chi_{\{\cdot\}},y),x'\Big\rangle+\Big\langle\int_{A\cap C_{y,x'}^c} f\, dB(\chi_{\{\cdot\}},y),x'\Big\rangle\Big)$$

$$\leq \sup_{A\in\Sigma, y\in B_Y, x'\in B_{E^*}}\|B(f\chi_{A\cap C_{y,x'}},y)\|_E+\sup_{A\in\Sigma, y\in B_Y, x'\in B_{E^*}}\|B(f\chi_{A\cap C_{y,x'}^c},y)\|_E$$

$$\leq 2\sup_{A, y\in B_Y}\|B(f\chi_A,y)\|_E=2|\|f\||_{L^1(\nu_{T_B})}$$

$$\leq 2\sup_{A\in\Sigma, y\in B_Y,\, x'\in B_{E^*}}\Big\langle\int_A f\, dB(\chi_{\{\cdot\}},y),x'\Big\rangle\leq 2\sup_{y\in B_Y,\, x'\in B_{E^*}}\int_\Omega |f|\, d|m_{B,y,x'}|$$

$$=2\,\|f\|_{L^1(m_B)}\leq 4\sup_{A\in\Sigma, y\in B_Y,\, x'\in B_{E^*}}\int_A f\, d|m_{B,y,x'}|.$$

Taking into account that simple functions are dense in both spaces, the result is obtained. These computations give also the inequalities among the norms. □

The next result is a version of Proposition 3.1 in [4]. We write it for the aim of completeness. Recall that it is said that $B$ is $\mu$-determined if for every $A\in\Sigma$, the set function $\sup_{C\subset A, y\in B_Y}\|B(\chi_C,y)\|_E$ equals 0 if and only if $\mu(A)=0$

**Lemma 3.** *Let $B:X(\mu)\times Y\to E$ be a Banach space valued bilinear map, where $X(\mu)$ is an order continuous Banach function space over a finite measure $\mu$. Consider the vector measure $\nu_{T_B}$. Then*

- (i) *there is a Rybakov measure $\eta$ for $\nu_{T_B}$ such that $\eta\ll\mu$, and*
- (ii) *the inclusion/quotient map $[i]:X(\mu)\to L^1(\nu_{T_B})$ given by $[i](f)=[f]_{\nu_{T_B}}$ for $f\in X(\mu)$, is well-defined and continuous, and $\|[i]\|\leq\|B\|$.*
- (iii) *If $\mu$ is equivalent to the Rybakov measure $\eta$ —that is, to $\nu_{T_B}$—, then $[i]$ is injective. This happens if and only if $B$ is $\mu$-determined.*

*Proof.* (i) We consider the vector measure $\nu_{T_B}:X(\mu)\to\mathcal{L}(Y,E)$. By Rybakov's Theorem (see Ch.IX in [11]) we have that for every $E$-valued countably additive vector measure $m$ we can find a measure $\eta=|\langle m,x'\rangle|$ for a certain $x'\in B_{E^*}$ such that $m$ and $\eta$ have the same null sets. Applying this to $\nu_{T_B}$ we get the result.

(ii) In general, we have that for measurable sets $A$ and $C$,

$$\|\nu_{T_B}\|(A) = \sup_{C \subseteq A} \|\nu_{T_B}(C)\|_{\mathcal{L}(Y,E)}$$

$$= \sup_{C \subseteq A} \|T_B(\chi_C)\|_{\mathcal{L}(Y,E)} = \sup_{C \subseteq A,\, y \in B_Y} \|B(\chi_C, y)\|_E \leq \sup_{C \subseteq A} \|B\| \|\chi_C\| \leq \|B\| \|\chi_A\|.$$

Thus, since $X(\mu)$ is a Banach function space we have that $\|\chi_A\| = 0$ if and only if $\mu(A) = 0$, and then $\|\nu_{T_B}\|(A) = 0$. Therefore, for a couple of measurable functions $f, g \in L^0(\mu)$ such that $f = g$ $\mu$-a.e., we have that $f = g$ also $\eta$-a.e., and so $[i](f) = [i](g)$. The map $[i]$ is then well-defined. Moreover, taking into account the equivalent norm for the space $L^1(\nu_{T_B})$, if $f \in X(\mu)$,

$$\|[i](f)\|_{L^1(\nu_{T_B})} = \sup_{A \in \Sigma} \|T_B(f\chi_A)\|_{\mathcal{L}(Y,E)} = \sup_{A \in \Sigma,\, y \in B_Y} \|B(f\chi_A, y)\|_E \leq \|B\| \|f\|_{X(\mu)}.$$

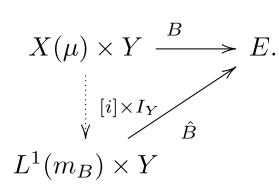
This proves (ii).

(iii) Just note that, if $\mu$ is equivalent to the Rybakov measure $\eta$, we have that for any couple of $\mu$-measurable functions, $f = g$ $\mu$-a.e. if and only in $f = g$ $\eta$-a.e. Then, since $\eta$ is equivalent to $\nu_{T_B}$, we have that $f = g$ $\mu$-a.e. if and only if $[f] = [g]$, what gives the injectivity of $[i]$.

□

We will write $I_Y : Y \to Y$ for the identity map.

**Proposition 4.** *Let $B : X(\mu) \times Y \to E$ be a Banach space valued bounded bilinear map, where $X(\mu)$ is an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$. Consider the vector measure $m_B$. Then $B$ can be factored through the following scheme,*




*Moreover, this factorization is optimal, in the sense that for any order continuous Banach function space $Z(\eta)$ over the measure space $(\Omega, \Sigma, \eta)$ such that*

1) $\eta \ll \mu$, *and*

2) $B$ *can be factored through* $Z(\eta) \times Y$ *by* $[i] \times I_Y$,

*we have that* $Z(\eta) \to_{[i_Z]} L^1(m_B)$.

*Proof.* This is a direct consequence of Lemma 3. Indeed, recall that by Lemma 2, $L^1(m_B) = L^1(\nu_{T_B})$, and write $B_Z$ for the bilinear map $B$ extended to $Z(\eta) \times Y$. Since the measurable space where both vector measures are defined is the same, we get that $\nu_{T_{B_Z}} = \nu_{T_B}$. By assumption, we have that the inclusion/quotient map $[i] : X(\mu) \to Z(\eta)$ is well-defined and continuous, and on the other hand, by Lemma 2 we have that the inclusion/quotient map $[i_Z] : Z(\eta) \to L^1(\nu_{T_{B_Z}}) = L^1(m_B)$ is also well-defined and continuous.

□

In general, we cannot translate the optimality of $L^1(m_B)$ for the factorization of $B$ in the sense of Proposition 4 to a factorization of a *linear* operator acting in a Köthe-Bochner space as $T : X(\mu, Y) \to E$. However, there are some cases for which we have that the factorization of the bilinear operator $B_T : X(\mu) \times Y \to E$ defined as

$$B_T(f, y) = T(f \cdot y), \quad f \in X(\mu), \;\; y \in Y,$$

can be used to construct a factorization for the operator $T$. We will write $T_B : X(\mu) \to \mathcal{L}(Y, E)$ for the linear operator induced by the bilinear map $B_T$. The next result shows that the required property is what we call to be right-dominated, and is written in terms of inequalities. As the reader will see, this is some sort of Banach space version of the positive operators for the case of operators between Banach lattices.

We will say that a bounded linear operator $T : X(\mu, Y) \to E$ is *right-dominated* if there is a constant $K > 0$ such that

$$\big\| T(f) \big\|_E \leq K \left\| T_B(\|f\|_Y) \right\|_{\mathcal{L}(Y,E)}, \quad f \in X(\mu, Y).$$

*Example* 5. Suppose that simple functions are dense in $X(\mu, Y)$. An easy example of a right-dominated operator is given by the linear operator $T_0 : X(\mu, Y) \to E$ defined for simple functions by $T_0(\sum_{i=1}^n \chi_{A_i} \cdot y_i) = \sum_{i=1}^n S(\chi_{A_i}) \cdot \langle y_i, y_0' \rangle$, where $E$ is a Banach lattice, $S : X(\mu) \to E$ is a positive operator and $y_0' Y^*$ is a norm one functional.

Indeed, take a simple function $f = \sum_{i=1}^n \chi_{A_i} \cdot y_i$. Then

$$\|T_0(f)\|_E = \| \sum_{i=1}^n S(\chi_{A_i}) \cdot \langle y_i, y_0' \rangle \|_E = \|S(\sum_{i=1}^n \chi_{A_i} \cdot \langle y_i, y_0' \rangle)\|_E$$

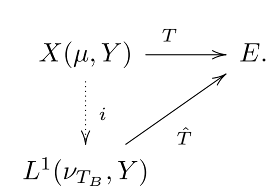
$$\leq \|S(\sum_{i=1}^n \chi_{A_i} \cdot \|y_i\|)\|_E = \sup_{y \in B_Y} \|S(\sum_{i=1}^n \chi_{A_i} \cdot \|y_i\|) \cdot \langle y, y_0' \rangle \|_E$$

$$= \sup_{y \in B_Y} \|B_{T_0}(\sum_{i=1}^n \chi_{A_i} \cdot \|y_i\|, y)\|_E = \|(T_0)_B(\|f(w)\|_Y)\|_{\mathcal{L}(Y,E)}.$$

Therefore, $T_0$ is right-dominated for $K = 1$.

Another relevant situation in which we can translate the optimality obtained for the factorization of bilinear maps to a linear operator acting in a Köthe-Bochner space is when $B_T$ is right-uniformly-compact. This will be explained later. Let us show first the optimal factorization for right-dominated operators. We will explain it for the case of $\mu$-determined operators. Another version is available without this assumption, just changing inclusions by inclusion/quotient maps, in the spirit of Lemma 3.

**Theorem 6.** *Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$. Let $T : X(\mu, Y) \to E$ be a $\mu$-determined right-dominated operator. Then $T$ can be factored through the Köthe-Bochner space $L^1(\nu_{T_B}, Y)$ as*



*Moreover, this factorization is optimal, in the sense that for any Köthe-Bochner space $Z(\eta, Y)$ such that $Z(\eta)$ is order continuous —for $\eta$ equivalent to $\mu$— and such that $T$ can be extended to $Z(\eta, Y)$, we have that $Z(\eta, Y) \subset L^1(\nu_{T_B}, Y)$.*

*Proof.* Note first that $T$ can be used to define a bilinear map $B : X(\mu) \times Y \to E$ as $B(f, y) := T(f \cdot y)$. The continuity of $T$ gives that

$$\|B(f,y)\|_E = \|T(f \cdot y)\|_E \leq \|T\| \, \Big\| f \, \|y\|_Y \Big\|_{X(\mu,Y)} = \|T\| \, \|f\|_{X(\mu)} \, \|y\|_Y,$$

and then $B$ is continuous. Thus, we can consider the associated measure $\nu_{T_B} : \Sigma \to \mathcal{L}(Y, E)$. Take a function $f \in X(\mu, Y)$. Then, by Lemma 3( (ii) and (iii) ), we have that

$$\|f\|_{L^1(\nu_{T_B},Y)} = \big\| \, \|f(w)\|_Y \big\|_{L^1(\nu_{T_B})} \leq \|B\| \, \big\| \, \|f(w)\|_Y \big\|_{X(\mu)} = \|B\| \, \|f\|_{X(\mu,Y)}.$$

On the other hand, note that by the order continuity of both spaces $X(\mu)$ and $L^1(\nu_{T_B})$, the first mentioned space is dense in the second one, and so we only need to define the extension $\hat{T}$ for $f \in X(\mu, Y)$. Then we can use the formula $\hat{T}(f) = T(f)$. Let us prove that it is continuous when it acts in $L^1(\nu_{T_B}, Y)$. Identifying the tensor product of the space $X(\mu) \otimes Y$ with a subspace of $X(\mu, Y)$, we can take a simple function $f = \sum_{i=1}^n \chi_{A_i} \otimes y_i \in X(\mu) \otimes Y \subseteq X(\mu, Y)$, for disjoint measurable sets $A_1, ..., A_n$. The tensor product is dense in the Köthe-Bochner space, and so simple functions are too. Then since $T$ is right-dominated, we have that

$$\|\hat{T}(f)\|_E = \|T(f)\|_E \leq K \, \big\| \, T_B(\|f(w)\|_Y) \, \big\|_{\mathcal{L}(Y,E)} = K \Big\| \sum_{i=1}^n T_B(\chi_{A_i} \cdot \|y_i\|_Y) \Big\|_{\mathcal{L}(Y,E)}$$

$$\leq K \sup_{A \in \Sigma} \Big\| \sum_{i=1}^n T_B(\chi_{A_i \cap A} \|y_i\|_Y) \Big\|_{\mathcal{L}(Y,E)} = K \sup_A \Big\| \sum_{i=1}^n \nu_{T_B}(A_i \cap A) \|y_i\|_Y \Big\|_{\mathcal{L}(Y,E)}$$

$$= K \sup_A \Big\| \int_A \Big( \sum_{i=1}^n \chi_{A_i} \|y_i\|_Y \Big) \, d\nu_{T_B} \Big\|_{\mathcal{L}(Y,E)} \leq K \, \|f\|_{L^1(\nu_{T_B},Y)}.$$

Thus, the inequality holds for every $f \in L^1(\nu_{T_B}, Y)$.

For the optimality of the factorization it is enough to take into account Lemma 3. If $X(\mu) \subset Z(\eta)$, then the vector measure $\nu_{T_{B_Z}}$ generated by the bilinear map $B$

when defined from $Z(\eta) \times Y$ coincides with $\nu_{T_B}$. Therefore, by Lemma 3 we have that $Z(\eta, Y) \subseteq L^1(\nu_{T_{B_Z}}) = L^1(\nu_{T_B})$, which finishes the proof. □

## 4. Maximal extensions of right-uniformly-compact operators acting in Köthe-Bochner spaces

We will show in what follows that better results —without the restriction for the operator to be right-dominated— can be obtained under some compactness assumptions. The next results show that in this case, we can extend operators acting in the class of Köthe-Bochner spaces to the spaces that are in a sense maximal in this class: Bochner spaces of integrable functions. Using our general result, we will prove some maximality theorems that concern the general theory of operators on spaces of vector valued functions, providing concrete results for relevant classical spaces.

Let $B : X(\mu) \times Y \to E$ a continuous bilinear map. Let us define a new topology for $X(\mu)$ given by the (semi)norm

$$\|f\|_B := \sup_{A \in \Sigma, y \in B_Y} \|B(f\chi_A, y)\|_E, \quad f \in X(\mu).$$

Let us assume in this section that $\|\cdot\|_B$ is a norm, what happens if $B$ is $\mu$-determined. The resulting space is then a normed space. Note that such a bilinear map $B$ can always be used to define a continuous bilinear map $\hat{B} : (X(\mu), \|\cdot\|_B) \times E$, $\hat{B} = B$. Indeed, the continuity of $B$ and the fact that $X(\mu)$ is a Banach function space give that for every $f \in X(\mu)$ and $y \in Y$,

$$\|\hat{B}(f, y)\|_E = \|B(f, y)\|_E \leq \sup_{A \in \Sigma, z \in B_Y} \|B(f\chi_A, z)\| \, \|y\| \leq \|f\|_B \, \|y\|.$$

Recall that an operator from a normed space to a Banach space is compact if it carries the unit ball inside a compact set.

Let us start by proving a result on the relation between the compactness properties of a bilinear map $B : X(\mu) \times Y \to E$ and the associated operator $T_B : X(\mu) \to$

$\mathcal{L}(Y,E)$. We will consider the following compactness-type property for a bilinear map $B$, that characterizes when the associated map $T_B$ is compact.

**Definition 7.** We will say that the bilinear operator $B : X \times Y \to E$ is *right-uniformly-compact* if for every sequence $(x_i) \subset B_X$ there is a subsequence $(x_{i_j})_j$ such that for every $\varepsilon > 0$ we find a natural number $j$ such that for every sequence as $\big((x_{i_j} - x_{i_k}, y_{i_k})\big)_{j \le k} \subset X \times B_Y$ we have that $\|B(x_{i_j} - x_{i_k}, y_{i_k})\|_E < \varepsilon$.

In the same way, we will say that the bilinear operator $B$ is *left-uniformly-compact* if the symmetric property holds, that is, if the bilinear map $B^t : Y \times X \to E$ defined as $B^t(y,x) = B(x,y)$, is right-uniformly-compact.

The next result shows that this property gives a characterization of the compactness of $T_B$. We write the standard proof for the aim of completeness.

**Lemma 8.** *Let $X, Y$ and $E$ be Banach spaces. Then the following statements hold for a bilinear operator $B : X \times Y \to E$.*

(i) *$B$ is right-uniformly-compact if and only if its associated linear operator $T_B : X \to \mathcal{L}(Y,E)$ is compact.*

(ii) *$B$ is left-uniformly-compact if and only if its associated linear operator $T_{B^t} : Y \to \mathcal{L}(X,E)$ is compact.*

(iii) *If $B$ is right-uniformly-compact and left-uniformly compact, then it is compact.*

*Proof.* (i) Suppose first that $B$ is right-compact. Take a sequence of elements $(x_i) \subset B_X$ and consider the subsequence $(x_{i_j})$ with the given properties. Fix $\varepsilon > 0$. Then there is $j_1$ associated to $\varepsilon/2$ such that there is a sequence $(y_k)_{j \le k} \subset B_Y$ satisfying that for every $k \ge j_1$,

$$\|T_B(x_{i_{j_1}}) - T_B(x_{i_k})\|_{\mathcal{L}(Y,E)} < \|B(x_{i_{j_1}} - x_{i_k}, y_{i_k})\|_E + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, $\big(T_B(x_{i_j})\big)$ is a Cauchy sequence, what means that $T_B$ is compact.

Conversely, take a sequence $(x_i) \subset X$. If $T_B(B_X)$ is relatively compact there is a subsequence $(x_{i_j})$ such that $\big(T_B(x_{i_j})\big)$ is a Cauchy sequence in $E$. Therefore for every $\varepsilon$ there is $j$ such that $\|T_B(x_{i_j}) - T_B(x_{i_k})\|_{\mathcal{L}(Y,E)} < \varepsilon$ for each $k \geq j$, that is, for every sequence $(y_k) \subset B_Y$, $\|B(x_{i_j} - x_{i_k}, y_k)\|_E < \varepsilon$. That is, $B$ is right-uniformly-compact.

The proof of (ii) is similar.

(iii) Take a sequence $(x_i, y_i)$ in $B_X \times B_Y$. Then, since $B$ is right-uniformly-compact, we have that there is a subsequence $(x_{i_j})$ with the explained properties. In particular, the subsequence $\big((x_{i_k}, y_{i_k})\big)_k$ satisfies that for every $\varepsilon > 0$ there is $j_0$ such that for $k \geq j_0$,

$$\|B(x_{i_{j_0}} - x_{i_k}, y_{i_k})\|_E < \varepsilon.$$

Note also that in fact $\|B(x_{i_{j_0}} - x_{i_k}, y)\|_E < \varepsilon$ for all $y \in B_Y$. Now we use the fact that $B$ is also left-uniformly-compact. Then there is a subsequence of $(y_{i_j})$ —we use the same notation for it—, such that for every $\varepsilon > 0$ there is $j_1$ such that $\|B(x_{i_k}, y_{i_{j_1}} - y_{i_k})\|_E < \varepsilon$ for each $k \geq j_1$.

Write $j_2 := \max\{j_0, j_1\}$. Then we have that for $k_1, k_2 \geq j_2$,

$$\|B(x_{i_{k_1}}, y_{i_{k_1}}) - B(x_{i_{k_2}}, y_{i_{k_2}})\|_E$$

$$\leq \big\|B(x_{i_{k_1}}, y_{j_{k_1}}) - B(x_{i_{j_2}}, y_{i_{k_1}}) + B(x_{i_{j_2}}, y_{i_{k_1}})$$

$$-B(x_{i_{j_2}}, y_{i_{k_2}}) + B(x_{i_{j_2}}, y_{i_{k_2}}) - B(x_{i_{k_2}}, y_{i_{k_2}})\big\|_E$$

$$\leq \|B(x_{i_{j_2}} - x_{i_{k_1}}, y_{j_{k_1}})\| + \|B(x_{i_{j_2}} - x_{i_{k_2}}, y_{i_{k_2}})\| + \|B(x_{i_{j_2}}, y_{i_{k_1}} - y_{i_{k_2}})\|$$

$$\leq \|B(x_{i_{j_2}} - x_{i_{j_0}}, y_{i_{k_1}})\| + \|B(x_{i_{j_0}} - x_{i_{k_1}}, y_{i_{k_1}})\| + \|B(x_{i_{j_2}} - x_{i_{j_0}}, y_{i_{k_2}})\|$$

$$+\|B(x_{i_{j_0}} - x_{i_{k_2}}, y_{i_{k_2}})\| + \|B(x_{i_{j_2}}, y_{i_{j_1}} - y_{i_{k_2}})\| + \|B(x_{i_{j_2}}, y_{i_{k_1}} - y_{i_{j_1}})\| < 6\varepsilon.$$

Therefore, $\big(B(x_{i_j}, y_{i_j})\big)$ is a Cauchy sequence in $E$, what implies that $B$ is a compact bilinear operator.

□

*Example* 9. Recall that if $f$ is a Bochner integrable function in $L^1(\mu, X)$, we can define its $k$-th Fourier coefficient by means of the vector valued integral

$$\hat{f}(k) := \int_\Omega f(t)e^{ikt}\, d\mu(t), \quad k \in \mathbb{Z}.$$

Let $\Pi$ be the torus, and consider Lebesgue measure $\mu$ on it. Note that for $f = h \cdot x$, $h \in L^p(\Pi)$ and $x \in X$, we have that $\hat{f}(k) = x \cdot \hat{h}_k$. Following the natural continuous Fourier operators from $L^p(\Pi)$ to $\ell^{p'}$ provided by the Hausdorff-Young inequality, a Banach space $X$ is said to have *Fourier type* $p$ for $1 < p \leq 2$ if the map $I_p$ : $L^p(\Pi, X) \to \ell^{p'}(X)$ given by

$$I_p(f) := \big(\hat{f}(k)\big)_{k\in\mathbb{Z}}, \quad f \in L^p(\Pi, X),$$

is well defined and continuous (see for example [8] and the references therein). Let $1 \leq s < p'$ and take the real number $r$ that satisfies $1/r + 1/p' = 1/s$. Take a positive sequence $0 < \alpha = (\alpha_k) \in \ell^r$, and consider the map $M_\alpha : \ell^{p'} \to \ell^s$ given by $M_\alpha((r_k)_k) = (\alpha_k r_k)_k$. It is clearly well-defined and continuous. Recall that Pitt's Theorem states that every operator from $\ell^{p'} \to \ell^s$, for $s < p'$, is compact (see [10] for example for this result and some generalizations). Consequently, $M_\alpha$ is compact. Let us write also $M_\alpha$ for the same operator but defined from $\ell^{p'}(X) \to \ell^s(X)$ in the natural way. We consider now the operator $M_\alpha \circ I_p : L^p(\Pi, X) \to \ell^s(X)$. The associated bilinear map $B_{p,s} : L^p(\Pi) \times X \to \ell^s(X)$ is given by $B_{p,s}(h, x) = (\alpha_k \hat{h}_k \cdot x)_k = (\alpha_k \hat{h}_k)_k \cdot x$. Notice that for $B_{p,s}$ we can consider the operator

$$T_{B_{p,s}} : L^p(\Pi) \to L(X, \ell^s(X)),$$

that is given by the formula

$$T_{B_{p,s}}(h)(x) := B_{p,s}(h, x) = (x\alpha_k \hat{h}_k)_k = x\,(\alpha_k \hat{h}_k)_k \in \ell^s(X), \quad h \in L^p(\Pi),\ x \in X.$$

Let us remark that the functions $h$ above are scalar, and so the map $h \mapsto (\alpha_k \hat{h}_k)_k$ is the standard Fourier operator $\mathcal{F}_p : L^p(\Pi) \to \ell^{p'}$ composed with the multiplication map $\alpha : \ell^{p'} \to \ell^s$. But this composition is compact. Indeed, note that for every

$h \in L^p(\Pi)$,

$$\|T_{B_{p,s}}(h)\|_{L(X,\ell^s(X))} = \sup_{x\in B_X} \|x \cdot (\alpha_k \hat{h}_k)_k\|_{\ell^s(X)} = \sup_{x\in B_X} \|x\| \cdot \|(\alpha_k \hat{h}_k)_k\|_{\ell^s}.$$

In other words, $T_{B_{p,s}}$ factors through the map $M_\alpha \circ \mathcal{F}_p : L^p(\Pi) \to \ell^s$. Since this map is compact, we get that $T_{B_{p,s}}$ is compact. Lemma 8 gives then that $B_{p,s}$ is right-uniformly-compact.

Let us explain now an easy example whose generalization will become the motivation of the rest of the paper. We are interested in determining when the optimal domain of a given operator acting in a Köthe-Bochner space is a Bochner space. The next case will show that this happens when the original bilinear map can be factored through a scalar functional in a certain way; we will show that this situation can be generalized to the case when this kind of factorization is done through a compact operator.

*Example* 10. Fix a finite measure space $(\Omega, \Sigma, \mu)$. Let $R : Y \to E$ be a norm one operator. Let $X(\mu)$ be an order continuous Banach function space—the particular case $X(\mu) = L^p(\mu)$ will be studied later on. Consider the bilinear map $B : X(\mu) \times Y \to E$ given by $B(f, y) := \Big(\int_\Omega f\, d\mu\Big) \cdot R(y)$, that is obviously continuous. Then the operator $T_B : X(\mu) \to \mathcal{L}(Y, E)$ is given by $T_B(f)(\cdot) := \Big(\int_\Omega f\, d\mu\Big) \cdot R(\cdot)$, $f \in X(\mu)$.

Note now that

$$\|f\|_B := \sup_{A\in\Sigma,\, y\in B_Y} \|B(f\chi_A, y)\| = \sup_{A\in\Sigma} \Big| \int_A f\, d\mu \Big| \cdot \sup_{y\in B_Y} \|R(y)\|$$

$$= \sup_{A\in\Sigma} \Big| \int_A f\, d\mu \Big|, \quad f \in X(\mu),$$

and this expression is equivalent to the standard norm in $L^1(\mu)$.

Note also that $\varphi(\cdot) := \int_\Omega (\cdot)\, d\mu$ defines a linear functional on $L^1(\mu)$, and so $T_B : (X(\mu), \|\cdot\|_B) \to \mathcal{L}(Y, E)$ can be factored as

$$(X(\mu), \|\cdot\|_B) \hookrightarrow L^1(\mu) \to_\varphi \mathbb{R} \to_S \mathcal{L}(Y, E),$$

where $S(r)(\cdot) = r \cdot R(\cdot)$ for all $r \in \mathbb{R}$. Therefore, $T_B$ acting in $(X(\mu), \|\cdot\|_B)$ is a compact map, what by Lemma 8 means that the extension of $B$, $\hat{B}$, is right-uniformly-compact.

On the other hand, it can be easily seen that $|\nu_{T_B}| = \mu$, and the original bilinear map $B$ can be factored as

$$X(\mu) \times Y \to_i L^1(\mu) \times Y \to_{\hat{B}} E,$$

where $\hat{B} : L^1(|\nu_{T_B}|) \times Y \to E$ is defined by continuity from $B$.

Let us give a general version of this result under the assumption of right-uniform-compactness of the bilinear map $B$ for the norm $\|\cdot\|_B$ in the left space of the Cartesian product.

**Proposition 11.** *Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$. Let $B : X(\mu) \times Y \to E$ be a Banach space valued bilinear map such that $B : (X(\mu), \|\cdot\|_B) \times Y \to E$ is right-uniformly-compact. Then $B$ can be factored through the bilinear map $\hat{B} : L^1(|\nu_{T_B}|) \times Y \to E$ defined by continuity from $B$, and this factorization is maximal.*

*Moreover, if $B : (X(\mu), \|\cdot\|_B) \times Y \to E$ is also left-uniformly-compact, or compact, its linearization*

$$L_{\hat{B}} : L^1(|\nu_{T_B}|)\hat{\otimes}_\pi Y \to E,$$

*is compact too.*

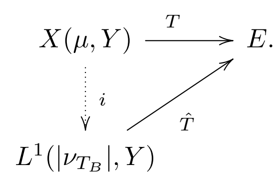
*Proof.* Since $B$ is right-uniformly-compact, we have by Lemma 8 that the operator $T_B : \overline{(X(\mu), \|\cdot\|_B)} \to \mathcal{L}(Y, E)$ is compact, and it is indeed the integration map for the vector measure $\nu_{T_B}$. Therefore, using Proposition 3.48 in [22], we get that $L^1(\nu_{T_B}) = L^1(|\nu_{T_B}|)$ (see [22, §3.3] for more explanations about this result). Proposition 4 gives the result.

We can get a proof of the last statement within the circle of ideas of the Krein-Milman Theorem. Indeed, first recall that $L^1(|\nu_{T_B}|, Y) = L^1(|\nu_{T_B}|)\hat{\otimes}_\pi Y$. Note that the unit ball of the projective tensor product is the closed convex hull of its extreme points, which are the single tensors as $f \otimes y$ for $\|f\| \leq 1$ and $\|y\| \leq 1$. Using Lemma 8, we have that the range of $\hat{B}$ is a relatively compact set, and its closed convex hull is then also compact. This set contains $\overline{L_{\hat{B}}(B_{L^1(|\nu_{T_B}|,Y)}}$. Thus, we get that $T_{\hat{B}}(L^1(|\nu_{T_B}|, Y))$ is relatively compact, and so the operator $T_{\hat{B}}$ is compact.

□

Using some known results on vector measure integration and the previous developments, we are ready to prove the main result of this section. Recall that, if $T : X(\mu, Y) \to E$ is an operator, the formula $\hat{B}_T(f, y) := T(f \cdot y)$, $f \in X(\mu)$, $y \in Y$, provides a continuous bilinear map from $(X(\mu), \|\cdot\|_{B_T}) \times Y$ to $E$.

**Theorem 12.** *Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$. Let $T : X(\mu, Y) \to E$ be a $\mu$-determined Banach space valued linear operator such that the associated bilinear map $\hat{B}_T : (X(\mu), \|\cdot\|_B) \times Y \to E$ is right-uniformly-compact. Then the operator $T$ can be factored through the Bochner space $L^1(|\nu_{T_B}|, Y)$ as*

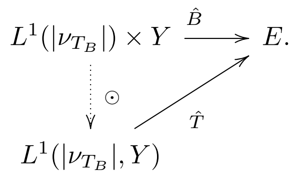



*Moreover, if $\hat{B}_T$ is also left-uniformly-compact, or $B_T$ is compact, this extension is compact and maximal.*

*In addition, there is an integral representation for $\hat{T}$, that is, there exists a $|\nu_{T_B}|$-Bochner integrable function $\phi : \Omega \to \mathcal{L}(Y, E)$ such that*

$$\hat{T}(f \cdot y) = \Big( \int_\Omega f \, \phi \, d|\nu_{T_B}| \Big)(y), \quad f \in L^1(|\nu_{T_B}|), \, y \in Y.$$

*Proof.* The proof is direct just using the previously obtained results. Given an operator $T : X(\mu, Y) \to E$, we can define a bilinear map $B_T : X(\mu) \times Y \to E$ by $B_T(f, y) = f \cdot y$, $f \in X(\mu)$, $y \in Y$. By Proposition 11, we get the optimal factorization for $B_T$ through $\hat{B} : L^1(|\nu_{T_B}|) \times Y \to E$. This map can be linearized through the projective tensor product, that coincides with the Bochner space $L^1(|\nu_{T_B}|, Y)$ as



This gives a factorization for the original operator $T$, that is clearly maximal: if there is any other Köthe-Bochner space $Z(\eta, Y)$ to which $T$ can be extended as $T_1$, the associated bilinear map factors through the scheme above. This, together with the fact that $Z(\eta, Y) \subseteq L^1(\nu_{T_B}, Y) = L^1(|\nu_{T_B}|, Y)$ by Lemma 3, gives that $T_1$ factors through $L^1(|\nu_{T_B}|, Y)$. Proposition 3.48 in [22] gives the integral representation of the operator written in the last part of the statement of the theorem. This finishes the proof.

□

*Example* 13. Following with the setting explained in Example 10, fix $X(\mu) = L^p(\mu)$, $1 < p < \infty$, and take as $R$ a norm one operator $R : Y \to E$. Consider the linear map $T : (\mathcal{S}, \| \cdot \|_{L^p(\mu,Y)}) \to E$, where $\mathcal{S}$ are the simple functions over $Y$, that is defined as $T(f) = T(\sum_{i=1}^n \chi_{A_i} y_i) = R(\sum_{i=1}^n \mu(A_i)\, y_i)$ for $f = \sum_{i=1}^n \chi_{A_i} y_i$. Note that

$$\|R(\sum_{i=1}^n \mu(A_i)\, y_i)\|_E \leq \| \sum_{i=1}^n \mu(A_i)\, y_i\|_Y \leq \sum_{i=1}^n \mu(A_i)\, \|y_i\|_Y$$
$$= \int_\Omega \| \sum_{i=1}^n \chi_{A_i} y_i \|_Y \, d\mu \leq \mu(\Omega)^{1/p'} \, \|f\|_{L^p(\mu,Y)},$$

and so the map is continuous, what implies that it can be extended by continuity to all the space $L^p(\mu, Y)$. Note also that the bilinear operator $B_T$ coincides with the map $B$ that was given in Example 10 and was shown to be right-uniformly-compact from $(L^p(\mu), \|\cdot\|_{B_T}) \times Y$ to $E$. It was also shown that $|\nu_{T_B}| = \mu$, and so Theorem 12 gives the factorization through $L^1(\mu, Y)$ and the maximality of such factorization.

The representation as a Bochner integral is given by the Radon-Nikodym derivative $\phi(w)(y) := \mathbf{1}(w)\, R(y) = R(y)$, $w \in \Omega$, $y \in Y$, that is

$$\hat{T}(f \cdot y) = \Big( \int_\Omega f \, \phi \, d\mu \Big)(y) = \big( \int_\Omega f d\mu \big) \cdot R(y), \quad f \in L^1(\mu),\, y \in Y.$$

This example provides a motivation for the following concrete result, that is up to our knowledge new, and is one of the main outcomes of the present paper: *every linear operator from a Bochner $L^p$-space $L^p(\mu, Y)$ with an associated right-uniformly-compact bilinear map can be extended optimally to a Bochner $L^1$-space. Moreover, this space can be chosen to be exactly $L^1(\mu, Y)$ if the inclusion is substituted by a multiplication operator in the factorization diagram.* For the aim of simplicity we write it for $\mu$-determined operators.

**Corollary 14.** *Let $1 \le p < \infty$, and consider a linear operator $T : L^p(\mu, Y) \to E$ such that it is $\mu$-determined and the associated bilinear map*

$$\hat{B}_T : (X(\mu), \| \cdot \|_B) \times Y \to E$$

*is right-uniformly-compact. Then $T$ has an optimal factorization as*

$$\begin{array}{ccc} L^p(\mu, Y) & \xrightarrow{T} & E. \\ {\scriptstyle M_g} \downarrow & \nearrow {\scriptstyle T_0} & \\ L^1(\mu, Y) & & \end{array}$$

*Proof.* By Theorem 12, we have a factorization as

$$\begin{array}{ccc} L^p(\mu, Y) & \xrightarrow{T} & E. \\ {\scriptstyle i} \downarrow & \nearrow {\scriptstyle \hat{T}} & \\ L^1(|\nu_{T_B}|, Y) & & \end{array}$$

On the other hand, since $|\nu_{T_B}| \ll \mu$, there is a $\mu$-integrable scalar function $g > 0$ —since the operator is $\mu$-determined—, such that $d|\nu_{T_B}| = g\, d\mu$. Therefore, we can factor the operator through the multiplication operator $M_g : L^p(\mu, Y) \to L^1(\mu, Y)$,

$M_g(f) := gf$, since

$$\int_\Omega \|M_g(f)\| d\mu = \int_\Omega \|f\| g d\mu = \int_\Omega \|f\| d|\nu_{T_B}|, \quad f \in L^p(\mu, Y),$$

closing the diagram with $T_0 = \hat{T}/g$. This gives the result. □

It can be easily seen that the function $g$ in the previous corollary has to belong to $L^{p'}(\mu)$. The same result can be adapted for other classical families of Banach function spaces, as Lorentz and Orlicz spaces.

**Corollary 15.** *Let $1 \le p, q < \infty$ and consider the Lorentz (Banach) space $L^{p,q}(\mu)$. Consider a linear operator $T : L^{p,q}(\mu, Y) \to E$ such that it is $\mu$-determined and the associated bilinear map $\hat{B}_T : (X(\mu), \| \cdot \|_B) \times Y \to E$ is right-uniformly-compact. Then $T$ has an optimal factorization through the Bochner space $L^1(\mu, Y)$ as the one given in Corollary 14.*

*Remark* 16. Since the factorization given in Theorem 12 depends only on the associated bilinear map $B_T$, the result can also be applied for spaces other than the Köthe-Bochner spaces, —for example spaces of vector valued integrable functions of Pettis type—, whenever we can prove that these spaces are included in $L^1(|\nu_{T_B}|, Y)$ and define the same bilinear map $B_T$.

## 5. Applications: Fourier type of Banach spaces and boundedness results for the vector valued Fourier coefficients of Bochner integrable functions

Recall from Example 9 that a Banach space $X$ is said to have *Fourier type $p$* for $1 < p \le 2$ if the map $I_p : L^p(\Pi, X) \to \ell^{p'}(X)$ given by $f \mapsto I_p(f) = \big(\hat{f}(k)\big)_{k \in \mathbb{Z}}$ is well defined and continuous. Let us use our results to provide two applications that give more information for the vector valued Fourier transform.

1) The bilinear map associated to $I_p$ is $B_p : L^p(\mu) \times X \to \ell^{p'}(X)$, given by $B_p(h, x) = (\hat{h}_k)_k \cdot x$, $h \in L^p(\Pi)$, $x \in X$. Note that it is always well-defined and

continuous as a consequence of the Hausdorff-Young inequality, without assuming the Fourier type $p$ for $X$. The linear operator $T_{B_p} : L^p(\Pi) \to \mathcal{L}(X, \ell^{p'})$ can also be defined; it can be easily checked, as in Example 9, that $\|T_{B_p}(h)\|_{\mathcal{L}(X,\ell^{p'})} = \|(\hat{h}_k)_k\|_{\ell^{p'}}$, $h \in L^p(\Pi)$. Therefore, the associated vector measure $\nu_{T_{B_p}}$ coincides with the vector measure $\nu_{F_p}$ defined by the Fourier operator $F_p : L^p(\Pi) \to \ell^{p'}$. The optimal domain for this operator is presented in [22], where it is shown that coincides with the space

$$\mathbb{F}_p(\Pi) := \Big\{ f \in L^1(\Pi) : (\widehat{f\chi_A}_k)_k \in \ell^{p'} \text{ for each Borel set } A \Big\},$$

and the norm is given by

$$\|f\|_{\mathbb{F}_p} := \sup_A \|(\widehat{f\chi_A}_k)_k\|_{\ell^{p'}}, \quad f \in \mathbb{F}_p(\Pi),$$

see [22, Proposition 7.13]. Therefore, we have the continuous inclusion $L^p(\Pi) \subseteq \mathbb{F}_p(\Pi)$, and $B_p$ can be extended to $\mathbb{F}_p(\Pi) \times X \to \ell^{p'}(X)$. Consequently, independently of the Fourier type of $X$, for every strongly measurable function $X$-valued function $f$ that can be written as a pointwise limit $f = \sum_{i=1}^\infty h_i \cdot x_i$ with $h_i \in \mathbb{F}_p(\Pi)$, $x_i \in X$ and $\sum_{i=1}^\infty \|h_i\|_{L^p(\Pi)} \|x_i\|_X < \infty$, the vector valued Fourier coefficients define a sequence $(\hat{f}_k)_k$ that is in $\ell^{p'}(X)$.

2) Fix now a natural number $n$ and consider the projection $P_n : \ell^{p'}(X) \to \ell^{p'}_n(X)$, $P_n((x_k)_{k=1}^\infty) := (x_k)_{k=1}^n$ for $(x_k)_{k=1}^\infty \in \ell^{p'}(X)$. This operator is not compact. However, if we define the bilinear map $B_{p,n} : L^p(\Pi) \times X \to \ell^{p'}_n(X)$ as $B_{p,n} = P_n \circ B_p$, we obtain that the associated operator

$$T_{B_{p,n}} : L^p(\Pi) \to L(X, \ell^{p'}_n(X)),$$

given by the formula $T_{B_{p,n}}(h)(x) := B_{p,n}(h, x) = (\hat{h}_k)_{k=1}^n \cdot x \in \ell^{p'}_n(X)$, $h \in L^p(\Pi)$, $x \in X$, has finite range, so it is compact, that is, $B_{p,n}$ is right-uniformly-compact (Lemma 8). The operator is not $\mu$-determined, but in this case a look to the proof of Lemma 14 shows that Corollary 14 holds also for a positive function $g_n$ that is not in general $g_n > 0$. Therefore, we obtain the next uniform domination result.

**Corollary 17.** *Suppose that $X$ has Fourier type $p$ for $1 < p \leq 2$. Let $n \in \mathbb{N}$ and assume that the Banach space $X$ has Fourier type equal to $p$. Then there is a function $0 \leq g_n \in L^{p'}(\Pi)$ such that the multiplication operator $M_{g_n} : L^p(\Pi, X) \to L^1(\Pi, X)$ is well-defined and continuous, and there is a constant $K_n$ such that for every $f \in L^1(\Pi, gd\mu, X)$,*

$$\left\| \left( \hat{f}_k \right)_{k=1}^n \right\|_{\ell_n^{p'}(X)} \leq K_n \int_\Pi \|f\|_X \, g \, d\mu.$$

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