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ON FINITE INVOLUTIVE YANG-BAXTER GROUPS

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Abstract. A group G is said to be an involutive Yang-Baxter group, or simply an IYB-group, if it is isomorphic to the permutation group of an involutive, non-degenerate set-theoretic solution of the Yang-Baxter equation. We give new sufficient conditions for a group that can be factorised as a product of two IYB-groups to be an IYB-group. Some earlier results are direct consequences of our main theorem.

1. Introduction

Following Drinfeld [5], we say that a *set-theoretic solution of the Yang-Baxter equation* is a pair (X, r), where X is a non-empty set and $r\colon X \times X \longrightarrow X \times X$ is a map such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

with the maps r_{12}, $r_{23}\colon X \times X \times X \longrightarrow X \times X \times X$ defined as $r_{12} = r \times \mathrm{id}_X$, $r_{23} = \mathrm{id}_X \times r$. For all x, $y \in X$, we define two maps $f_x\colon X \longrightarrow X$ and $g_y\colon X \longrightarrow X$ by setting $r(x, y) = (f_x(y), g_y(x))$. We say that the solution (X, r) is *involutive* if $r^2 = \mathrm{id}_{X\times X}$, and that (X, r) is *non-degenerate* if f_x, g_y are bijective maps for all x, $y \in X$. By a solution of the Yang-Baxter equation, or simply a solution of the YBE, we will understand an involutive, non-degenerate set-theoretic solution of the Yang-Baxter equation.

Let (X, r) be a solution of the YBE. The *permutation group* of (X, r) is the subgroup $\mathcal{G}(X, r)$ of $\mathrm{Sym}(X)$ generated by the bijections f_x for all $x \in X$, that is,

$$\mathcal{G}(X, r) = \langle f_x \mid x \in X\rangle \leqslant \mathrm{Sym}(X).$$

Following [3], a finite group G is called an *involutive Yang-Baxter group*, or simply an *IYB-group*, if there exists an involutive non-degenerate solution of the Yang-Baxter equation (X, r) such that $G \cong \mathcal{G}(X, r)$.

On the other hand, Rump [7] introduced a new algebraic structure as a generalisation of radical rings that turns out to be an important tool to study the solutions of the YBE. This structure is called *left brace* and it is defined as a set B with two binary operations, $+$ and $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group and

$$a \cdot (b + c) = a \cdot b + a \cdot c - a,$$

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for all a, b, $c \in B$. A right brace is defined similarly and a two-sided brace is a left and right brace (with the same operations).

The starting point of the results of this paper is the following characterisation of finite IYB-groups (see [3, Theorem 2.1]).

Theorem 1.1. *The following statements about a finite group G are pairwise equivalent:*

(1) G is an IYB-group.
(2) G is isomorphic to the multiplicative group of a left brace.
(3) There exists a (left) G-module V and a bijective 1-cocycle $\pi\colon G \longrightarrow V$.

As in [6], we call the pair (V, π) an *IYB-structure* on the group G.

Recall that a 1-*cocycle* or *derivation* of a G-module V is a map $\pi\colon G \longrightarrow V$ such that $\pi(gh) = \pi(g) + g\pi(h)$ for every g, $h \in G$.

Let G be a group with an IYB-structure (V, π). Then every Hall subgroup W of V is G-invariant, $H = \pi^{-1}(W)$ is a subgroup of G and (W, π_H), where π_H is the restriction of π to H, is an IYB-structure on H (see [3, Corollary 3.1]). Therefore every IYB-group is soluble and is a product of two IYB-groups.

Unfortunately the converse is not true. Bachiller [2] shows that there exist a prime p and a p-group G of order p^{10} and nilpotency class 9 that is not a IYB-group. Then G has a subgroup H which is not an IYB-group but all its proper subgroups are IYB-groups. Since every abelian group is an IYB-group, it follows that H is a product of two maximal subgroups which are IYB-groups. As a consequence, the following question is of interest.

Question 1.2. Let $G = HK$ be a finite group which is the product of the subgroups N and H. Assume that N and H are IYB-groups and N is normal in G. Under which conditions can we ensure that G is an IYB-group?

In this context, Cedó, Jespers, and del Río proved the following interesting theorem.

Theorem 1.3 ([3, Theorem 3.3])**.** *Let G be a finite group such that $G = AH$, where A is an abelian normal subgroup of G and H is an IYB-subgroup of G with associated IYB-structure (B, π) such that $H \cap A$ acts trivially on B. Then G is an IYB-group. In particular, every semidirect product $A \rtimes H$ of a finite abelian group A by an IYB-group H is an IYB-group.*

The notion of equivariant IYB-structure introduced by Eisele in [6] is quite useful to study IYB-groups.

Suppose that a group A acts on a IYB-group G with an IYB-structure (V, π). If a A and $g \in G$, we denote with ${}^a g \in G$ the result of the action of $a \in A$ on $g \in G$.

We call the IYB-structure (V, π) A-*equivariant* if there exists a group action of A on V, for which we denote with av the result of the action of $a \in A$ on $v \in V$, such that $\pi({}^a g) = a\pi(g)$ for all $a \in A$, $g \in G$. In fact, since π is bijective, such action of A on V is uniquely determined by the action of A on G by means of $av = \pi({}^a \pi^{-1}(v))$ for every $a \in A$, $v \in V$.

It is not difficult to see that (V, π) is an A-equivariant IYB-structure on G if and only if it is an A/K-equivariant IYB-structure on G, where $K = \operatorname{Ker}(A \operatorname{on} G)$ is the kernel of the action of A on G.

An IYB-structure (V,π) an a group G is called *fully equivariant* if (V,π) is $\mathrm{Aut}(G)$-equivariant (under the natural action of $\mathrm{Aut}(G)$ on G), which implies that (V,π) is A-equivariant for every action of a group A on G.

The following proposition shows that a semidirect product of an IYB-group H with a group N having an H-equivariant structure is an IYB-group.

Theorem 1.4 ([6, Proposition 2.2]). *Let $G = N \rtimes H$ be a finite group. If H is an IYB-group and N has an H-equivariant IYB-structure, then G is an IYB-group.*

Our main result in this paper significantly improves Theorem 1.3 and 1.4 by removing the abelianity condition on N and the requirement for the group G to be a semidirect product.

Theorem A. *Suppose that the group A acts on the group $G = NH$, where N and H are A-invariant subgroups of G and $N \trianglelefteq G$. Suppose that N and H are IYB-groups with A-equivariant IYB-structures (U,π_N) and (V,π_H), respectively, satisfying the following conditions:*

(C1) $N \cap H \subseteq \mathrm{Ker}(\mathrm{Z}(N) \,\mathrm{on}\, U) \cap \mathrm{Ker}(H \,\mathrm{on}\, V)$.

(C2) *(U,π_N) is also an H-equivariant IYB-structure on N with respect to the action by conjugation of H on N: ${}^h n = hnh^{-1}$ for $n \in N$, $h \in H$,*

Then G has an A-equivariant IYB-structure (W,π) such that

$$\mathrm{Ker}(N \,\mathrm{on}\, U)\, \mathrm{C}_{\mathrm{Ker}(H \,\mathrm{on}\, V)}(N) \subseteq \mathrm{Ker}(G \,\mathrm{on}\, W).$$

The proof of Theorem A appears in Section 3. We use some previous results needed that will be collected in Section 2. We present in Section 4 some applications of Theorem A to obtain new families of IYB-groups. Finally, we construct in Section 5 a family of IYB-groups that appear as a consequence of our results, but cannot appear as a consequence of the results of [3] or [6].

In the sequel, all groups considered will be finite.

2. Preliminary results

Lemma 2.1. *Let $(G,\cdot)$ be an IYB-group with IYB-structure (V,π) and let $A \leq \mathrm{Aut}(G)$. Note that $(G,+,\cdot)$ is a left brace with an addition defined by means of the following law:*

$$g + h \triangleq \pi^{-1}(\pi(g) + \pi(h)) \quad \text{for all } g,\, h \in G.$$

Then (V,π) is A-equivariant if and only if A is a group of automorphisms of the left brace G.

Proof. Suppose that (V,π) is A-equivariant. Then there exists an action of A on V, whose result is denoted by av for $a \in A$, $v \in V$, such that

$$\pi({}^a g) = a\pi(g) \quad \text{for all } a \in A,\, g \in G.$$

Given g, $h \in G$ and $a \in A$,

$$\begin{aligned}\pi({}^a(g+h)) &= a\pi(g+h) = a(\pi(g)+\pi(h)) = a\pi(g) + a\pi(h)\\ &= \pi({}^a g) + \pi({}^a h) = \pi({}^a g + {}^a h).\end{aligned}$$

This implies that ${}^a(g+h) = {}^a g + {}^a h$. Hence the action of A on G preserves the addition, as desired.

Conversely, suppose that A is a group of automorphisms of the left brace G. Let $a \in A$, $v \in V$. Since

$$\begin{aligned}\pi({}^{a}(\pi^{-1}(v) + \pi^{-1}(w))) &= \pi({}^{a}\pi^{-1}(v) + {}^{a}\pi^{-1}(w)) \\ &= \pi({}^{a}\pi^{-1}(v)) + \pi({}^{a}\pi^{-1}(w))\end{aligned}$$

we have that the assignment $av = \pi({}^{a}\pi^{-1}(v))$, $a \in A$, $v \in V$, defines a group action of A on V. Moreover, given $a \in A$, $g \in G$, as $\pi(g) \in V$, we have that

$$a\pi(g) = \pi({}^{a}\pi^{-1}(\pi(g))) = \pi({}^{a}g),$$

which implies that (V, π) is A-equivariant. $\square$

Example 2.2. Suppose that G is an abelian group. Let $V = G$ considered as a trivial G-module and $\pi = \mathrm{id}_G$. Obviously (V, π) is fully equivariant and $G = \mathrm{Ker}(G \,\mathrm{on}\, V)$.

Example 2.3 ([6, Remark 2.7])**.** Suppose that $(G, \cdot)$ is an odd order nilpotent group of class two. Then for every element $g \in G$ there exists a unique element $h = \sqrt{g}$ such that $h^2 = g$. We define an addition $+$ on G by means of $g_1 + g_2 \triangleq g_1 g_2 \sqrt{[g_2, g_1]}$. It is easy to check that $(G, +)$ is an abelian group. We give $V = (G, +)$ a structure of G-module by means of the law

$${}^{g}v \triangleq gv + g^{-1},$$

and set $\pi = \mathrm{id}_G$. Then (V, π) is fully equivariant and $\mathrm{Z}(G) = \mathrm{Ker}(G \,\mathrm{on}\, V)$.

The following example is a special case of [1].

Example 2.4. Suppose that $(G, \cdot)$ is a nilpotent group of class two. Set $Z = \mathrm{Z}(G)$ and write $G/Z = \langle a_1 Z\rangle \times \cdots \times \langle a_n Z\rangle$. Thus every element of G can be written in the form $a_1^{t_1} \cdots a_n^{t_n} z$, where $z \in Z$. We can define an addition on G by means of

$$a_1^{t_1} \cdots a_n^{t_n} z + a_1^{s_1} \cdots a_n^{s_n} z' = a_1^{t_1+s_1} \cdots a_n^{t_n+s_n} zz'.$$

It is not difficult to check that $(G, +, \cdot)$ is a two-side brace. We give $V = (G, +)$ a structure of G-module by means of the following law:

$${}^{g}v \triangleq gv - g = v \prod_{1 \le j < i \le n} [a_i, a_j]^{t_i s_j},$$

where $g = a_1^{t_1} \cdots a_n^{t_n} z \in G$ and $v = a_1^{s_1} \cdots a_n^{s_n} z' \in V$. Set $\pi = \mathrm{id}_G$. We have that (V, π) is an IYB-structure on G.

Recall that an automorphism α of a group G is called *central* if ${}^{\alpha}g g^{-1} \in \mathrm{Z}(G)$ for all $g \in G$, where ${}^{\alpha}g$ denotes the image of g by α. The set $\mathrm{Aut}_c(G)$ of all central automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$ (for example, see [8]).

Proposition 2.5. *Let $(G, \cdot)$ be a nilpotent group of class two. There exists an IYB-structure (V, π) on G such that (V, π) is $\mathrm{Aut}_c(G)$-equivariant and $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \,\mathrm{on}\, V)$.*

Proof. Write $A = \mathrm{Aut}_c(G)$ and choose the IYB-structure (V, π) on G as defined in Example 2.4. It is not difficult to see that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \,\mathrm{on}\, V)$. We only must show that (V, π) is A-equivariant. By Lemma 2.1, it suffices to show that every central automorphism preserves the addition on G defined in Example 2.4. Let

$g = a_1^{t_1} \cdots a_n^{t_n} z$, $h = a_1^{s_1} \cdots a_n^{s_n} z' \in G$, where $z, z' \in \mathrm{Z}(G)$ and $\alpha \in A$. As α is central, we may assume that ${}^{\alpha}a_i = a_i z_i$, where $z_i \in \mathrm{Z}(G)$, $i = 1, \ldots, n$.

$$\begin{aligned}
{}^{\alpha}(g+h) &= {}^{\alpha}(a_1^{t_1+s_1} \cdots a_n^{t_n+s_n} zz') \\
&= ({}^{\alpha}a_1)^{t_1+s_1} \cdots ({}^{\alpha}a_n)^{t_n+s_n} ({}^{\alpha}z)({}^{\alpha}z') \\
&= (a_1 z_1)^{t_1+s_1} \cdots (a_n z_n)^{t_n+s_n} ({}^{\alpha}z)({}^{\alpha}z') \\
&= a_1^{t_1+s_1} \cdots a_n^{t_n+s_n} (z_1^{t_1} \cdots z_n^{t_n} ({}^{\alpha}z))(z_1^{s_1} \cdots z_n^{s_n} ({}^{\alpha}z')) \\
&= a_1^{t_1} \cdots a_n^{t_n} (z_1^{t_1} \cdots z_n^{t_n} ({}^{\alpha}z)) + a_1^{s_1} \cdots a_n^{s_n} (z_1^{s_1} \cdots z_n^{s_n} ({}^{\alpha}z')) \\
&= (a_1 z_1)^{t_1} \cdots (a_n z_n)^{t_n} ({}^{\alpha}z) + (a_1 z_1)^{s_1} \cdots (a_n z_n)^{s_n} ({}^{\alpha}z') \\
&= ({}^{\alpha}a_1)^{t_1} \cdots ({}^{\alpha}a_n)^{t_n} ({}^{\alpha}z) + ({}^{\alpha}a_1)^{s_1} \cdots ({}^{\alpha}a_n)^{s_n} ({}^{\alpha}z') \\
&= {}^{\alpha}g + {}^{\alpha}h.
\end{aligned}$$

as desired. □

Lemma 2.6. *Let π be a 1-cocycle of the G-module V. Suppose that $x \in \mathrm{Ker}(G \,\mathrm{on}\, V)$ and $g \in G$. Then*

(1) $\pi(xg) = \pi(x) + \pi(g)$;
(2) $\pi(gxg^{-1}) = g\pi(x)$.

Proof. As x acts trivially on V, it is easy to see that $\pi(xg) = \pi(x) + x\pi(g) = \pi(x) + \pi(g)$ and Statement 1 follows. Now we prove Statement 2.

$$\begin{aligned}
\pi(gxg^{-1}) &= \pi(g) + g\pi(xg^{-1}) \\
&= \pi(g) + g(\pi(x) + \pi(g^{-1})) \\
&= \pi(g) + g\pi(g^{-1}) + g\pi(x) \\
&= \pi(gg^{-1}) + g\pi(x) = g\pi(x),
\end{aligned}$$

as desired. □

Lemma 2.7. *Suppose that the group A acts on a group G with A-equivariant IYB-structure (V, π), which determines the unique action of A on V. Then for every $a \in A$, $g \in G$ and $v \in V$,*

$$({}^{a}g)v = a(g(a^{-1}v)).$$

Proof. Since $a^{-1}v \in V$ and π is bijective, we may assume that $\pi(x) = a^{-1}v$ for some $x \in G$. Note that $g\pi(x) = \pi(gx) - \pi(g)$. Hence we have

$$\begin{aligned}
a(g(\pi(x))) &= a\pi(gx) - a\pi(g) \\
&= \pi({}^{a}(gx)) - \pi({}^{a}x) \\
&= \pi(({}^{a}g)({}^{a}x)) - \pi({}^{a}g) \\
&= ({}^{a}g)\pi({}^{a}x) = ({}^{a}g)(a\pi(x)).
\end{aligned}$$

Note that $a\pi(x) = v$. It implies that $({}^{a}g)v = a(g(a^{-1}v))$, as desired. □

3. Proof of the main theorem

Proof of Theorem A. Note that there exist actions of A on U and V such that $\pi_N({}^{a}n) = a\pi_N(n)$ and $\pi_H({}^{a}h) = a\pi_H(h)$ for all $a \in A, n \in N$ and $h \in H$. Thus we can view $U \oplus V$ as an A-module via the law:

$$a(u, v) = (au, av), a \in A, (u, v) \in U \oplus V.$$

Let $X = \{(\pi_N(x^{-1}), \pi_H(x)) \in U \oplus V : x \in H \cap N\}$. By hypothesis (C1), $N \cap H$ acts trivially on U and V, and $N \cap H \subseteq \mathrm{Z}(N)$. For every x, $y \in N \cap H$, it follows from Lemma 2.6 (1) that

$$\begin{aligned}(\pi_N(x^{-1}), \pi_H(x)) + (\pi_N(y^{-1}), \pi_H(y)) &= (\pi_N(x^{-1}y^{-1}), \pi_H(xy)) \\ &= (\pi_N((xy)^{-1}), \pi_H(xy)) \in X,\end{aligned}$$

moreover,

$$a(\pi_N(x^{-1}), \pi_H(x)) = (a\pi_N(x^{-1}), a\pi_H(x)) = (\pi_N(({}^a x)^{-1}), \pi_H({}^a x)) \in X.$$

It implies that X is an A-submodule of $U \oplus V$.

Consider the quotient A-module $W = (U \oplus V)/X$. By hypothesis (C2), there exists a unique action of H on U such that $\pi_N({}^h n) = h\pi_H(n)$ for every $h \in H$, $n \in N$, where hu denotes the result of the action of $h \in H$ on $u \in U$. Now we consider the assignment $G \times W \longrightarrow W$ given by

$$(g, (u, v) + X) \mapsto g((u, v) + X) \triangleq (n(hu), hv) + X,$$

where $g = nh$, $n \in N$, $h \in H$ and $(u, v) \in U \oplus V$. We first prove that this is a map and it is indeed an action of G on W. Let $g = nh = n'h'$ and suppose that $(u, v) + X = (u', v') + X$, where $n' \in N$, $h' \in H$, $(u', v') \in U \oplus V$. It suffices to show that

$$(n(hu), hv) + X = (n'(h'u'), h'v') + X.$$

Write $t = n^{-1}n' = h(h')^{-1} \in N \cap H$ and so t acts trivially on U and V. Thus $h'u' = (t^{-1}h)u' = t^{-1}(hu') = hu'$ and $h'v' = t^{-1}(hv') = hv'$. Furthermore, $n'(h'u') = n(t(hu')) = n(hu')$. Hence it is enough to show that

$$(n(h(u - u')), h(v - v')) \in X.$$

Recall that $(u - u', v - v') \in X$. Then we may assume that $u - u' = \pi_N(x^{-1})$ and $v - v' = \pi_H(x)$ for some $x \in N \cap H$. By hypothesis (C2), $h\pi_N(x^{-1}) = \pi_N(hx^{-1}h^{-1})$. Note that $hx^{-1}h^{-1}$ and x act trivially on U and V. It follows from Lemma 2.6 (2) that $n\pi_N(hx^{-1}h^{-1}) = \pi_N(nhx^{-1}h^{-1}n^{-1})$ and $h\pi_H(x) = \pi_H(hxh^{-1})$.

As $hxh^{-1} \in \mathrm{Z}(N)$, we can conclude that

$$\begin{aligned}(n(h(u - u')), h(v - v')) &= (n(h\pi_N(x^{-1})), h\pi_H(x)) \\ &= (\pi_N((hxh^{-1})^{-1}), \pi_H(hxh^{-1})) \in X,\end{aligned}$$

so this assignment is a map from $G \times W$ to W. Now let $g_1 = n_1h_1$ and $g_2 = n_2h_2$ with $n_i \in N$ and $h_i \in H$, and $(u, v) + X \in W$. It follows that

$$\begin{aligned}(g_1g_2)((u, v) + X) &= (n_1h_1n_2h_1^{-1}h_1h_2)((u, v) + X) \\ &= ((n_1h_1n_2h_1^{-1})((h_1h_2)u), (h_1h_2)v) + X \\ &= (n_1(h_1(n_2(h_2u))), h_1(h_2v)) + X \qquad \text{(by Lemma 2.7)} \\ &= g_1((n_2(h_2u), h_2v) + X) \\ &= g_1(g_2((u, v) + X)).\end{aligned}$$

Hence this map is an action of G on W and it is easy to see that $N \cap H \subseteq \mathrm{Ker}(G \,\mathrm{on}\, W)$.

Consider the assignment $\pi\colon G \longrightarrow W$ given by

$$\pi(g) = (\pi_N(n), \pi_H(h))X,$$

where $g = nh$, $n \in N$, $h \in H$. Note that if $g = nh = n'h'$ with n, $n' \in N$ and $h, h' \in H$, we have that $z = n^{-1}n' = h((h')^{-1}) \in N \cap H$. As $z \in \mathrm{Z}(N)$, $z^{-1} = n'^{-1}n = n'(n')^{-1}n(n')^{-1} = n(n')^{-1}$. Since $H \cap N$ acts trivially on U and V, it implies that

$$\begin{aligned}
\pi_N(z^{-1}) &= \pi_N(n(n')^{-1}) \\
&= \pi_N(n) + n\pi_N((n')^{-1}) \\
&= \pi_N(n) + n'(z^{-1}\pi_N((n')^{-1})) \\
&= \pi_N(n) + n'\pi_N((n')^{-1}) \\
&= \pi_N(n) - \pi_N(n'),
\end{aligned}$$

and by a similar calculation, we have that $\pi_H(z) = \pi_H(h) - \pi_H(h')$. It follows that the assignment π is a map between G and W. Given $(u, v)+X \in W$, as π_N and π_H are bijective, we can take $g = \pi_N^{-1}(u)\pi_H^{-1}(v)$ and clearly $\pi(g) = (u, v) + X$. Hence π is surjective. Furthermore, as

$$|G| = \frac{|N||H|}{|N \cap H|} = \frac{|U||V|}{|X|} = |W|,$$

we conclude that π is bijective.

Now we prove that π is a 1-cocycle of the G-module W. Let $g_1 = n_1h_1$ and $g_2 = n_2h_2$, with $n_i \in N$ and $h_i \in H$. Then

$$\begin{aligned}
\pi(g_1g_2) &= \pi(n_1h_1n_2h_2) = \pi(n_1h_1n_2h_1^{-1}h_1h_2) \\
&= (\pi_N(n_1h_1n_2h_1^{-1}), \pi_H(h_1h_2)) + X \\
&= ((\pi_N(n_1), \pi_H(h_1)) + X) + ((n_1\pi_N(h_1n_2h_1^{-1}), h_1\pi_H(h_2)) + X) \\
&= \pi(g_1) + ((n_1(h_1\pi_N(n_2)), h_1\pi_H(h_2)) + X) \\
&= \pi(g_1) + g_1((\pi_N(n_2), \pi_H(h_2)) + X) \\
&= \pi(g_1) + g_1\pi(g_2).
\end{aligned}$$

Hence (W, π) is an IYB-structure on G. The last part is to show that (W, π) is A-equivariant. Let $g = nh \in G$ with $n \in N, h \in H$ and $a \in A$. Recall the action of A on W above. It follows that

$$\begin{aligned}
a\pi(g) &= a(\pi_N(n), \pi_H(h))X \\
&= (a\pi_N(n), a\pi_H(h))X \\
&= (\pi_N({}^an), \pi_H({}^ah)) \\
&= \pi({}^an{}^ah) = \pi({}^ag),
\end{aligned}$$

as desired. Hence the theorem is proved. □

4. Some applications

Our first corollary shows that the direct product case follows directly from Theorem A.

Corollary 4.1. *Let a group A act on a group $G = N \times H$ which is the direct product of two A-invariant subgroups N and H. Suppose that N, and H are IYB-groups with A-equivariant IYB-structures (U, π_N) and (V, π_H), respectively. Then G has an A-equivariant IYB-structures (W, π_G) such that*

$$\mathrm{Ker}(N \,\mathrm{on}\, U)\,\mathrm{Ker}(H \,\mathrm{on}\, V) \subseteq \mathrm{Ker}(G \,\mathrm{on}\, W).$$

The next result appears as a consequence of Corollary 4.1

Corollary 4.2. *Let G be a nilpotent group of class two with an abelian Sylow 2-subgroup. Then G has a fully equivariant IYB-structure (W, π_G) such that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \,\mathrm{on}\, W)$.*

The following corollary is an extension of Theorem 1.3..

Corollary 4.3. *Let a group $G = NH$ such that N is a nilpotent normal subgroup of class two and H is an IYB-group with IYB-structure (V, π). Assume that the following conditions hold:*

(1) $N \cap H \subseteq \mathrm{Z}(N)$;
(2) $[H, \mathrm{O}_2(N)] \subseteq \mathrm{Z}(N)$;
(3) $H \cap N$ acts trivially on V.

Then G is an IYB-group.

Proof. Let $N_1 = \mathrm{O}_2(N)$ and $N_2 = \mathrm{O}_{2'}(N)$. Note that $N = N_1 \times N_2$. Consider the action H on N via conjugate. Then N_1, N_2 are both H-invariant. As N_2 is nilpotent of class two with odd order, by Example 2.3, there exists a fully equivariant (of course, H-equivariant) IYB-structure (U_2, π_{N_2}) on N_2 such that $\mathrm{Z}(N_2) \subseteq \mathrm{Ker}(N_2 \,\mathrm{on}\, U_2)$. Note that $[H, N_2] \subseteq \mathrm{Z}(N) \cap N_2 = \mathrm{Z}(N_2)$, which means that every element of H acts on N_2 as an central automorphism. By Example 2.4 and Proposition 2.5, there exists an H-equivariant IYB-structure (U_1, π_{N_1}) on N_1 such that $\mathrm{Z}(N_1) \subseteq \mathrm{Ker}(N_1 \,\mathrm{on}\, U_1)$. Applying Corollary 4.1, we obtain that N has an H-equivariant IYB-structure, (U, π_N) say, such that $\mathrm{Z}(N) = \mathrm{Z}(N_1)\,\mathrm{Z}(N_2) \subseteq \mathrm{Ker}(N \,\mathrm{on}\, U)$.

Since $N \cap H$ is contained in $\mathrm{Z}(N)$ and acts trivially on V, we have that $N \cap H \subseteq \mathrm{Ker}(\mathrm{Z}(N) \,\mathrm{on}\, U) \cap \mathrm{Ker}(H \,\mathrm{on}\, V)$. Applying Theorem A for $A = 1$, we conclude that G is an IYB-group. □

Note that [3, Corollary 3.10] is a special case of the following result.

Corollary 4.4. *Let a group $G = NH$ such that N, H are two nilpotent subgroup of class two and N is normal in G. If $N \cap H \subseteq \mathrm{Z}(G)$ and $[H, \mathrm{O}_2(N)] \subseteq \mathrm{Z}(N)$, then G is an IYB-group.*

Proof. As H is a nilpotent group of class two, it follows from Example 2.4 and Proposition 2.5 that there exist an IYB-structure (V, π_H) on H such that $\mathrm{Z}(H) \subseteq \mathrm{Ker}(H \,\mathrm{on}\, V)$. Since $N \cap H \subseteq \mathrm{Z}(G)$, we have that $N \cap H$ is contained in $\mathrm{Z}(N)$ and $\mathrm{Z}(H)$, which acts trivially on V. By Corollary 4.3, G is an IYB-group. □

Corollary 4.5. *Let a group $G = N_1 N_2 \cdots N_s$ the product of subgroups $N_1, \dots, N_s$. Suppose that*

(1) N_i is a nilpotent group of class two with an abelian Sylow 2-subgroup, $i = 1, \dots, s$;
(2) N_i is normalised by N_j, for all $1 \le i < j \le s$;
(3) $N_1 \cdots N_i \cap N_{i+1} = \mathrm{Z}(G)$, $i = 1, \dots, s-1$.

Then G is an IYB-group.

Proof. Write $X_i = N_1 \cdots N_i$ and $H_i = N_{i+1} \cdots N_s$ for all i, where $H_s = N_{s+1} = 1$. In order to show that G is an IYB-group, we use induction on i to prove the following result: *X_i has an H_i-equivariant IYB-structure (U_i, π_i) such that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(X_i \,\mathrm{on}\, U_i)$.*

For $i = 1$, it is a consequence of Corollary 4.2. Hence we assume it is true for case $i-1$, i.e., X_{i-1} has an H_{i-1}-equivariant IYB-structure (U_{i-1}, π_{i-1}) such that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(X_{i-1} \,\mathrm{on}\, U_{i-1})$. As $H_i \leq H_{i-1}$, we have that (U_{i-1}, π_{i-1}) is H_i-equivariant. Note that H_i acts on the group $X_i = X_{i-1}N_i$, where $X_{i-1} \trianglelefteq X_i$ and X_{i-1}, N_i are H_i-invariant. By Corollary 4.2, N_i has a fully equivariant IYB-structure (V_i, ϕ_i) such that $\mathrm{Z}(N_i) \subseteq \mathrm{Ker}(N_i \,\mathrm{on}\, V_i)$. Since

$$X_{i-1} \cap N_i = \mathrm{Z}(G) \subseteq \mathrm{Ker}(\mathrm{Z}(X_{i-1}) \,\mathrm{on}\, U_{i-1}) \cap \mathrm{Ker}(N_i \,\mathrm{on}\, V_i),$$

it follows from Theorem A that X_i has an H_i-equivariant IYB-structure (U_i, π_i) such that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(\mathrm{Z}(X_{i-1}) \,\mathrm{on}\, U_{i-1}) \subseteq \mathrm{Ker}(X_i \,\mathrm{on}\, U_i)$, as desired. □

5. An example

The following example shows that Theorem A improves Theorem 1.3 and Theorem 1.4.

Example 5.1. Let $p \geq 3$ be a prime, let $m \geq 2$ be a natural number and let G be the group with the following presentation

$$G = \langle a, b, c \mid a^{p^m} = b^{p^m} = 1,\ c^{p^m} = a^{p^{m-1}},\ a^b = a^{1+p^{m-1}},$$
$$a^c = aa^{-p}b^{-p},\ b^c = ba\rangle.$$

Then G is a group of order p^{3m} and nilpotency class $2m$ with derived subgroup $G' = \langle b^p, a\rangle$ and Frattini subgroup $\Phi(G) = \langle c^p, b^p, a\rangle$. Let $N = \langle a, b\rangle$ and let $H = \langle c\rangle$. Then $G = NH$, N is a normal subgroup of G, N is nilpotent of class two (in fact, a minimal non-abelian group) and $N \cap H = \langle c^{p^m}\rangle \subseteq \mathrm{Z}(G)$. By Corollary 4.4, G is an IYB-group.

Claim 1. The group G cannot be expressed as the product of an abelian normal subgroup of G and a proper supplement.

It will be enough to show that every abelian normal subgroup of G is contained in $\Phi(G)$. Let T be an abelian normal subgroup of G. Since T is abelian, for every $g \in G$ we have that the map $t \mapsto [t, g] = t^{-1}t^g$, $t \in T$, is an endomorphism of T. Note that $[a, b] = a^{p^{m-1}}$, $[a, c] = a^{-p}b^{-p}$, $[b, c] = a$, and that a^p, $b^p \in \mathrm{Z}(N)$. Every element of G has the form $c^k b^l a^r$ for suitable integers k, l, r. Suppose that $c^k b^l a^r \in T \setminus \Phi(G)$. Then $p \nmid k$ or $p \nmid l$.

Suppose first that $p \nmid k$. Then $[c^k b^l a^r, c] = [b, c]^l [a, c]^r = a^l (a^{-p}b^{-p})^r = a^{l-pr}b^{-pr} \in T$. Since $\gcd(l - pr, p^m) = 1$, there exist λ, $\mu \in \mathbb{Z}$ such that $\lambda(l - pr) + \mu p^m = 1$. Therefore $(a^{l-pr}b^{-pr})^{\lambda} = ab^{-\lambda pr} \in T$. Since T is abelian,

$$\begin{aligned}
1 &= [c^k b^l a^r, ab^{-\lambda pr}] = [c, ab^{-\lambda pr}]^k [b, ab^{-\lambda pr}]^r [a, ab^{-\lambda pr}]^r \\
&= ([a, b^{-\lambda pr}][a, c])^k [b, a]^r = a^{-pk} b^{-pk} a^{-rp^{m-1}} \\
&= a^{-pk-rp^{m-1}} b^{-pk}.
\end{aligned}$$

It follows that $a^{-pk-rp^{m-1}} = b^{-pk} = 1$. Therefore $p^m \mid pk$, in particular, $p \mid k$, against our hypothesis on k.

Suppose now that $p \nmid l$. Then

$$[c^k, b^l, a^r, b] = [c, b]^k [b, b]^l [a, b]^r = a^{-k} a^{rp^{m-1}} = a^{-k+rp^{m-1}} \in T.$$

Since $\gcd(-k + rp^{m-1}, p^m) = 1$, we conclude that $a \in T$. Therefore

$$\begin{aligned} 1 = [c^k b^l a^r, a] &= [c, a]^k [b, a]^l [a, a]^r = a^{pk} b^{pk} a^{-lp^{m-1}} \\ &= a^{pk-lp^{m-1}} b^{pk}. \end{aligned}$$

It follows that $a^{pk-lp^{m-1}} = b^{pk} = 1$. Consequently $p^m \mid pk$ and $p^m \mid pk - lp^{m-1}$, which implies that $p^m \mid lp^{m-1}$ and so $p \mid l$, against our hypothesis on l.

We conclude that all abelian normal subgroups of G are contained in $\Phi(G)$ and so the fact that G is an IYB-group cannot be obtained as a consequence of the results of [3].

Claim 2. The group G cannot be expressed as a non-trivial semidirect product of a normal subgroup and a complement.

Suppose that the result is false. Then there exists a normal subgroup N with a complement. In particular, N is not contained in $\Phi(G) = \langle c^p, b^p, a\rangle$.

Step 2.1. Let us prove that $\langle a, b^p\rangle \leq N$.

Suppose that $c^i b^j a^k \in N \setminus \Phi(G)$. Assume first that $p \nmid i$. By taking a suitable power, we can assume that $i = 1$. Therefore

$$\begin{aligned} [cb^j a^k, b] &= a^{-k} b^{-j} c^{-1} b^{-1} cb^j a^k b = a^{-k} b^{-j} a^{-1} b^{-1} b^j a^k b \\ &= a^{-k} a^{-1-jp^{m-1}} a^{k+kp^{m-1}} = a^{-1+(k-j)p^{m-1}} \in N. \end{aligned}$$

This element is a generator of $\langle a\rangle$, consequently $a \in N$. We conclude that $[a, c] = a^{-p} b^{-p} \in N$, and since $a \in N$, we obtain that $b^p \in N$. In particular, $\langle a, b^p\rangle \leq N$.

Suppose now that $p \nmid j$. Then

$$\begin{aligned} [c^i b^j a^k, c] &= [b^j a^k, c] = a^{-k} b^{-j} c^{-1} b^j a^k c \\ &= a^{-k} b^{-j} (ba)^j a^k a^{-pk} b^{-pk} = a^{-k} b^{-j} b^j a^{j+j(j-1)p^{m-1}/2} a^k a^{-pk} b^{-pk} \\ &= a^{j+j(j-1)p^{m-1}/2-pk} b^{-pk} \in N \end{aligned}$$

and p does not divide the exponent of a. Hence we can assume that N possesses an element of the form $c^i b^l$ with $p \nmid l$. Consequently $[c^i b^l, c] = a^{l+l(l-1)p^{m-1}/2} \in N$, and so $a \in N$. As above, since $[a, c] = a^{-p} b^{-p} \in N$ and $a \in N$, we have that $b^p \in N$ and again $\langle a, b^p\rangle \leq N$.

Step 2.2. Let us prove that N has no elements of the form $cb^j a^k$.

Since $G' = \langle a, b^p\rangle$ has order p^{2m-1} and $N \not\leq \Phi(G)$, we conclude that $|G/N| \leq p^m$. Suppose that $cb^j a^k \in N$, then $N\langle b\rangle = G$ and so N has a cyclic complement of order p. Suppose that $c^i b^l a^r$ is a generator of this complement. We can check by induction that, for $u \in \mathbb{N}$,

$$b^{c^u} = b^{\sum_{w=0}^{u-1} (-1)^w \binom{u+w-1}{2w} p^w} a^{\sum_{w=0}^{u-1} (-1)^w \binom{u+w}{2w+1} p^w}.$$

Now we have that

$$1 = (c^i b^l a^r)^p = c^{ip} (b^l a^r)^{c^{i(p-1)}} \cdots (b^l a^r)^{c^i} (b^l a^r). \tag{5.1}$$

We obtain that $c^{ip} \in \langle c\rangle \cap \langle a, b\rangle = \langle a^{p^{m-1}}\rangle$ and so $p^{m-1} \mid i$, that is, $i = tp^{m-1}$ for an integer t. Since $c^i b^l a^r$ cannot be in $\Phi(G) = \langle c^p, b^p, a\rangle$, we conclude that p does not divide l. The exponent s of b in the right hand side of Equation (5.1) satisfies that

$$\begin{aligned} s &\equiv l\left(p - \sum_{t=0}^{p-1}\binom{tp^{m-1}}{2}p\right) \pmod{p^2} \\ &\equiv l\left(p - \sum_{t=0}^{p-1}\frac{tp^m(tp^{m-1}-1)}{2}\right) \pmod{p^2} \\ &\equiv lp \pmod{p^2}, \end{aligned}$$

but $s \equiv 0 \pmod{p^2}$, and so $p \mid l$, against the previous remark. Hence no element of the form $cb^j a^k$ belongs to N.

Step 2.3. Final contradiction

Take $C = \langle c^r b^s a^t \rangle$ a complement to N in G. Since $c \in NC$, we have a power of $c^r b^s a^t$ in which the exponent of c is equal to 1. In other words, we can assume that $r = 1$ and $cb^s a^t \in C$. Note that $(cb^s a^t)^{p^k} \in \langle c^{p^k}, b^{p^k}, a^{p^k} \rangle$ for k natural, and so $(cb^s a^t)^{p^m} = c^{p^m} = a^{p^{m-1}} \in C \cap N$ with $c^{p^m} \neq 1$. This contradicts that C is a complement to N in G.

Therefore, the fact that G is an IYB-group cannot be obtained from the results of [6].

Since these groups have nilpotency class at least 4, they cannot be obtained as a consequence of the results of [4].

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