# AGGREGATION OF FUZZY QUASI-METRICS

TATIANA PEDRAZA, JESÚS RODRÍGUEZ-LÓPEZ AND ÓSCAR VALERO

Abstract. In the last years fuzzy (quasi-)metrics and indistinguishability operators have been used as a mathematical tool in order to develop appropriate models useful in applied sciences as, for instance, image processing, clustering analysis and multi-criteria decision making. The both aforesaid similarities allow us to fuzzify the crisp notion of equivalence relation when a certain degree of similarity can be only provided between the compared objects. However, the applicability of fuzzy (quasi-)metrics is reduced because the difficulty of generating examples. One technique to generate new fuzzy binary relations is based on merging a collection of them into a new one by means of the use of a function. Inspired, in part, by the preceding fact, this paper is devoted to study which functions allow us to merge a collection of fuzzy (quasi-)metrics into a single one. We present a characterization of such functions in terms of $*$-triangular triplets and also in terms of isotonicity and $*$-supmultiplicativity, where $*$ is a t-norm. We also show that this characterization does not depend on the symmetry of the fuzzy quasi-metrics. The same problem for stationary fuzzy (quasi-)metrics is studied and, as a consequence, characterizations of those functions aggregating fuzzy preorders and indistinguishability operators are obtained.

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## 1. Introduction

Nowadays, aggregation functions have become an important area of research. The necessity of merging the information, usually represented by means of numerical values, contained in a collection of pieces of information into a single one for making decisions in applied sciences, has lead to a growing interest on studying numerical functions which allow this aggregation. A sample of the importance of this topic is given by the fact that three monographs devoted to aggregation function theory and its applications have been published in the last years (see [3, 4, 21]).

A prototypical example where aggregation function theory has been successfully applied and, in addition, fuzzy binary relations have shown to be particularly very useful is provided by the so-called fuzzy databases, i.e. databases where uncertain information can be managed [41]. In this way, one of the basic features of a database is the retrieval of data. However, a database system may not always be able to satisfy a query. Then it is natural that a database affords flexible queries. Thus if a query has not an exact result in the database, then it is desirable that the querying system gives a collection of alternative results ordered by means of a certain distance which measures how close is a data to match the criteria under consideration [41]. Following [46], given a query of $n$ expressions $(q_1, \ldots, q_n)$ each one corresponding

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to a different field, and given a data record $(x_1, \ldots, x_n)$, where $q_i, x_i \in X_i$ for all $i \in \{1, \ldots, n\}$, then it is necessary to combine the degrees of matching of each $q_i$ with respect to $x_i$ to obtain an overall degree of matching computed by the querying system. The degree of matching of each field is usually measured by means (see for example [5, 6]) of a fuzzy binary relation $R_i : X_i \times X_i \to [0,1]$, $i = 1, \ldots, n$, and the overall degree is obtained by a fuzzy binary relation $\widetilde{R} : (\prod_{i=1}^n X_i)^2 \to [0,1]$ defined as

$$\widetilde{R}((q_1, \ldots, q_n), (x_1, \ldots, x_n)) = F(R_1(q_1, x_1), \ldots, R_n(q_n, x_n))$$

where $F : [0,1]^n \to [0,1]$ is a function which reduces $n$ values into a unique representative (or meaningful) one.

In the light of this fact, it seems natural to wonder whether the aggregation process must preserve a certain property when all fuzzy binary relations $R_i$ to be merged satisfy the same property and, thus, that the aggregated fuzzy binary relation $\widetilde{R}$ also verifies the aforesaid property. This fact led to Saminger, Mesiar and Bodenhofer to study those conditions over an aggregation function $F$ which preserve the $*$-transitivity of the fuzzy binary relations $R_i$ to be merged, where $*$ is a t-norm. Thus, in [46], they proved a characterization of those aggregation functions that preserve $*$-transitivity in terms of domination.

In [14], a related problem was considered by Drewniak and Dudziak. Concretely, given a finite family of fuzzy binary relations $R_i : X \times X \to [0,1]$, $i = 1, \ldots, n$, on a fixed set $X$ and a t-norm $*$, they studied under which conditions the fuzzy binary relation $R : X \times X \to [0,1]$ given by

$$R(q, x) = F(R_1(q, x), \ldots, R_n(q, x))$$

is $*$-transitive whenever $R_i$ is also $*$-transitive for all $i \in \{1, \ldots, n\}$.

Observe that the fuzzy binary relation $R$ is the so-called aggregated fuzzy relation in the sense of [17]. Drewniak and Dudziak did a deep study on those conditions over $F$ which preserve the main properties of the fuzzy binary relations to be merged (properties that are of great interest in multiple-criteria decision making) (see [13, 14, 15]). Again, in the aforesaid reference, a characterization of those aggregation functions that preserve $*$-transitivity in terms of domination was provided. However, this time the operation $*$ was considered more general than a t-norm and the function $F$ is not required to be an aggregation function.

When fuzzy equivalence relations (or indistinguishability operators) are under consideration, a characterization of those functions that merge this particular case of $*$-transitive fuzzy binary relations was given by Mayor and Recasens in terms of $*$-triangular triplets (see [31]). Moreover, in the particular case in which the t-norm $*$ is continuous and Archimedean, characterizations of those functions that allow us to aggregate fuzzy equivalence relations were given by means of additive generators and functions preserving generalized metrics in [39, 42].

It must be stressed that the numerical value provided by a fuzzy equivalence relation when applied to two elements, matches up with the degree of indistinguishability between them. However in those cases in which the similarity degree between the elements must be measured with respect to a positive real parameter, fuzzy equivalence relations are not a suitable tool. In order to avoid this handicap, the notion of fuzzy (pseudo-)metric was introduced in the literature (see, [19, 27]). This new type of similarity measures can be interpreted in a similar way as fuzzy equivalence relations but in this case the obtained measures are always relative to the parameter value.

In the light of the exposed facts a natural question appears: if $\{R_i : i = 1, \ldots, n\}$ is a collection of fuzzy metrics, instead of indistinguishability operators, what are the conditions that $F$ must verify in order to guarantee that $\widetilde{R}$ or $R$ is a fuzzy metric? The aim of the present paper is to answer this question. It must be pointed out that the applicability of fuzzy metrics is reduced because the difficulty of generating examples and, thus, the absence of them in the literature. Thus a response to the posed question is not only interesting from the aggregation theory viewpoint, but also by the fact that the process of aggregation would allow us the introduction of techniques to generate new fuzzy metrics from old ones.

We observe that this problem has been addressed in the crisp framework by several authors. In particular, for metrics it has been mainly studied by Borsík and Doboš (see [8, 44]) and by Castiñeira, Pradera and Trillas (see [39, 40]). In [32] and [38], Mayor and Valero and Miñana and Valero obtained a characterization of those functions which aggregate a collection of quasi-metrics into a single one in the spirit of Borsík and Doboš (see also [30]). It should be pointed out that a function which aggregates quasi-metrics (on products) also aggregates metrics (on products) but the converse is not true in general (see Corollary 7, Example 8 in [32]). Nevertheless, these two concepts coincide in the fuzzy context as we will demonstrate.

The structure of the paper is as follows. In Section 2 we recall basic facts about t-norms and fuzzy (quasi-)metrics providing several illustrative examples. Section 3 is devoted to study functions which preserve a certain properties of fuzzy binary relations. Particular attention is paid to those functions which preserve $*$-transitivity of fuzzy binary relations, since fuzzy (quasi-)metrics satisfy a property near to $*$-transitivity. Therefore we review a few results of [14, 15, 17, 31, 46] characterizing these functions in terms of isotonicity and $*$-supmultiplicativity, or in terms of $*$-triangular triplets. In these papers the authors have considered different notions of a function preserving $*$-transitivity of fuzzy binary relations. In order to distinguish them we have added "on products" or "on sets" to these notions. Nevertheless, we will show that there is no difference between these two concepts. Moreover, we obtain related results that will be useful in Section 4. Among others, we will provide a new characterization of these functions in terms of an asymmetric version of the concept of $*$-triangular triplet given in [31].

Last section, Section 4, treats the main objective of the paper which is to characterize those functions which aggregate a family of fuzzy (quasi-)metrics into a single one. We stress that in contrast with the crisp case, fuzzy metric aggregation functions (on products or on sets) are also fuzzy quasi-metric aggregation functions (on products or on sets). We also characterize functions which aggregate stationary fuzzy (quasi-pseudo)metrics into a single one. From this, we deduce results about aggregation of fuzzy preorders and indistinguishability operators.

## 2. Fuzzy (quasi-)metrics

In this section we collect some basic facts about fuzzy (quasi-)metrics in the sense of Kramosil and Michalek [24, 27]. We start recalling the concept of a triangular norm as well as its associated residuum which will play a central role in our subsequent discussions.

**Definition 2.1** ([26])**.** A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called a *triangular norm* or a *t-norm* if $([0,1],*)$ is an Abelian monoid with unit 1, such that $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$, with $a,b,c,d\in[0,1]$.

If $*$ is also continuous we will say that it is a *continuous t-norm.*

Triangular norms are considered as the translation of the conjunction logic operator to fuzzy logic and it is a fundamental tool in fuzzy set theory. According to [26], the next example introduces the four basic t-norms in the literature.

**Example 2.2.** The following are examples of continuous t-norms:

- $x\wedge y:=\min\{x,y\}$ (minimum t-norm)
- $x *_P y := x\cdot y$ (product t-norm)
- $x *_{\text{Ł}} y := \max\{x+y-1,0\}$ (Łukasiewicz t-norm)
- $x *_D y := \begin{cases}\min\{x,y\} & \text{if } x=1 \text{ or } y=1\\ 0 & \text{otherwise}\end{cases}$ (drastic t-norm)

**Remark 2.3.** Given two t-norms $*$ and $\star$ we will write $*\leq\star$ whenever $a*b\leq a\star b$ for every $a,b\in[0,1]$. Notice that for any t-norm $*$ we have that $*_D\leq *\leq\wedge$.

Next we recall the residuum associated to a t-norm.

**Definition 2.4** ([2, 26])**.** For a given t-norm $*$, its *residuation* or *residuation implication* is the function $\overset{*}{\to}\colon[0,1]\times[0,1]\to[0,1]$ given by

$$x\overset{*}{\to}y=\sup\{z\in[0,1]: x*z\leq y\},\qquad \text{for all } x,y\in[0,1].$$

The following example introduces the residuum associated to the four basic t-norms exposed in Example 2.2.

**Example 2.5.** The residuations associated with the t-norms of Example 2.2 are given by:

- $x\overset{\wedge}{\to}y=\begin{cases}1 & \text{if } x\leq y\\ y & \text{if } x>y\end{cases}$;
- $x\overset{*_P}{\to}y=\begin{cases}1 & \text{if } x\leq y\\ \frac{y}{x} & \text{if } x>y\end{cases}$;
- $x\overset{*_{\text{Ł}}}{\to}y=\min\{1-x+y,1\}$;
- $x\overset{*_D}{\to}y=\begin{cases}1 & \text{if } x\neq 1\\ y & \text{if } x=1\end{cases}$.

Related to the residuation associated to a t-norm one can found the notion of biresiduation.

**Definition 2.6** ([42])**.** The biresiduation $\overset{*}{\leftrightarrow}$ of a t-norm $*$ is the map $\overset{*}{\leftrightarrow}\colon[0,1]\times[0,1]\to[0,1]$ given by

$$x\overset{*}{\leftrightarrow}y=\min\{x\overset{*}{\to}y,y\overset{*}{\to}x\}\qquad\text{for all } x,y\in X.$$

**Example 2.7.** Biresiduations of t-norms given in of Example 2.2 are

- $x\overset{\wedge}{\longleftrightarrow}y=\begin{cases}1 & \text{if } x=y\\ x\wedge y & \text{if } x\neq y\end{cases}$;

- $x \stackrel{*_P}{\longleftrightarrow} y = \dfrac{x \wedge y}{x \vee y}$;
- $x \stackrel{*_L}{\longleftrightarrow} y = 1 - |x - y|$;
- $x \stackrel{*_D}{\longleftrightarrow} y = \begin{cases} 1 & \text{if } x \neq 1, y \neq 1 \\ x \wedge y & \text{otherwise} \end{cases}$.

**Definition 2.8** ([24])**.** A *fuzzy quasi-pseudometric* (in the sense of Kramosil and Michalek) on a nonempty set $X$ is a pair $(M, *)$ such that $*$ is a t-norm and $M$ is a fuzzy set in $X \times X \times [0, +\infty)$ such that for every $x, y, z \in X$ and $t, s > 0$ it verifies

(FQM1) $M(x, y, 0) = 0$;
(FQM2) $M(x, x, t) = 1$ for all $t > 0$;
(FQM3) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FQM4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous.

If $M$ also verifies

(FQM2') $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$ if and only if $x = y$;

then $(M, *)$ is called a *fuzzy quasi-metric*.

If a fuzzy quasi-(pseudo)metric $(M, *)$ also satisfies

(FM) $M(x, y, t) = M(y, x, t)$

for all $x, y \in X$ and all $t \geq 0$ then $(M, *)$ is said to be a fuzzy (pseudo)metric on $X$.

A *fuzzy (quasi-pseudo)metric* space is a triple $(X, M, *)$ such that $X$ is a nonempty set and $(M, *)$ is a fuzzy (quasi-pseudo)metric on $X$.

**Remark 2.9.** Notice that if $(X, M, *)$ is a fuzzy quasi-pseudometric space, it is customary to ask the t-norm $*$ to be continuous. This condition is needed in order to construct a topology $\tau_M$ on $X$ by means of the fuzzy quasi-pseudometric $(M, *)$. Indeed, it can be easily proved that the family of sets $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$ where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ is a base for the topology $\tau_M$ on $X$ [19, 24].

Nevertheless, since we will not treat with topology in this paper we have removed the condition of continuity for the t-norm $*$ in the previous definition.

**Remark 2.10.** Observe that if $(X, M, *)$ is a fuzzy quasi-pseudometric space then by (FQM3) the function $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is isotone for every $x, y \in X$.

The next example provides an illustrative instance of fuzzy quasi-pseudometric.

**Example 2.11** ([43, Example 3])**.** Given a (quasi-pseudo)metric space $(X, d)$, let $M_d$ be the fuzzy set on $X \times X \times [0, \infty)$ defined by

$$M_d(x, y, t) = \begin{cases} \dfrac{t}{t + d(x, y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every t-norm $*$, $(M_d, *)$ is a fuzzy (quasi-pseudo)metric on $X$ which is called the *standard fuzzy (quasi-pseudo)metric* induced by $d$.

A prominent type of fuzzy (quasi-pseudo)metrics are the so-called stationary fuzzy (quasi-pseudo)metrics defined as follows.

**Definition 2.12** ([24])**.** A fuzzy (quasi-pseudo)metric space $(X, M, *)$ is said to be *stationary* (or $(M, *)$ is a *stationary fuzzy (quasi-pseudo)metric* on $X$) if the function $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is constant for every $x, y \in X$.

Following examples yield instances of stationary fuzzy (quasi-)metrics.

**Example 2.13.** Let $*$ be a t-norm. Define $M^{\overset{*}{\rightarrow}}, M^{\overset{*}{\leftrightarrow}} : [0,1] \times [0,1] \times [0,+\infty) \to [0,1]$ by

$$M^{\overset{*}{\rightarrow}}(x,y,t) = \begin{cases} x \overset{*}{\rightarrow} y & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

$$M^{\overset{*}{\leftrightarrow}}(x,y,t) = \begin{cases} x \overset{*}{\leftrightarrow} y & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x, y \in [0,1]$ and all $t \geq 0$. Then $(M^{\overset{*}{\rightarrow}}, *)$ is a stationary fuzzy quasi-metric on $[0,1]$ (Proposition 2.45 in [42]) and $(M^{\overset{*}{\leftrightarrow}}, *)$ is a stationary fuzzy metric on $[0,1]$ (Proposition 2.50 in [42]).

**Example 2.14** (Example 18 in [23])**.** Given a t-norm $*$, let us define $M^* : [0,1] \times [0,1] \times [0,+\infty) \to [0,1]$ by

$$M^*(x,y,t) = \begin{cases} 1 & \text{if } x = y \text{ and } t > 0 \\ x * y & \text{if } x \neq y \text{ and } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then $(M^*, *)$ is a stationary fuzzy metric on $[0,1]$.

## 3. Functions preserving properties of fuzzy binary relations

In [14, 17, 31, 46], it is studied the problem of characterizing those functions which allow us to merge transitive fuzzy binary relations into a single transitive fuzzy binary relation. This research is important for the aim of this paper since condition (FQM3) of a fuzzy (quasi-)metric is a certain type of transitive property. Then we start going over some results which appear in the aforementioned references and obtaining some new facts which will be useful in our characterization of those functions that merge fuzzy quasi-metrics given in the next section. We start recalling some definitions.

**Definition 3.1** ([17, 42, 50])**.** Let $X$ be a nonempty set and $*$ be a t-norm. A *fuzzy binary relation* on $X$ is a map $E : X \times X \to [0,1]$.

A fuzzy binary relation $E$ is said to:

(1) satisfy the *reflexivity* property if $E(x,x) = 1$ for all $x \in X$;
(2) satisfy the $*$-*transitivity* property if $E(x,y) * E(y,z) \leq E(x,z)$ for all $x, y, z \in X$;
(3) satisfy the *symmetry* property if $E(x,y) = E(y,x)$ for all $x, y \in X$;
(4) *separate points* if $E(x,y) = 1$ implies $x = y$ for all $x, y \in X$;
(5) *asymmetrically separate points* if $E(x,y) = E(y,x) = 1$ implies $x = y$ for all $x, y \in X$.

Observe that property (5) is called antisymmetry in [20]. Nevertheless, we have preferred to rename it since the term antisymmetry is normally used in the sense of Zadeh [50].

We next consider special kinds of fuzzy binary relations which satisfy some of the previous conditions. We must mention that there is not a unified terminology for these fuzzy binary relations in the literature so some of them receive different names.

**Definition 3.2.** Let $*$ be a t-norm. A fuzzy binary relation $E$ on a nonempty set $X$ is said to be:

- an $*$-*fuzzy preorder* if it is reflexive and $*$-transitive [29];
- an $*$-*fuzzy partial order* if it is an $*$-fuzzy preorder which asymmetrically separates points [20, 29];
- an $*$-*indistinguishability operator* if it is a symmetric $*$-fuzzy preorder [42, 50];
- an $*$-*equality* if it is an $*$-indistinguishability operator which separate points [11, 26, 42].

**Remark 3.3.** Notice that, since we have avoided the continuity of the t-norm in the axiomatic of fuzzy pseudometrics, we have that stationary fuzzy pseudometrics and indistinguishability operators are equivalent concepts. Indeed, if $R$ is an $*$-indistinguishability operator on a nonempty set $X$ then the function $M_R : X \times X \times [0,+\infty) \to [0,1]$ given by

$$M_R(x,y,t) = \begin{cases} R(x,y) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

for all $x, y \in X$, $t \geq 0$, determines a stationary fuzzy pseudometric $(M_R, *)$ on $X$. Conversely, if $(X, M, *)$ is a stationary fuzzy pseudometric space then the fuzzy binary relation $R_M : X \times X \to [0,1]$ given by $R_M(x,y) = M(x,y,1)$ is an $*$-indistinguishability operator on $X$.

In a similar way, stationary fuzzy metrics and equalities (indistinguishability operators which separate points) are also equivalent. Moreover, we can also establish an equivalence between fuzzy preorders and stationary fuzzy quasi-pseudometrics.

We refer the reader to [10, 11, 22, 25, 42, 45, 49] for a comprehensive study on the dual relationship between metrics and indistinguishability operators (see also [18, 36, 37] for a recent treatment of this topic).

We next recall the concept of an (infinitary) aggregation function (see [1, 21, 35]).

Let $I$ be a set of indices. We will denote the elements of $[0,1]^I$ by boldface letters $\boldsymbol{a}$ and we will write $\boldsymbol{a}_i$ instead of $\boldsymbol{a}(i)$ for all $i \in I$. Furthermore, $[0,1]^I$ becomes a partially ordered set endowed with the partial order $\preceq$ given by $\boldsymbol{a} \preceq \boldsymbol{b}$ if $\boldsymbol{a}_i \leq \boldsymbol{b}_i$ for all $i \in I$. We will denote by $\mathbf{1}$ and $\mathbf{0}$ the elements of $[0,1]^I$ such that $\mathbf{1}_i = 1$ and $\mathbf{0}_i = 0$ for all $i \in I$.

Notice that if $*$ is a t-norm, then we can define as usual an operation $*^I$ on $[0,1]^I$ given by $(\boldsymbol{a} *^I \boldsymbol{b})_i = \boldsymbol{a}_i * \boldsymbol{b}_i$ for all $i \in I$.

**Definition 3.4** (cf. [1, 21, 35])**.** Let $I$ be a set of indices. An *$I$-aggregation function* is a function $F : [0,1]^I \to [0,1]$ such that:

(1) $F$ is isotone, i. e. if $\boldsymbol{a} \preceq \boldsymbol{b}$ then $F(\boldsymbol{a}) \leq F(\boldsymbol{b})$;
(2) $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$.

In general, we will omit the reference to the set of indices $I$ when we consider an $I$-aggregation function.

We next introduce two different notions of functions which merge a family of fuzzy binary relations into a single one.

**Definition 3.5** (cf. [17, 31, 39, 46])**.** Let $I$ be a set of indices and $F : [0,1]^I \to [0,1]$ be a function. We say that:

- *$F$ preserves properties $P_1, \ldots, P_n$ of fuzzy binary relations on products* if whenever $\{(X_i, E_i) : i \in I\}$ is a family of nonempty sets $X_i$ endowed with fuzzy binary relations $E_i$ satisfying all the properties $P_1, \ldots, P_n$ for all $i \in I$, then $F \circ \widetilde{\boldsymbol{E}}$ is a fuzzy binary relation on $\prod_{i\in I} X_i$ also satisfying all the properties $P_1, \ldots, P_n$ where $\widetilde{\boldsymbol{E}} : (\prod_{i\in I} X_i)^2 \to [0,1]^I$ is given by
$$\widetilde{\boldsymbol{E}}(\boldsymbol{a}, \boldsymbol{b})_i = E_i(\boldsymbol{a}_i, \boldsymbol{b}_i)$$
for all $i \in I$.[1]
- *$F$ preserves properties $P_1, \ldots, P_n$ of fuzzy binary relations on sets* if whenever $\{E_i : i \in I\}$ is a family of fuzzy binary relations $E_i$ on a fixed nonempty set $X$ satisfying all the properties $P_1, \ldots, P_n$ for all $i \in I$, then $F \circ \boldsymbol{E}$ is a fuzzy binary relation on $X$ also satisfying all the properties $P_1, \ldots, P_n$ where $\boldsymbol{E} : X^2 \to [0,1]^I$ is given by
$$\boldsymbol{E}(a, b)_i = E_i(a, b)$$
for all $i \in I$.

**Remark 3.6.** Let $P_1, \ldots, P_n$ be properties of fuzzy binary relations which are hereditary, i.e. if $E$ is a fuzzy binary relation on $X$ satisfying all the properties $P_1, \ldots, P_n$ then its restriction to any subset of $X$ also satisfies all the properties $P_1, \ldots, P_n$. Notice that if $F : [0,1]^I \to [0,1]$ preserves the properties $P_1, \ldots, P_n$ of fuzzy binary relations on products then it preserves the properties $P_1, \ldots, P_n$ of fuzzy binary relations on sets. In fact, let $E_i$ be a fuzzy binary relation on a nonempty set $X$ satisfying all the properties $P_1, \ldots, P_n$ for all $i \in I$. Given $a, b \in X$ we have that
$$\boldsymbol{E}(a, b)_i = E_i(a, b) = \widetilde{\boldsymbol{E}}(\boldsymbol{a}, \boldsymbol{b})_i$$
where $\boldsymbol{a}, \boldsymbol{b} \in X^I$ and $\boldsymbol{a}_i = a$, $\boldsymbol{b}_i = b$ for all $i \in I$. Since $F \circ \widetilde{\boldsymbol{E}}|_{\Delta\times\Delta} = F \circ \boldsymbol{E}$ then $F \circ \boldsymbol{E}$ satisfies properties $P_1, \ldots, P_n$, where $\Delta = \{\boldsymbol{x} \in \prod_{i\in I} X : \boldsymbol{x}_i = \boldsymbol{x}_j \text{ for all } i, j \in I\}$.

As exposed before, in [46] those aggregation functions which preserve $*$-transitivity of fuzzy binary relations on products were characterized (see also [14]). Meanwhile, in [31] it was analyzed which functions merge a family of $*$-indistinguishability operators into a single one, which provided a characterization of those functions preserving $*$-transitivity of fuzzy binary relations on sets. Surprisingly, we will prove in Proposition 3.26 that there is no difference between those functions which preserve $*$-transitivity of fuzzy binary relations on products and those which preserve $*$-transitivity of fuzzy binary relations on sets (see Definition 3.5). Moreover, although the results of [31, 46] (see also [14, 17]) consider functions defined on finite products, their results are also valid for infinite products, so we state them in this level of generality.

The following concept was introduced in [46] in order to characterize aggregation functions which preserve $*$-transitivity of fuzzy binary relations on products.

**Definition 3.7** (cf. [46])**.** Let $F : [0,1]^I \to [0,1]$ and $G : [0,1]^J \to [0,1]$ be two aggregation functions. We say that $F$ *dominates* $G$ if for all $x_{i,j} \in [0,1]$ with $i \in I$ and $j \in J$ then
$$G((F((x_{i,j})_{i\in I}))_{j\in J}) \le F((G((x_{i,j})_{j\in J}))_{i\in I}).$$

[1]Notice that $\widetilde{\boldsymbol{E}}$ depends on the family $\{(X_i, E_i) : i \in I\}$. Nevertheless, in order to not overload the notation we haven't specified this dependence. We will do it throughout the paper.

**Remark 3.8.** Let $F : [0,1]^I \to [0,1]$ be an aggregation function and $*$ be a t-norm, which is also an aggregation function. Then $F$ dominates $*$ whenever

$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b})$$

for every $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$.

Then we will say that an arbitrary function $F : [0,1]^I \to [0,1]$ which satisfies the above inequality is $*$-*supmultiplicative* (the terminology comes from the classical property of superadditivity). If $F$ is $*$-supmultiplicative for every t-norm $*$ then it will be called *supmultiplicative*.

$*$-supmultiplicative functions appear in different contexts. For example, isotone functions which are ($*$-)supmultiplicative are called ($*$-)closed, see Definition 7.1 in [16]. Moreover, if $F$ is $*$-supmultiplicative and $F(\mathbf{1}) = 1$ then it is called an $*$-fuzzy monoid [7].

Next we provide a few examples of $*$-supmultiplicative functions.

**Example 3.9.** Let $I$ be a set of indices.

- Given $i \in I$ the projection function $P_i : [0,1]^I \to [0,1]$ given by $P_i(\boldsymbol{x}) = \boldsymbol{x}_i$ is $*$-closed for every t-norm $*$.
- The function $\mathsf{Inf} : [0,1]^I \to [0,1]$ given by $\mathsf{Inf}(\boldsymbol{x}) = \inf_{i\in I} \boldsymbol{x}_i$ is $*$-closed for every t-norm $*$.

  Nevertheless, the function $\mathsf{Sup} : [0,1]^I \to [0,1]$ given by $\mathsf{Sup}(\boldsymbol{x}) = \sup_{i\in I} \boldsymbol{x}_i$ not $*$-supmultiplicative for any t-norm $*$ whenever $|I| > 1$ (otherwise is the identity function). Indeed, let $j, k \in I$ with $j \neq k$ and let $\boldsymbol{x}, \boldsymbol{y} \in [0,1]$ such that $\boldsymbol{x}_i = 0$ for all $i \in I\backslash\{j\}$ and $\boldsymbol{x}_j = 1$ meanwhile $\boldsymbol{y}_i = 0$ for all $i \in I\backslash\{k\}$ and $\boldsymbol{y}_k = 1$. Then $\mathsf{Sup}(\boldsymbol{x}) * \mathsf{Sup}(\boldsymbol{y}) = 1 \not\leq \mathsf{Sup}(\boldsymbol{x} *^I \boldsymbol{y}) = 0$.
- Given a t-norm $*$, $n \in \mathbb{N}$ and $k \in [0,1]$, the function $T_*^k : [0,1]^n \to [0,1]$ given by $T_*(\boldsymbol{x}) = k * \boldsymbol{x}_1 * \ldots * \boldsymbol{x}_n$ is $*$-closed.
- Given $k > 0$, the function $F_k : [0,1]^I \to [0,1]$ given by $F_k(\boldsymbol{a}) = \prod_{i\in I} \boldsymbol{a}_i^k$ is $*_P$-closed.
- Let $n \in \mathbb{N}$ and consider functions $f_i : [0,1] \to [0,1]$ which are isotone for all $i \in \{1, \ldots, n\}$. Then the function $F : [0,1]^n \to [0,1]$ given by $F(\boldsymbol{x}) = \bigwedge_{i=1}^n f_i(\boldsymbol{x}_i)$ is $\wedge$-closed (in fact, it is minitive [21]).

The following result characterizes aggregation functions which preserve $*$-transitivity of fuzzy binary relations on products. It is a modification of [46, Theorem 9] (see also [14, Theorem 10]), where the result was for aggregation functions defined over $\cup_{n\in\mathbb{N}}[0,1]^n$ and for fuzzy binary relations defined over a predetermined set $X$. Nevertheless, the same proof is valid when the aggregation function is defined over an arbitrary product $[0,1]^I$ and when the universe where the fuzzy binary relations are defined is not the same and it is not fixed beforehand. This avoids to add the assumption $|X| \geq 3$ which appears in Theorem 9 in [46] (and in Theorem 10 in [14]).

**Theorem 3.10** (cf. [46, Theorem 9])**.** *Let $F : [0,1]^I \to [0,1]$ be an aggregation function and $*$ be a t-norm. Then $F$ preserves $*$-transitivity of fuzzy binary relations on products if and only if $F$ dominates $*$ ($F$ is $*$-supmultiplicative).*

In [13, 14, 15], it is studied in depth the problem of characterizing those functions (not necessarily aggregation functions) which preserve basic properties of fuzzy binary relations defined over the same set as reflexivity, symmetry and $*$-transitivity, where $*$ is an operation which is not necessarily a t-norm. In particular, among others, a version of Theorem 3.10 was obtained in this framework removing the conditions $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$, implicitly included in the definition of aggregation function (Theorem 10 in [14]).

Recently, in [31], the problem of characterizing those functions (not necessarily aggregation functions) which preserve reflexivity, symmetry and $*$-transitivity of fuzzy binary relations defined over the same set was studied, i. e. it was studied those functions which aggregate $*$-indistinguishability operators into a single one. This characterization makes use of the concept of $*$-triangular triplet. This notion is an adaption to the fuzzy context of the concept of triangular triplet which allows us to characterize metric preserving functions, i. e. those functions whose composition with an arbitrary metric is again a metric (see [44]). In this way, a triplet $(a, b, c)$ whose elements belong to $[0, +\infty)$ is a *triangular triplet* if $a \leq b + c$, $b \leq a + c$ and $c \leq a + b$. If $(a, b, c)$ only satisfies that $a \leq b + c$ then we say that it is an *asymmetric triangular triplet* (see [32]). Notice that if $(X, d)$ is a metric space then $(d(x, y), d(y, z), d(x, z))$ is a triangular triplet for every $x, y, z \in X$.

Let us recall the notion of $*$-triangular triplet in the sense of [31].

**Definition 3.11** ([31])**.** Let $*$ be a t-norm and $I$ be a set of indices. A triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0, 1]^I)^3$ is said to be $*$-*triangular* if

$$\boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}, \quad \boldsymbol{a} *^I \boldsymbol{c} \preceq \boldsymbol{b} \quad \text{and} \quad \boldsymbol{b} *^I \boldsymbol{c} \preceq \boldsymbol{a},$$

i. e.

$$a_i * b_i \leq c_i, \quad a_i * c_i \leq b_i \quad \text{and} \quad b_i * c_i \leq a_i$$

for every $i \in I$. If $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is $*$-triangular for every t-norm $*$ then we say that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a *triangular triplet*.

The following examples provide instances of $*$-triangular triplets.

**Example 3.12.** Given a t-norm $*$, the triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} *^I \boldsymbol{b})$ is $*$-triangular for every $\boldsymbol{a}, \boldsymbol{b} \in [0, 1]^I$. Furthermore, the triplet $(\boldsymbol{a}, \boldsymbol{a}, \mathbf{1})$ is $*$-triangular for every $\boldsymbol{a} \in [0, 1]^I$ and every t-norm $*$ so it is triangular.

**Example 3.13.** Let $*$ be a t-norm and let $E$ be a fuzzy binary relation on a nonempty set $X$ such that $E$ is symmetric and $*$-transitive. Given $x, y, z \in X$ then $(E(x, y), E(y, z), E(x, z))$ is an $*$-triangular triplet.

**Remark 3.14** ([31, Remark 1])**.** Let $*$ and $\star$ be two t-norms such that $* \leq \star$, i. e. $a * b \leq a \star b$ for every $a, b \in [0, 1]$. Then every $\star$-triangular triplet is an $*$-triangular triplet. Consequently, if we denote by $\mathcal{T}_*$ the set of all $*$-triangular triplets then $\mathcal{T}_\wedge \subseteq \mathcal{T}_* \subseteq \mathcal{T}_{*_D}$ for any t-norm $*$. If $|I| = 1$ it is shown in Examples 1 and 2 in [31] that

- $\mathcal{T}_\wedge = \{(a, b, c) \in [0, 1]^3$ : two of the elements are equal and the other is greater than the other two$\}$;
- $\mathcal{T}_{*_D} = \{(a, b, c) \in [0, 1]^3 : a, b, c$ are different from 1 or one of them is 1 and the other two coincide$\}$.

See Figure 1 for a graphical representation of $\mathcal{T}_\wedge$, $\mathcal{T}_{*_P}$, $\mathcal{T}_{*_L}$, $\mathcal{T}_{*_D}$ when $|I| = 1$.

The next result will be useful later on.

**Lemma 3.15.** *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0, 1]^I$ with $\boldsymbol{a} \preceq \boldsymbol{b} \preceq \boldsymbol{c}$ and let $*$ be a t-norm. Then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an $*$-triangular triplet if and only if $\boldsymbol{b} *^I \boldsymbol{c} \preceq \boldsymbol{a}$.*

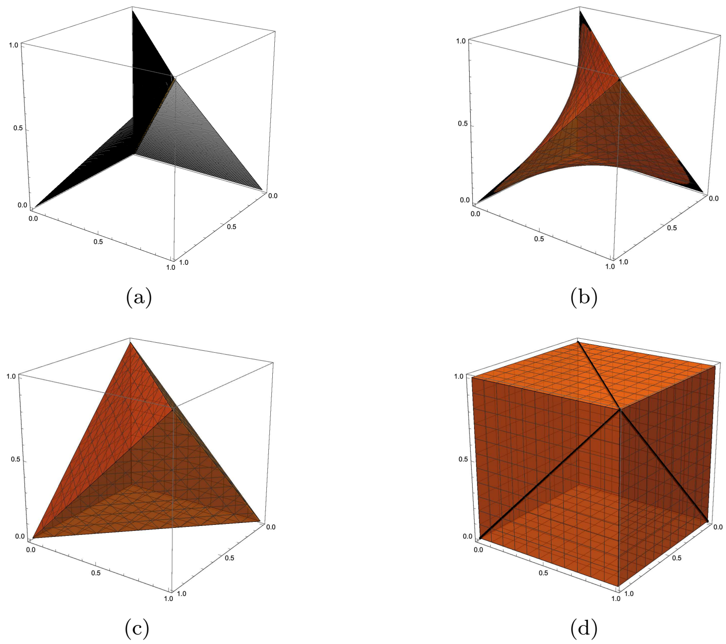

FIGURE 1. (a) $\wedge$-triangular triplets (b) $*_P$-triangular triplets (c) $*_{\mathrm{L}}$-triangular triplets (d) $*_D$-triangular triplets

*Proof.* Necessity is obvious. Suppose now that $\boldsymbol{b} *^I \boldsymbol{c} \preceq \boldsymbol{a}$. Since $\boldsymbol{a} \preceq \boldsymbol{b} \preceq \boldsymbol{c}$ then $\boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}$ and $\boldsymbol{a} *^I \boldsymbol{c} \preceq \boldsymbol{b}$ so $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an $*$-triangular triplet. □

**Remark 3.16.** Notice that a triplet $(a, b, c)$ in $[0, 1]^3$ is $*$-triangular if and only if any reordering of the triplet is also $*$-triangular. Since $[0, 1]$ is linearly ordered, we can apply the above lemma to determine if a triplet in $[0, 1]^3$ is $*$-triangular only by ordering the numbers.

An asymmetric version of the previous concept will be crucial in the next section for our characterization of those functions which merge fuzzy (quasi-)metrics into a new one.

**Definition 3.17.** Let $*$ be a t-norm and $I$ be a set of indices. A triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0, 1]^I)^3$ is said to be *asymmetric $*$-triangular* if $\boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}$.

If the triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is asymmetric $*$-triangular for every t-norm $*$ then we say that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an asymmetric triangular triplet.

Observe that the preceding notion is an adaption to the fuzzy context of the concept of asymmetric triangular triplet introduced in [32].

The following examples provide instances of asymmetric $*$-triangular triplets.

**Example 3.18.** For any t-norm $*$, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{1})$ is always an asymmetric $*$-triangular triplet for any $\boldsymbol{a}, \boldsymbol{b} \in [0, 1]^I$ so it is an asymmetric triangular triplet. Furthermore, it is an $*$-triangular triplet if and only if $\boldsymbol{a} = \boldsymbol{b}$. This shows that for any t-norm $*$, the set of $*$-triangular triplets is strictly included in the set of asymmetric $*$-triangular triplets.

**Example 3.19.** Let $(X, M, *)$ be a fuzzy (quasi-)metric space. Given $x, y, z \in X$ and $s, t > 0$ we have that $(M(x, y, t), M(y, z, s), M(x, z, t + s))$ is an asymmetric $*$-triangular triplet.

Observe that this behaviour is different from the crisp case since, as we have previously observed, if $(X, d)$ is a metric space then $(d(x, y), d(y, z), d(x, z))$ is always a triangular triplet.

**Remark 3.20.** Notice that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a triangular (an asymmetric triangular) triplet if and only if it is a $\wedge$-triangular (an asymmetric $\wedge$-triangular) triplet.

**Example 3.21.** Given a t-norm $*$, the set $\mathcal{AT}_*$ of asymmetric $*$-triangular triplets is the epigraph of the function $F_* : [0,1]^I \times [0,1]^I \to [0,1]^I$ given by $F_*(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} *^I \boldsymbol{b}$ for every $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$, i. e.

$$\mathcal{AT}_* = \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,1]^I)^3 : \boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}\} = \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in ([0,1]^I)^3 : F_*(\boldsymbol{a}, \boldsymbol{b}) \preceq \boldsymbol{c}\}.$$

Therefore, when $|I| = 1$ we can draw these sets for some usual t-norms:

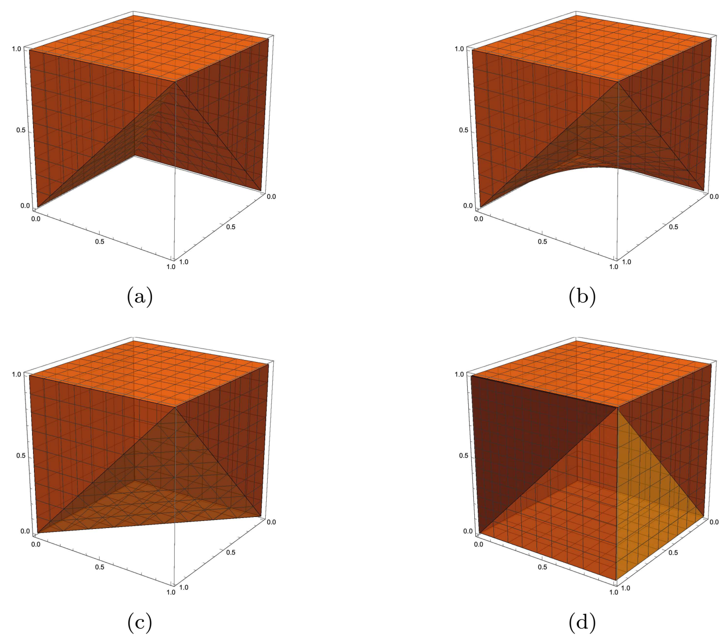


FIGURE 2. (a) asymmetric $\wedge$-triangular triplets (b) asymmetric $*_P$-triangular triplets (c) asymmetric $*_{\mathrm{L}}$-triangular triplets (d) asymmetric $*_D$-triangular triplets

The next notion will play a central role in our subsequent study.

**Definition 3.22.** Let $*$ be a t-norm and $I$ be a set of indices. A function $F : [0,1]^I \to [0,1]$ *preserves $*$-triangular (asymmetric $*$-triangular) triplets* if $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an $*$-triangular (an asymmetric $*$-triangular) triplet whenever $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ so is, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,1]^I$.

If $F$ preserves $*$-triangular (asymmetric $*$-triangular) triplets for every t-norm $*$ we will simply say that $F$ *preserves triangular (asymmetric triangular) triplets*.

It is clear that if a function $F$ preserves asymmetric $*$-triangular triplets then it preserves $*$-triangular triplets. However, in general the converse is only true for the t-norm $\wedge$ (see Corollary 3.33) but not for any other t-norm as the next example shows.

**Example 3.23.** Let $*$ be a t-norm different from $\wedge$. Then $*$ is not idempotent (see [26]) so there exists $\alpha \in ]0,1[$ such that $\alpha * \alpha < \alpha$. Let us consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < \frac{1}{2} \\ \alpha * \alpha & \text{if } \frac{1}{2} \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

Let us check that $f$ preserves $*$-triangular triplets. Let $(a,b,c)$ be an arbitrary triplet with all its elements different from 1. Then the triplet $(f(a), f(b), f(c))$ is some of the following triplets or a permutation of them: $(\alpha * \alpha, \alpha, \alpha)$, $(\alpha * \alpha, \alpha * \alpha, \alpha)$, $(\alpha * \alpha, \alpha * \alpha, \alpha * \alpha)$, $(\alpha, \alpha, \alpha)$. Notice that all these triplets are $*$-triangular. Furthermore, if for example $a = 1$ and $(a,b,c)$ is an $*$-triangular triplet then $b = c$ so $(f(1), f(b), f(c)) = (1, f(b), f(b))$ also is an $*$-triangular triplet. Hence $f$ preserves $*$-triangular triplets.

However, $(1, \frac{1}{3}, \frac{1}{2})$ is an asymmetric $*$-triangular triplet but $(f(1), f(\frac{1}{3}), f(\frac{1}{2})) = (1, \alpha, \alpha * \alpha)$ is not an asymmetric $*$-triangular triplet since $\alpha * \alpha < \alpha$.

We can characterize the functions of one variable which preserve (asymmetric) $*_D$-triangular triplets as follows.

**Proposition 3.24.** *Let $f : [0,1] \to [0,1]$.*

*(1) $f$ preserves $*_D$-triangular triplets if and only if either $f^{-1}(1) = [0,1]$ or $f^{-1}(1) \subseteq \{1\}$.*

*(2) $f$ preserves asymmetric $*_D$-triangular triplets if and only if one of the following conditions is satisfied:*

(a) $f^{-1}(1) = [0,1]$;

(b) $f^{-1}(1) = \varnothing$;

(c) $f^{-1}(1) = \{1\}$ *and $f$ is isotone.*

*Proof.* (1) Suppose that $f$ preserves $*_D$-triangular triplets. If $f^{-1}(1) \neq [0,1]$ and $f^{-1}(1) \nsubseteq \{1\}$ we can find $a \in [0,1[$ such that $f(a) = 1$ and $b \in [0,1]$ such that $f(b) \neq 1$. By Remark 3.14 $(a,a,b)$ is an $*_D$-triangular triplet so $(f(a), f(a), f(b)) = (1, 1, f(b))$ must be an $*_D$-triangular triplet which is a contradiction since $1 *_D 1 = 1 \nleq f(b)$.

The converse is straightforward.

(2) Suppose that $f$ preserves asymmetric $*_D$-triangular triplets. In particular it preserves $*_D$-triangular triplets so, by (1), $f^{-1}(1) = [0,1]$ or $f^{-1}(1) \subseteq \{1\}$. Suppose that $f^{-1}(1) = \{1\}$. Let $a, b \in [0,1]$ with $a \leq b$. Then $(1, a, b)$ is an asymmetric $*_D$-triangular triplet so $(f(1), f(a), f(b))$ so is, i. e. $f(1) *_D f(a) = f(a) \leq f(b)$. Hence $f$ is isotone.

Conversely, if either $f^{-1}(1) = [0,1]$ or $f^{-1}(1) = \varnothing$ then $f$ preserves asymmetric $*_D$-triangular triplets trivially. Suppose that $f^{-1}(1) = 1$ and $f$ is isotone. Let $(a,b,c)$ be an asymmetric $*_D$-triangular triplet. If $a \neq 1, b \neq 1$ then $f(a) \neq 1, f(b) \neq 1$ so $f(a) *_D f(b) = 0$ and $(f(a), f(b), f(c))$ is an asymmetric $*_D$-triangular triplet. Suppose that $a = 1$ so $f(a) = 1$. Since $(1, b, c)$ is an asymmetric $*_D$-triangular triplet then $b \leq c$ so $(f(a), f(b), f(c)) = (1, f(b), f(c))$ is also an asymmetric $*_D$-triangular triplet due to the isotonicity of $f$. If $b = 1$ we can reason in a similar way. □

Next we introduce the announced characterization, given in [31], of those functions which aggregate $*$-indistinguishability operators, i. e. functions which combine a collection of $*$-indistinguishability operators defined on the same set into a single one, in terms of the preservation of $*$-triangular triplets.

**Theorem 3.25** (Proposition 4 in [31])**.** *Let $F : [0,1]^I \to [0,1]$ be a function and let $*$ be a t-norm. Then $F$ preserves reflexivity, symmetry and $*$-transitivity of fuzzy binary relations on sets if and only if $F(\mathbf{1}) = 1$ and $F$ preserves $*$-triangular triplets.*

If we pay attention on functions aggregating fuzzy binary relations which satisfy each of the previous properties separately, we obtain the following characterizations, including also the situation on products. In particular, we obtain a characterization of functions which preserve $*$-transitivity of fuzzy binary relations in terms of the preservation of asymmetric $*$-triangular triplets.

**Proposition 3.26.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. Then:*

*(1) The following statements are equivalent:*
  - *(a) $F$ preserves reflexivity of fuzzy binary relations on products;*
  - *(b) $F$ preserves reflexivity of fuzzy binary relations on sets;*
  - *(c) $F(\mathbf{1}) = 1$.*

*(2) $F$ preserves symmetry of fuzzy binary relations on products and symmetry of fuzzy binary relations on sets.*

*(3) The following statements are equivalent:*
  - *(a) $F$ preserves $*$-transitivity of fuzzy binary relations on products (resp. $*$-transitivity and symmetry of fuzzy binary relations on products);*
  - *(b) $F$ preserves $*$-transitivity of fuzzy binary relations on sets (resp. $*$-transitivity and symmetry of fuzzy binary relations on sets);*
  - *(c) $F$ preserves asymmetric $*$-triangular triplets (resp. $*$-triangular triplets).*

*Proof.* (1) and (2) are extremely easy so they are omitted.

Let us check (3). That (a) implies (b) is obvious by Remark 3.6. To prove (b)⇒(c), let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ be an asymmetric $*$-triangular triplet. Let us consider a set $X = \{x, y\}$ with two different elements and the $*$-transitive fuzzy binary relations $E_i : X^2 \to [0,1]$ given by $E_i(x,y) = \boldsymbol{a}_i$, $E_i(y,x) = \boldsymbol{b}_i$ and $E_i(x,x) = E_i(y,y) = \boldsymbol{c}_i$ for all $i \in I$. Then $\{E_i : i \in I\}$ is a family of $*$-transitive fuzzy binary relations on $X$. By assumption $F \circ \boldsymbol{E}$ is an $*$-transitive fuzzy binary relation on $X$ so

$$F \circ \boldsymbol{E}(x,y) * F \circ \boldsymbol{E}(y,x) \leq F \circ \boldsymbol{E}(x,x)$$
$$F((E_i(x,y))_{i\in I}) * F((E_i(y,x))_{i\in I}) \leq F((E_i(x,x))_{i\in I})$$
$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{c}).$$

Hence $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an asymmetric $*$-triangular triplet.

In case that $F$ preserves $*$-transitivity and symmetry of fuzzy binary relations on sets, let us consider an $*$-triangular triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. Define a set $X = \{x, y, z\}$ with three different elements and the $*$-transitive and symmetric fuzzy binary relations $E_i$ on $X$ given by $E_i(x,x) = E_i(y,y) = E_i(z,z) = 1$, $E_i(x,y) = E_i(y,x) = \boldsymbol{a}_i$,

$E_i(y,z) = E_i(z,y) = \boldsymbol{b}_i$ and $E_i(x,z) = E_i(z,x) = \boldsymbol{c}_i$ for all $i \in I$. The proof to conclude that $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an $*$-triangular triplet follows as in the asymmetric case.

Let us prove (c) $\Rightarrow$ (a). Let $\{(X_i, E_i) : i \in I\}$ be a family of nonempty sets $X_i$ endowed with $*$-transitive fuzzy binary relations $E_i$ for all $i \in I$. Let us check that $F \circ \widetilde{\boldsymbol{E}}$ is an $*$-transitive fuzzy binary relation on $\prod_{i\in I} X_i$ (if $E_i$ is also symmetric for all $i \in I$, then $F \circ \widetilde{\boldsymbol{E}}$ is symmetric by (2)). Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \prod_{i\in I} X_i$. Since $E_i$ is $*$-transitive (resp. $*$-transitive and symmetric) then $(E_i(\boldsymbol{x}_i, \boldsymbol{y}_i), E_i(\boldsymbol{y}_i, \boldsymbol{z}_i), E_i(\boldsymbol{x}_i, \boldsymbol{z}_i))$ is an asymmetric $*$-triangular triplet (resp. $*$-triangular triplet) for every $i \in I$. Consequently, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an asymmetric $*$-triangular triplet (resp. $*$-triangular triplet) where $\boldsymbol{a}_i = E_i(\boldsymbol{x}_i, \boldsymbol{y}_i)$, $\boldsymbol{b}_i = E_i(\boldsymbol{y}_i, \boldsymbol{z}_i)$, $\boldsymbol{c}_i = E_i(\boldsymbol{x}_i, \boldsymbol{z}_i)$ for all $i \in I$. By assumption $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is also an asymmetric $*$-triangular triplet (resp. $*$-triangular triplet) so

$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{c})$$
$$F((E_i(\boldsymbol{x}_i, \boldsymbol{y}_i))_{i\in I}) * F((E_i(\boldsymbol{y}_i, \boldsymbol{z}_i))_{i\in I}) \leq F((E_i(\boldsymbol{x}_i, \boldsymbol{z}_i))_{i\in I})$$
$$F \circ \widetilde{\boldsymbol{E}}(\boldsymbol{x}, \boldsymbol{y}) * F \circ \widetilde{\boldsymbol{E}}(\boldsymbol{y}, \boldsymbol{z}) \leq F \circ \widetilde{\boldsymbol{E}}(\boldsymbol{x}, \boldsymbol{z})$$

so $F \circ \widetilde{\boldsymbol{E}}$ is $*$-transitive. □

**Remark 3.27.** Observe that the equivalence between (b) and (c) in (1) as well as (2), were proved in [13] when $I$ has finite cardinality.

In the light of the preceding result, it is natural to wonder whether there exists a property of fuzzy binary relations which is preserved on sets by a function but not on products. The next result gives the answer.

**Proposition 3.28.** *Let $F : [0,1]^I \to [0,1]$ be a function.*

*(1) $F$ preserves the property of separating points of fuzzy binary relations on products if and only if $F^{-1}(1) \subseteq \{\boldsymbol{1}\}$.*

*(2) $F$ preserves the property of separating points of fuzzy binary relations on sets if and only if $F^{-1}(1) \subseteq \bigcup_{i\in I}\{\boldsymbol{a} \in [0,1]^I : \boldsymbol{a}_i = 1\}$.*

*Proof.* (1) Suppose that $F$ preserves the property of separating points of fuzzy binary relations on products. Suppose also that there exists an element $\boldsymbol{a} \in [0,1]^I$, $\boldsymbol{a} \neq \boldsymbol{1}$ such that $F(\boldsymbol{a}) = 1$. Let us define $I_1 = \{i \in I : \boldsymbol{a}_i = 1\}$ and $I_2 = \{i \in I : \boldsymbol{a}_i \neq 1\}$ which is nonempty. Let us consider two different elements $x, y$ and define

$$X_i = \begin{cases} \{x\} & \text{if } i \in I_1 \\ \{x,y\} & \text{if } i \in I_2 \end{cases}.$$

If $i \in I_1$ define the symmetric fuzzy binary relation $E_i$ on $X_i$ as $E_i(x,x) = 1 = \boldsymbol{a}_i$ meanwhile if $i \in I_2$ define the symmetric fuzzy binary relation $E_i$ on $X_i$ as $E_i(x,x) = E_i(y,y) = 1$ and $E_i(x,y) = E_i(y,x) = \boldsymbol{a}_i$. It is clear that $\{(X_i, E_i) : i \in I\}$ is a family of sets endowed with symmetric fuzzy binary relations which separate points. Hence, by hypothesis, $F \circ \widetilde{\boldsymbol{E}}$ is a symmetric fuzzy binary relation on $\prod_{i\in I} X_i$ which also separates points. Let us consider $\boldsymbol{z}, \boldsymbol{t} \in \prod_{i\in I} X_i$ such that $\boldsymbol{z}_i = x$ for all $i \in I$ and

$$\boldsymbol{t}_i = \begin{cases} x & \text{if } i \in I_1 \\ y & \text{if } i \in I_2 \end{cases}.$$

Since $I_2$ is nonempty then $\boldsymbol{z} \neq \boldsymbol{t}$. Nevertheless

$$F \circ \widetilde{\boldsymbol{E}}(\boldsymbol{z}, \boldsymbol{t}) = F((E_i(\boldsymbol{z}_i, \boldsymbol{t}_i))_{i \in I}) = F(\boldsymbol{a}) = 1$$

so $F \circ \widetilde{\boldsymbol{E}}$ does not separate points, which is a contradiction. Therefore $F^{-1}(1) \subseteq \{\boldsymbol{1}\}$.

The converse is obvious.

(2) Suppose now that $F$ preserves the property of separating points of fuzzy binary relations on sets. If we could find $\boldsymbol{a} \in [0,1]$ such that $F(\boldsymbol{a}) = 1$ but $\boldsymbol{a}_i \neq 1$ for all $i \in I$, let us consider a set $X = \{x, y\}$ with two different elements and the symmetric fuzzy binary relations $E_i$ on $X$ given by $E_i(x,x) = E_i(y,y) = 1$ and $E_i(x,y) = E_i(y,x) = \boldsymbol{a}_i$ for all $i \in I$. Then $\{E_i : i \in I\}$ is a family of symmetric fuzzy binary relations on $X$ which separate points. By assumption, $F \circ \boldsymbol{E}$ is a symmetric fuzzy binary relation on $X$ which separates points but $F \circ \boldsymbol{E}(x,y) = F((E_i(x,y))_{i \in I}) = F(\boldsymbol{a}) = 1$ which is a contradiction since $x \neq y$. Therefore, there must exists $i \in I$ such that $\boldsymbol{a}_i = 1$.

Conversely, let $X$ be a set and let $\{E_i : i \in I\}$ be a family of fuzzy binary relations on $X$ which separate points. Let $x, y \in X$ such that $F \circ \boldsymbol{E}(x,y) = F((E_i(x,y))_{i \in I}) = 1$. By hypothesis there exists $j \in I$ such that $E_j(x,y) = 1$. Since $E_j$ separates points then $x = y$. Hence $F \circ \boldsymbol{E}$ separates points. □

**Remark 3.29.** Notice that (1) implies (2) in the above proposition but the converse is not true in general (if $|I| = 1$ then (2) implies (1)). You can consider the function $F : [0,1] \times [0,1] \to [0,1]$ given by $F(x,y) = \max\{x,y\}$. Even more, it can be easy to construct a function $F$ which preserves the property of separating points of fuzzy binary relations on sets and $F(\boldsymbol{1}) \neq 1$. For example, you can consider $F : [0,1] \times [0,1] \to [0,1]$ given by $F(1,0) = 1$ and $F(x,y) = 0$ whenever $(x,y) \neq (1,0)$.

As pointed out before, in [46] it is proved a characterization of those aggregation functions which preserve $*$-transitivity of fuzzy binary relations on products (see Theorem 3.10) in terms of domination of $*$. On its part, Proposition 3.26 characterizes functions preserving the same property in terms of $*$-triangular triplets but without assuming any property on the function. Nevertheless, this induces to think that there must be a relationship between the property of preserving (asymmetric) $*$-triangular triplets and the property of domination. We discuss it in the following.

**Proposition 3.30** (cf. Proposition 7 in [31])**.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. Each of the following statements implies its successor:*

*(1) $F$ preserves asymmetric $*$-triangular triplets;*
*(2) $F$ preserves $*$-triangular triplets;*
*(3) $F$ dominates $*$, i. e. $F$ is $*$-supmultiplicative.*

*If $F$ is isotone then all the above statements are equivalent.*

*Proof.* (1) $\Rightarrow$ (2). This is straightforward.

(2) $\Rightarrow$ (3). Let $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$. Then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} *^I \boldsymbol{b})$ is an $*$-triangular triplet. Hence $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{a} *^I \boldsymbol{b}))$ so that is, i.e. $F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b})$. Then $F$ is $*$-supmultiplicative.

Suppose that $F$ is isotone. Let us prove (3) $\Rightarrow$ (1). Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ be an asymmetric $*-$triangular triplet, i.e. $\boldsymbol{a} *^I \boldsymbol{b} \preceq \boldsymbol{c}$. By assumption, $F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b})$. Since $F$ is isotone then $F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a} *^I \boldsymbol{b}) \leq F(\boldsymbol{c})$ so $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an asymmetric $*-$triangular triplet. □

**Remark 3.31.** As a consequence of Proposition 3.26 and the previous result, we deduce that Theorem 3.10 remains true if we remove the assumptions $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$ as it was first proved in Theorem 10 in [14] when $|I|$ is finite.

Notice that none of the converse of the implications of Proposition 3.30 are true in general. A counterexample for (2)$\Rightarrow$ (1) is given in Example 3.23. Next example shows that in general, (3) does not imply (2) for any t-norm different from $\wedge$.

**Example 3.32.** Let $*$ be a t-norm different from $\wedge$ so there exists $\alpha \in\, ]0,1[$ such that $\alpha * \alpha < \alpha$. Let us consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \alpha \\ 0 & \text{if } \alpha < x \leq 1 \end{cases}.$$

It is easy to check that $f$ is $*$-supmultiplicative. In fact, given $a, b \in [0,1]$ if $a > \alpha$ or $b > \alpha$ then $f(a) * f(b) = 0 \leq f(a * b)$. If $a, b$ are less than or equal to $\alpha$ then $a * b \leq \alpha * \alpha < \alpha$ so $f(a) * f(b) = 1 * 1 = 1 = f(a * b)$.

Nevertheless, $(\alpha, \alpha, 1)$ is an $*$-triangular triplet but $(f(\alpha), f(\alpha), f(1)) = (1,1,0)$ is not.

Although we have just shown that the statements of Proposition 3.30 are not equivalent in general, when one considers the t-norm $\wedge$ we can prove that a function $F$ preserves $\wedge$-triangular triplets if and only if it preserves asymmetric $\wedge$-triangular triplets.

**Corollary 3.33.** *Let $F : [0,1]^I \to [0,1]$ be a function. The following statements are equivalent:*

*(1) $F$ preserves asymmetric $\wedge$-triangular triplets;*
*(2) $F$ preserves $\wedge$-triangular triplets;*
*(3) $F$ is isotone and $\wedge$-supmultiplicative ($\wedge$-closed).*

*If $|I| = 1$ then $\wedge$-supmultiplicativity can be removed.*

*Proof.* By Proposition 3.30 we only have to show that if $F$ preserves $\wedge$-triangular triplets then it is isotone. Let $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$ with $\boldsymbol{a} \preceq \boldsymbol{b}$. Then $(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{b})$ is a $\wedge$-triangular triplet so $(F(\boldsymbol{a}), F(\boldsymbol{a}), F(\boldsymbol{b}))$ so is, i. e. $F(\boldsymbol{a}) \wedge F(\boldsymbol{a}) = F(\boldsymbol{a}) \leq F(\boldsymbol{b})$ so $F$ is isotone.

If $|I| = 1$ and if $F$ is isotone then $F(a \wedge b) = F(a) \wedge F(b)$ for all $a, b \in [0,1]$. Hence $F$ is $\wedge$-supmultiplicative. $\square$

**Remark 3.34.** Although isotonicity implies $\wedge$-supmultiplicativity if $|I| = 1$, this is not true in general. To show this, consider the function $F : [0,1]^2 \to [0,1]$ given by $F(x,y) = x \cdot y$. It is clear that $F$ is isotone. Nevertheless, it is not $\wedge$-supmultiplicative. Indeed, take for example $(\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{2}) \in [0,1]^2$. Then

$$F(\tfrac{1}{2}, \tfrac{1}{3}) \wedge F(\tfrac{1}{3}, \tfrac{1}{2}) = \tfrac{1}{6} \not\leq F((\tfrac{1}{2}, \tfrac{1}{3}) \wedge (\tfrac{1}{3}, \tfrac{1}{2})) = F(\tfrac{1}{3}, \tfrac{1}{3}) = \tfrac{1}{9}.$$

Notice that the above result holds due to the fact that t-norm $\wedge$ is the only idempotent t-norm [26] (see Example 3.23 for a counterexample). Nevertheless, we can also obtain the following result for an arbitrary t-norm.

**Corollary 3.35.** *Let $F : [0,1]^I \to [0,1]$ such that $F^{-1}(1) \neq \varnothing$ and let $*$ be a t-norm. Then $F$ preserves asymmetric $*$-triangular triplets if and only if it is $*$-closed.*

*Proof.* From Proposition 3.30 we only need to show that if $F$ preserves asymmetric $*$-triangular triplets then it is isotone. Let $\boldsymbol{x} \in [0,1]^I$ such that $F(\boldsymbol{x}) = 1$. Given $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$ verifying $\boldsymbol{a} \preceq \boldsymbol{b}$ then $(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{b})$ is an asymmetric $*-$triangular triplet so $(F(\boldsymbol{a}), F(\boldsymbol{x}), F(\boldsymbol{b}))$ also is. Hence $F(\boldsymbol{a}) * F(\boldsymbol{x}) = F(\boldsymbol{a}) \leq F(\boldsymbol{b})$ and, therefore, $F$ is isotone. □

The previous result is not true if we replace "asymmetric $*$-triangular triplets" by "$*$-triangular triplets" (see Example 3.23). Furthermore, next example, which is a slight modification of Example 3.23, shows that the assumption $F^{-1}(1) \neq \varnothing$ is crucial in the previous corollary.

**Example 3.36** (cf. Example 3.23)**.** Let $*$ be a t-norm different from $\wedge$ so there exists $\alpha \in ]0,1[$ such that $\alpha * \alpha < \alpha$. Let us consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < \frac{1}{2} \\ \alpha * \alpha & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}.$$

We can easily check that $(f(a), f(b), f(c))$ is $*$-triangular for every arbitrary triplet $(a, b, c)$. Therefore $f$ preserves asymmetric $*$-triangular triplets. However, since $\alpha * \alpha < \alpha$, $f$ is not isotone.

**Remark 3.37.** For the sake of clarity, we summarize the results obtained about the preservation of $*$-transitivity. Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. Consider the following statements:

(1) $F$ preserves $*$-transitivity of fuzzy binary relations on products;
(2) $F$ preserves $*$-transitivity of fuzzy binary relations on sets;
(3) $F$ preserves asymmetric $*$-triangular triplets;
(4) $F$ preserves $*$-triangular triplets;
(5) $F$ preserves $*$-transitivity and symmetry of fuzzy binary relations on products;
(6) $F$ preserves $*$-transitivity and symmetry of fuzzy binary relations on sets;
(7) $F$ dominates $*$ ($F$ is $*$-supmultiplicative).

From Proposition 3.26 we have that $(1), (2), (3)$ are equivalent. Moreover, if $F^{-1}(1) \neq \varnothing$, by Corollary 3.35, these three statements are equivalent to "$F$ is $*$-closed". Furthermore, (3) implies (4) which in turn is equivalent to (5) and (6). Besides, (6) implies (7) by Proposition 3.30. But if $F$ is isotone, we have that all the previous statements are equivalent.

## 4. Aggregation of fuzzy quasi-metrics

Let $f : [0, +\infty) \to [0, +\infty)$ be a function. Given a metric space $(X, d)$ we could wonder when $f \circ d$ is also a metric. Although this problem goes back to a paper of Wilson in 1936, and several results about it are scattered in the literature, Doboš gathered together all of them in a booklet [44] including fundamental results due to him and his collaborators [8]. This problem can be generalized by studying those functions which merge several metrics into a single one [44]. We observe that functions which aggregate metrics have been applied to image segmentation [12].

All this theory was extended to quasi-metrics in [32], which differs from the symmetric one as we will next see. In this section, we focus our efforts on studying this question in the fuzzy context. As observed before, in [31, 46] it is given a characterization of those functions which preserve transitivity of fuzzy binary relations. Let us recall that condition (FQM3) of a fuzzy quasi-metric is near to transitivity of fuzzy binary relations. Hence, these characterizations will play an important role in our subsequent study.

We first recall some important facts about metric and quasi-metric aggregation functions on products.

**Definition 4.1** ([8, 32, 44])**.** Given a set of indices $I$, a function $f : [0,+\infty)^I \to [0,+\infty)$ is called a *(quasi-)metric aggregation function on products* if whenever $(X_i, d_i)$ is a (quasi-)metric space for all $i \in I$, then the function $d_f = f \circ \widetilde{\boldsymbol{d}}$ is a (quasi-)metric on $\prod_{i\in I} X_i$ where $\widetilde{\boldsymbol{d}} : (\prod_{i\in I} X_i)^2 \to [0,+\infty)^I$ is given by

$$(\widetilde{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}))_i = d_i(x_i, y_i)$$

for all $i \in I$, $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$.

The next result provides sufficient conditions which make that a function aggregates metrics.

**Theorem 4.2** ([44])**.** *If $f : [0,+\infty)^I \to [0,+\infty)$ is isotone, subadditive and $f^{-1}(0) = \{\mathbf{0}\}$ then it is a metric aggregation function on products.*

However, the converse of the above proposition is not true in general [44, Example 4] although it can be easily checked that every metric aggregation function on products is subadditive. The following result characterizes metric aggregation functions on products.

**Theorem 4.3** ([8])**.** *Let $f : [0,+\infty)^I \to [0,+\infty)$. Then $f$ is a metric aggregation function on products if and only if the following assertions hold:*

- $f^{-1}(0) = \{\mathbf{0}\}$;
- *$f$ preserves triangular triplets.*

The following characterization of quasi-metric aggregation functions on products was obtained in [32].

**Theorem 4.4** ([32])**.** *Let $f : [0,+\infty)^I \to [0,+\infty)$. The following statements are equivalent:*

*(1) $f$ is a quasi-metric aggregation function on products;*
*(2) $f^{-1}(0) = \{\mathbf{0}\}$ and $f$ preserves asymmetric triangular triplets;*
*(3) $f^{-1}(0) = \{\mathbf{0}\}$, $f$ is subadditive and isotone.*

**Remark 4.5.** Notice that every quasi-metric aggregation function on products is a metric aggregation function on products. Nevertheless the converse is not true in general (see Example 8 in [32]).

Now we focus our attention on the same problem but in the fuzzy context. Notice that the concept of a (quasi-)metric aggregation function on products $f : [0,+\infty)^I \to [0,+\infty)$ means that if $\{(X_i, d_i) : i \in I\}$ is a family of (quasi-)metric spaces then we can obtain a (quasi-)metric on the product $\prod_{i\in I} X_i$. We could also wonder what happens if $X_i = X$ for all $i \in I$ and we try to construct a metric on $X$ by means of $f$ (we refer the reader to [33, 38] for a fuller treatment of this case). Then, for our discussion in the fuzzy context, we will consider two different notions of fuzzy (quasi-)metric aggregation functions.

**Definition 4.6.** A function $F : [0,1]^I \to [0,1]$ is said to be:

- a *fuzzy (quasi-)metric aggregation function on products* if whenever $*$ is a t-norm and $\{(X_i, M_i, *) : i \in I\}$ is a family of fuzzy (quasi-)metric spaces then $(F \circ \widetilde{\boldsymbol{M}}, *)$ is a fuzzy (quasi-)metric on $\prod_{i\in I} X_i$ where $\widetilde{\boldsymbol{M}} : (\prod_{i\in I} X_i)^2 \times [0, +\infty) \to [0, 1]^I$ is given by
$$(\widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{y}, t))_i = M_i(x_i, y_i, t)$$
for every $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$ and $t \geq 0$.
If $F$ only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-fuzzy (quasi-)metric aggregation function on products.
- a *fuzzy (quasi-)metric aggregation function on sets* if whenever $*$ is a t-norm and $\{(M_i, *) : i \in I\}$ is a family of fuzzy (quasi-)metrics on the same set $X$ then $(F \circ \boldsymbol{M}, *)$ is a fuzzy (quasi-)metric on $X$ where $\boldsymbol{M} : X^2 \times [0, +\infty) \to [0, 1]^I$ is given by
$$(\boldsymbol{M}(x, y, t))_i = M_i(x, y, t)$$
for every $x, y \in X$ and $t \geq 0$.
If $F$ only satisfies the above condition for a fixed t-norm $*$ then it is said to be an $*$-fuzzy (quasi-)metric aggregation function on sets.

**Remark 4.7.** It is clear that if $F : [0, 1]^I \to [0, 1]$ is a fuzzy (quasi-)metric aggregation function on products then it is a fuzzy (quasi-)metric aggregation function on sets. Moreover, if $|I| = 1$ then the two concepts coincide. In this case, if $(X, M, *)$ is a fuzzy (quasi-)metric space then $(F \circ M, *)$ is a fuzzy (quasi-)metric on $X$ and we will say that $F$ is a *fuzzy (quasi-)metric preserving function.*

Nevertheless, in general, if $|I| > 1$ then the concepts of fuzzy (quasi-)metric aggregation function on products and fuzzy (quasi-)metric aggregation function on sets are different as the next example shows.

**Example 4.8.**

- if $P_i : [0, 1]^I \to [0, 1]$ denotes the $i$-th projection as in Example 3.9, then $P_i$ is a fuzzy (quasi-)metric aggregation function on sets but not on products (it is immediate to check that $F \circ \widetilde{\boldsymbol{M}}$ does not verify (FQM2') for every family of fuzzy metrics).
- if $*$ is a t-norm and $F_* : [0, 1]^n \to [0, 1]$ is given by $F_*(a_1, \ldots, a_n) = a_1 * \ldots * a_n$ then $F_*$ is a fuzzy (quasi-)metric aggregation function on products;
- The function $f : [0, 1] \to [0, 1]$ given by
$$f(x) = \begin{cases} \max\{k, x\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
where $k \in (0, 1)$ is a fuzzy metric preserving function (Proposition 3.2 in [47]).
- The function $f : [0, 1] \to [0, 1]$ given by
$$f(x) = \begin{cases} \dfrac{k + x}{k + 1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$
where $k > 0$, is an $*_P$-fuzzy metric preserving function (cf. Example 15 in [23] and Proposition 3.3 in [47]).

Notice that one of the requirements of the characterization of a (quasi-)metric aggregation function $f$ is $f^{-1}(0) = \{\mathbf{0}\}$. As we will see, in the fuzzy context, this requirement will be replaced by certain conditions on the core, that we next define.

**Definition 4.9.** Let $I$ be a set of indices. The *core* of a function $F : [0,1]^I \to [0,1]$ is the set $F^{-1}(1)$.

Observe that the core has played an important role in the preservation of the property of separating points under an aggregation process of fuzzy relations (see Proposition 3.28).

From now on, for a given $\boldsymbol{x} \in [0,1]^I$, we will set $I_{\boldsymbol{x}} := \{i \in I : \boldsymbol{x}_i = 1\}$.

We next introduce several conditions on the core that will be crucial in our study.

**Definition 4.10.** Let $F : [0,1]^I \to [0,1]$ be a function such that $F^{-1}(1) \neq \varnothing$. Then we say that:

- $F$ has *trivial core* if $F^{-1}(1) = \{\mathbf{1}\}$.
- The *core of F is included in a unitary face* if there exists $i \in I$ such that $F^{-1}(1) \subseteq \{\boldsymbol{x} \in [0,1]^I : \boldsymbol{x}_i = 1\}$, that is $\bigcap_{\boldsymbol{x} \in F^{-1}(1)} I_{\boldsymbol{x}} \neq \varnothing$.
- The *core of F is countably included in a unitary face* if for a given $\{\boldsymbol{x}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ there exists $i \in I$ such that $(\boldsymbol{x}_n)_i = 1$ for all $n \in \mathbb{N}$, that is $\bigcap_{n \in \mathbb{N}} I_{\boldsymbol{x}_n} \neq \varnothing$.
- The *core of F is finitely included in a unitary face* if for a given $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq F^{-1}(1)$ there exists $i \in I$ such that $(\boldsymbol{x}_m)_i = 1$ for all $m \in \{1, \ldots, n\}$, that is $\bigcap_{m=1}^{n} I_{\boldsymbol{x}_m} \neq \varnothing$.
- The *core of F is pairwise included in a unitary face* if for any $\boldsymbol{x}, \boldsymbol{y} \in F^{-1}(1)$ there exists $i \in I$ such that $\boldsymbol{x}_i = \boldsymbol{y}_i = 1$, that is $I_{\boldsymbol{x}} \cap I_{\boldsymbol{y}} \neq \varnothing$.
- The *core of F is included in the unitary boundary* if $F^{-1}(1) \subseteq \bigcup_{i \in I} \{\boldsymbol{x} \in [0,1]^I : \boldsymbol{x}_i = 1\}$, that is $I_{\boldsymbol{x}} \neq \varnothing$ for every $\boldsymbol{x} \in F^{-1}(1)$.

Obviously, every of the above concepts imply its successor and if $|I| = 1$ all of them are equal. Nevertheless, if $|I| > 1$ we can find examples showing that all these notions are different.

**Example 4.11.**

- *A function whose core is included in a unitary face but it is not trivial.*
  Let us consider $P_i : [0,1]^I \to [0,1]$, the $i$-th projection where $i \in I$ and $|I| > 1$. Then $P_i^{-1}(1) = \{\boldsymbol{x} \in [0,1]^I : \boldsymbol{x}_i = 1\} \neq \{\mathbf{1}\}$.
- *A function whose core is countably included in a unitary face but it is not included in a unitary face.*
  Let $I$ with $|I| > \aleph_0$. Let $F : [0,1]^I \to [0,1]$ given by
  $$F(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\{i \in I : \boldsymbol{x}_i \neq 1\}| \leq \aleph_0 \\ 0 & \text{otherwise} \end{cases}.$$
  Let us show that the core of $F$ is countably included in a unitary face, i. e. if $\{\boldsymbol{x}_n : n \in \mathbf{N}\} \subseteq F^{-1}(1)$ then $\bigcap_{n \in \mathbb{N}} I_{\boldsymbol{x}_n} \neq \varnothing$. Otherwise $I = I \backslash \bigcap_{n \in \mathbb{N}} I_{\boldsymbol{x}_n} = \bigcup_{n \in \mathbb{N}} I \backslash I_{\boldsymbol{x}_n}$ but since $F(\boldsymbol{x}_n) = 1$ then $|I \backslash I_{\boldsymbol{x}_n}| \leq \aleph_0$ for all $n \in \mathbb{N}$ so $|I| = |\bigcup_{n \in \mathbb{N}} I \backslash I_{\boldsymbol{x}_n}| \leq \aleph_0$ which is a contradiction since $|I| > \aleph_0$.
  We next check that the core of $F$ is not included in a unitary face. Given $i \in I$ let us consider $\boldsymbol{a}_i \in [0,1]^I$ such that $(\boldsymbol{a}_i)_j = 0$ if $j = i$ and $(\boldsymbol{a}_i)_j = 1$ if $i \neq j$ for all $j \in I$. Then $\{\boldsymbol{a}_i : i \in I\} \subseteq F^{-1}(1)$ and $\bigcap_{i \in I} I_{\boldsymbol{a}_i} = \varnothing$ so $\bigcap_{\boldsymbol{x} \in F^{-1}(1)} I_{\boldsymbol{x}} = \varnothing$.

- *A function whose core is finitely included in a unitary face but it is not countably included in a unitary face.*
  We only have to take $I$ with $|I| = \aleph_0$ and define $F : [0,1]^I \to [0,1]$ as
  $$F(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\{i \in I : \boldsymbol{x}_i \neq 1\}| \text{ is finite} \\ 0 & \text{otherwise} \end{cases}.$$
  Arguing similarly as in the previous example, it can be proved that the core of $F$ is finitely included in a unitary face but it is not countably included in a unitary face.
- *A function whose core is pairwise included in a unitary face but it is not finitely included in a unitary face.*
  Let us consider $F : [0,1]^3 \to [0,1]$ given by
  $$F(x,y,z) = \begin{cases} 1 & \text{if } x = y = 1 \text{ or } y = z = 1 \text{ or } x = z = 1 \\ 0 & \text{otherwise} \end{cases}.$$
  It is obvious that the core of $F$ is pairwise included in a unitary face. Nevertheless, $I_{(1,1,0)} \cap I_{(1,0,1)} \cap I_{(0,1,1)} = \varnothing$ so the core is not finitely included in a unitary face.
- *A function whose core is included in the unitary boundary but it is not pairwise included in a unitary face.*
  Let us consider $F : [0,1]^2 \to [0,1]$ given by
  $$F(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise} \end{cases}.$$
  Obviously the core of $F$ is included in the unitary boundary but, for instance, $I_{(1,0)} \cap I_{(0,1)} = \varnothing$ so the core is not pairwise included in a unitary face.

However, if we restrict the cardinality of $I$ or we impose conditions on the function $F$, some of the above definitions coincide.

**Proposition 4.12.** *Let $I$ be a set of indices and $F : [0,1]^I \to [0,1]$ be a function.*

*(1) If $|I|$ is countable, then the core of $F$ is included in a unitary face if and only if it is countably included in a unitary face.*
*(2) If $|I|$ is finite, then the core of $F$ is countably included in a unitary face if and only if it is finitely included in a unitary face (which is equivalent to that the core is included in a unitary face).*
*(3) If $|I| = 1$ then all the concepts of Definition 4.10 coincide for $F$.*
*(4) If $F$ is $*$-supmultiplicative then the core of $F$ is finitely included in a unitary face if and only if the core of $F$ is included in the unitary boundary.*

*Proof.*

(1) We only prove the sufficiency since necessity is obvious. Suppose that we cannot find $j \in I$ with $F^{-1}(1) \subseteq \{\boldsymbol{x} \in [0,1]^I : \boldsymbol{x}_j = 1\}$. Then for each $j \in I$ there exists $\boldsymbol{x}^j \in F^{-1}(1)$ such that $(\boldsymbol{x}^j)_j \neq 1$. Hence $\{\boldsymbol{x}^j : j \in I\} \subseteq F^{-1}(1)$ is a countable set but for every $i \in I$ we have that $(\boldsymbol{x}^i)_i \neq 1$ which contradicts our assumption.

(2) As above we only prove the sufficiency. Let $\{\boldsymbol{x}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$. Since the core of $F$ is finitely included in a unitary face then $I_n := \bigcap_{m=1}^{n} I_{\boldsymbol{x}_m} \neq \varnothing$

for all $n \in \mathbb{N}$. Then $I_n$ is nonempty, finite and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Hence $\bigcap_{n\in\mathbb{N}} I_n = \bigcap_{n\in\mathbb{N}} I_{\boldsymbol{x}_n} \neq \varnothing$ so the core of $F$ is countably included in a unitary face.

(3) This is trivial.

(4) We only have to prove the sufficiency. Suppose that the core of $F$ is included in the unitary boundary and let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq F^{-1}(1)$. Since $F$ is $*$-supmultiplicative then $F(\boldsymbol{x}_1 *^I \ldots *^I \boldsymbol{x}_n) \geq F(\boldsymbol{x}_1) * \ldots * F(\boldsymbol{x}_n) = 1 * \ldots * 1 = 1$ so $\boldsymbol{x}_1 *^I \ldots *^I \boldsymbol{x}_n \in F^{-1}(1)$. By assumption $I_{\boldsymbol{x}_1 *^I \ldots *^I \boldsymbol{x}_n} \neq \varnothing$. Since $I_{\boldsymbol{x}_1 *^I \ldots *^I \boldsymbol{x}_n} = \bigcap_{m=1}^{n} I_{\boldsymbol{x}_m}$ we conclude that the core of $F$ is finitely included in a unitary face.

□

**Remark 4.13.** Observe that from the proof of (4) we can also deduce that if $F$ is $*$-supmultiplicative then the core of $F$ is an abelian semigroup with the operation $*$.

**Definition 4.14** ([21, 35])**.** Let $I$ be a set of indices. A function $F : [0,1]^I \to [0,1]$ is said to be *left-continuous* if whenever $(\boldsymbol{x}_n)_n$ is nondecreasing sequence in $[0,1]^I$ converging to $\boldsymbol{x} \in [0,1]^I$ in the product topology then $(F(\boldsymbol{x}_n))_n$ converges to $F(\boldsymbol{x})$.

We next characterize the fuzzy quasi-metric aggregation functions on products. Surprisingly they are the same than the fuzzy metric aggregation functions on products and, in addition, these functions turn to be aggregation functions, i. e. isotone functions verifying that $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$ (see Definition 3.4).

**Theorem 4.15.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy quasi-metric aggregation function on products;*
*(2) $F$ is a ($*$-)fuzzy metric aggregation function on products;*
*(3) $F$ is an aggregation function, ($*$-)supmultiplicative, left-continuous and $F$ has trivial core;*
*(4) $F(\mathbf{0}) = 0$, $F$ is left-continuous with trivial core and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* (1)⇒ (2). This is obvious.

(2)⇒ (3). We first check that $F$ is an aggregation function. We begin proving that $F$ is isotone. Let $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$. Let us consider $X = \{c, d\}$ a set with two different elements and for every $i \in I$, let us define a fuzzy metric $(N_i, *)$ on $X$ as

$$N_i(x,y,t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } x = y \text{ and } t > 0 \\ 0 & \text{if } x \neq y \text{ and } 0 < t \leq 1 \\ \boldsymbol{a}_i & \text{if } x \neq y \text{ and } 1 < t \leq 2 \\ \boldsymbol{b}_i & \text{if } x \neq y \text{ and } 2 < t \end{cases}.$$

Then $\{(X, N_i, *) : i \in I\}$ is a family of fuzzy metric spaces so $(F \circ \widetilde{\boldsymbol{N}}, *)$ is a fuzzy metric on $X^I$. Consequently, if we fix $\boldsymbol{x}, \boldsymbol{y} \in X^I$ then $F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{x}, \boldsymbol{y}, \cdot)$ is an increasing function. In particular, if we choose $\boldsymbol{c}, \boldsymbol{d} \in X^I$ such that $\boldsymbol{c}_i = c$ and

$\boldsymbol{d}_i = d$ for every $i \in I$, we have that

$$\begin{aligned} F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{c}, \boldsymbol{d}, 2) &\leq F \circ \widetilde{\boldsymbol{N}}(\boldsymbol{c}, \boldsymbol{d}, 3) \\ F((N_i(\boldsymbol{c}_i, \boldsymbol{d}_i, 2))_{i\in I}) &\leq F((N_i(\boldsymbol{c}_i, \boldsymbol{d}_i, 3))_{i\in I}) \\ F(\boldsymbol{a}) &\leq F(\boldsymbol{b}). \end{aligned}$$

Therefore, $F$ is isotone.

We check now that $F(\mathbf{0}) = 0$ and $F$ has trivial core, i. e. $F^{-1}(1) = \{\mathbf{1}\}$. Let $(X, M, *)$ be an arbitrary fuzzy metric space, $x \in X$ and $t > 0$. Considering the family of fuzzy metric spaces $\{(X_i, M_i, *) : i \in I\}$ where $(X_i, M_i, *) = (X, M, *)$ for all $i \in I$, we have that $(F \circ \widetilde{\boldsymbol{M}}, *)$ is a fuzzy metric on $X^I$ so given $\boldsymbol{x} \in X^I$ with $\boldsymbol{x}_i = x$ for all $i \in I$ and $t > 0$ then $1 = F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{x}, t) = F((M(x, x, t))_{i\in I}) = F(\mathbf{1})$. Furthermore, $0 = F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{x}, 0) = F(\mathbf{0})$. Suppose that we can find $\boldsymbol{a} \in [0, 1]^I$ such that $F(\boldsymbol{a}) = 1$ but $\boldsymbol{a} \neq \mathbf{1}$. Consider the family of fuzzy metric spaces $\{(X_i, M_i, *) : i \in I\}$ where $X_i = [0, 1]$ and $(M_i, *) = (M^*, *)$ is the fuzzy metric of Example 2.14. Then $(F \circ \widetilde{\boldsymbol{M}^*}, *)$ is a fuzzy metric on $[0, 1]^I$, but given $t > 0$ we have that $F \circ \widetilde{\boldsymbol{M}^*}(\mathbf{1}, \boldsymbol{a}, t) = F((M^*(1, a_i, t))_{i\in I}) = F(\boldsymbol{a}) = 1$. Therefore $\boldsymbol{a} = \mathbf{1}$ by (FQM2'). This contradicts our assumption, so $F$ is an aggregation function with trivial core.

We next prove that $F$ is left-continuous. Let us consider a sequence $(\boldsymbol{t}_n)_{n\in\mathbb{N}}$ in $[0, 1]^I$ such that $\boldsymbol{t}_n \preceq \boldsymbol{t}_{n+1}$ for all $n \in \mathbb{N}$. We also denote by $\boldsymbol{s}$ the supremum of this sequence. Then $(t_{n,i})_{n\in\mathbb{N}} := ((\boldsymbol{t}_n)_i)_{n\in\mathbb{N}}$ converges to $\boldsymbol{s}_i$ for all $i \in I$. Let $X = \{a, b\}$ be a set with two different elements. Then given $i \in I$, define $M_i : X \times X \times [0, +\infty) \to [0, 1]$ as

$$M_i(x, y, t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } x = y,\ t > 0 \\ 0 & \text{if } x \neq y,\ 0 < t \leq \frac{1}{2} \\ t_{n,i} & \text{if } x \neq y, 1 - \frac{1}{n+1} < t \leq 1 - \frac{1}{n+2} \\ \boldsymbol{s}_i & \text{if } x \neq y, t \geq 1 \end{cases}.$$

It is easy to check that $(X, M_i, *)$ is a fuzzy metric space for all $i \in I$. Then $(F \circ \widetilde{\boldsymbol{M}}, *)$ is a fuzzy metric on $X^I$ associated with the family of fuzzy metric spaces $\{(X, M_i, *) : i \in I\}$. Consequently $F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{b}, \cdot)$ is left-continuous where $\boldsymbol{a}_i = a$ and $\boldsymbol{b}_i = b$ for all $i \in I$. Hence $(F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{b}, 1 - \frac{1}{n+2}))_{n\in\mathbb{N}}$ converges to $F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{b}, 1)$, but

$$F \circ \widetilde{\boldsymbol{M}} \left( \boldsymbol{a}, \boldsymbol{b}, 1 - \frac{1}{n+2} \right) = F \left( \left( M_i \left( a, b, 1 - \frac{1}{n+2} \right) \right)_{i\in I} \right) = F((t_{n,i})_{i\in I}) = F(\boldsymbol{t}_n)$$

for every $n \in \mathbb{N}$ and

$$F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{b}, 1) = F((M_i(a, b, 1))_{i\in I}) = F((\boldsymbol{s}_i)_{i\in I}) = F(\boldsymbol{s}).$$

Consequently $F$ is left-continuous.

Let us check that $F$ is $*$-supmultiplicative. Let $\boldsymbol{a}, \boldsymbol{b} \in [0, 1]^I$. Define $I_1 = \{i \in I : \boldsymbol{a}_i \neq 1, \boldsymbol{b}_i \neq 1\}$, $I_2 = \{i \in I : \boldsymbol{a}_i \neq 1, \boldsymbol{b}_i = 1\}$, $I_3 = \{i \in I : \boldsymbol{a}_i = 1, \boldsymbol{b}_i \neq 1\}$, $I_4 = \{i \in I : \boldsymbol{a}_i = \boldsymbol{b}_i = 1\}$. Following a similar idea to the proof of Theorem 9 in [46], let us consider a fixed set $X = \{x_1, x_2, x_3\}$ with three different elements and a fuzzy metric $(M_i, *)$ on $X$ given by

- if $i \in I_1$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t=0 \\ 1 & \text{if } x=y,\ t>0 \\ \boldsymbol{a}_i & \text{if } x=x_1,\ y=x_2,\ t>0 \\ \boldsymbol{b}_i & \text{if } x=x_2,\ y=x_3,\ t>0 \\ \boldsymbol{a}_i * \boldsymbol{b}_i & \text{if } x=x_1,\ y=x_3,\ t>0 \end{cases};$$

- if $i \in I_2$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t=0 \\ 1 & \text{if } x=y,\ t>0 \\ \boldsymbol{a}_i & \text{if } x=x_1,\ y=x_2,\ t>0 \\ \boldsymbol{a}_i * \boldsymbol{a}_i & \text{if } x=x_2,\ y=x_3,\ 0<t\leq 1 \\ 1 & \text{if } x=x_2,\ y=x_3,\ t>1 \\ \boldsymbol{a}_i & \text{if } x=x_1,\ y=x_3,\ t>0 \end{cases};$$

- if $i \in I_3$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t=0 \\ 1 & \text{if } x=y,\ t>0 \\ \boldsymbol{b}_i * \boldsymbol{b}_i & \text{if } x=x_1,\ y=x_2,\ 0<t\leq 1 \\ 1 & \text{if } x=x_1,\ y=x_2,\ t>1 \\ \boldsymbol{b}_i & \text{if } x=x_2,\ y=x_3,\ t>0 \\ \boldsymbol{b}_i & \text{if } x=x_1,\ y=x_3,\ t>0 \end{cases};$$

- if $i \in I_4$ then

$$M_i(x,y,t) = M_i(y,x,t) = \begin{cases} 0 & \text{if } t=0 \\ 0 & \text{if } x\neq y,\ 0<t\leq 1 \\ 1 & \text{if } x\neq y,\ t>1 \\ 1 & \text{if } x=y,\ t>0 \end{cases}.$$

Then $\{(X, M_i, *) : i \in I\}$ is a family of fuzzy metric spaces. Hence $F \circ \widetilde{\boldsymbol{M}}$ is a fuzzy metric on $X^I$. Let us define $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in X^I$ such that $(\boldsymbol{x}_i)_j = x_i$ for all $j \in I$ and $i = 1, 2, 3$. Then we have that

$$\begin{aligned} (F \circ \widetilde{\boldsymbol{M}})(\boldsymbol{x}_1, \boldsymbol{x}_2, 2) * (F \circ \widetilde{\boldsymbol{M}})(\boldsymbol{x}_2, \boldsymbol{x}_3, 2) &\leq F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}_1, \boldsymbol{x}_3, 4) \\ F((M_i(x_1, x_2, 2))_{i\in I}) * F((M_i(x_2, x_3, 2))_{i\in I}) &\leq F((M_i(x_1, x_3, 4))_{i\in I}) \\ F(\boldsymbol{a}) * F(\boldsymbol{b}) &\leq F(\boldsymbol{a} *^I \boldsymbol{b}) \end{aligned}$$

and, thus, $F$ is $*$-supmultiplicative.

(3) $\Rightarrow$ (4) This is a direct consequence of Corollary 3.35.

(4) $\Rightarrow$ (1) Let $\{(X_i, M_i, *) : i \in I\}$ be a family of fuzzy quasi-metric spaces. Let us check that $(F \circ \widetilde{\boldsymbol{M}}, *)$ is a fuzzy quasi-metric on $\prod_{i\in I} X_i$.

Since $F$ is an aggregation function then $F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{y}, 0) = F((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, 0))_{i\in I}) = F(\boldsymbol{0}) = 0$ for every $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$ so (FQM1) holds. Moreover, if $t > 0$ then $F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{x}, t) = F((M_i(\boldsymbol{x}_i, \boldsymbol{x}_i, t))_{i\in I}) = F(\boldsymbol{1}) = 1$ for all $\boldsymbol{x} \in \prod_{i\in I} X_i$. On the other hand, given $\boldsymbol{x}, \boldsymbol{y} \in \prod_{i\in I} X_i$, if $F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{y}, t) = F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{y}, \boldsymbol{x}, t) = 1$ for all

$t > 0$ then $F((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t))_{i\in I}) = F((M_i(\boldsymbol{y}_i, \boldsymbol{x}_i, t))_{i\in I}) = 1$ for all $t > 0$. Since $F$ has trivial core we have that for every $i \in I$, $M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t) = M_i(\boldsymbol{y}_i, \boldsymbol{x}_i, t) = 1$ for all $t > 0$ so $\boldsymbol{x}_i = \boldsymbol{y}_i$ for all $i \in I$, i. e. $\boldsymbol{x} = \boldsymbol{y}$. Thus (FQM2') holds.

Moreover, given $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \prod_{i\in I} X_i$ and $t, s > 0$ we have that $M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t) * M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s) \leq M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s)$ for all $i \in I$, i. e. $((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t))_{i\in I}, (M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s))_{i\in I}, (M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s))_{i\in I})$ is an asymmetric $*$-triangular triplet. Then, by assumption, $(F((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t))_{i\in I}), F((M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s))_{i\in I}), F((M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s))_{i\in I}))$ is an asymmetric $*$-triangular triplet so

$$F((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t))_{i\in I}) * F((M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s))_{i\in I}) \leq F((M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s))_{i\in I})$$

and, thus, (FQM3) holds for $F \circ \widetilde{\boldsymbol{M}}$.

Finally, (FQM4) is clear since $F$ is left-continuous and $M_i(x, y, \cdot)$ is left-continuous and isotone for every $x, y \in X$ and every $i \in I$. □

**Remark 4.16.** The above result shows that, in contrast with the crisp case, fuzzy metric aggregation functions on products coincides with fuzzy quasi-metric aggregation functions on products.

We also observe that aggregation functions with trivial core are called in [34] aggregation functions which have no unit multipliers.

**Example 4.17.**

- Given a set of indices $I$, the function $\mathsf{Inf} : [0,1]^I \to [0,1]$ given by $\mathsf{Inf}(\boldsymbol{x}) = \inf_{i\in I} \boldsymbol{x}_i$ is a left-continuous ($*$-)supmultiplicative aggregation function with trivial core for every t-norm $*$. Then it is a fuzzy (quasi-)metric aggregation function on products.
- Given a t-norm $*$ and $n \in \mathbb{N}$, the function $F_* : [0,1]^n \to [0,1]$ given by $F_*(a_1, \ldots, a_n) = a_1 * \ldots * a_n$ is a left-continuous ($*$-)supmultiplicative aggregation function with trivial core. So its is an $*$-fuzzy (quasi-)metric aggregation function on products.
- The function $F : [0,1] \to [0,1]$ given by $F(x) = \sqrt{x}$ is a continuous $*_P$-supmultiplicative aggregation function with trivial core. Consequently, it is an $*_P$-fuzzy (quasi-)metric preserving function.
- The function $F : [0,1] \to [0,1]$ given by $F(x) = x^2$ is a continuous $*_{\mathrm{L}}$-supmultiplicative aggregation function with trivial core. Hence it is an $*_{\mathrm{L}}$-fuzzy (quasi-)metric preserving function.

The following result shows that there is a huge quantity of functions which are $\wedge$-fuzzy (quasi-)metric preserving.

**Corollary 4.18.** *Let $F : [0,1] \to [0,1]$ be a function. Then $F$ is a $\wedge-$fuzzy (quasi-)metric preserving function if and only if $F$ is an aggregation function, left-continuous and $F$ has trivial core.*

*Proof.* This follows from Theorem 4.15 and Corollary 3.33. □

Next we give a characterization of fuzzy (quasi-)metric aggregation functions on sets.

**Theorem 4.19.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy quasi-metric aggregation function on sets;*
*(2) $F$ is a ($*$-)fuzzy metric aggregation function on sets;*

*(3) $F$ is an aggregation function, ($*$-)supmultiplicative, left-continuous and the core of $F$ is countably included in a unitary face;*

*(4) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, $F$ is left-continuous, the core of $F$ is countably included in a unitary face and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (3). Everything follows with a simple adaptation of the proof of Theorem 4.15 except for proving that the core of $F$ is countably included in a unitary face. Assume, for the sake of contradiction, that we can find a sequence $\{\boldsymbol{x}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$ such that for any $i \in I$ we could find $n_i \in \mathbb{N}$ such that $(\boldsymbol{x}_{n_i})_i \neq 1$. Let us consider a set $X = \{a, b\}$ with two different elements and for each $i \in I$ define $M_i : X \times X \times [0, +\infty) \to [0, 1]$ as

$$M_i(x,y,t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } x = y, t > 0 \\ (\boldsymbol{x}_1)_i & \text{if } x \neq y, t > 1 \\ (\boldsymbol{x}_1)_i * \ldots * (\boldsymbol{x}_{n+1})_i & \text{if } x \neq y, \frac{1}{n+1} < t \leq \frac{1}{n} \end{cases}.$$

Notice that $(X, M_i, *)$ is a fuzzy metric space for all $i \in I$. We only show that (FQM2') holds since the other conditions follow trivially. If $M_i(x, y, t) = 1$ for all $t > 0$ then $x = y$. Otherwise, we can suppose that $x = a$ and $y = b$ so $M_i(a, b, t) = 1$ for all $t > 0$. Nevertheless, this is impossible because $M_i(a, b, \frac{1}{n_i}) = (\boldsymbol{x}_1)_i * \ldots * (\boldsymbol{x}_{n_i+1})_i \neq 1$ since $(\boldsymbol{x}_{n_i})_i \neq 1$ (recall that for any t-norm, $a * b = 1$ if and only if $a = b = 1$).

By assumption, $(F \circ \boldsymbol{M}, *)$ is a fuzzy metric on $X$. However, given $t > 0$ and since $F$ is ($*$-)supmultiplicative we have that

$$\begin{aligned} F \circ \boldsymbol{M}(a,b,t) &= F((M_i(a,b,t)_{i\in I})) = F(\boldsymbol{x}_1 *^I \ldots *^I \boldsymbol{x}_{n+1}) \geq F(\boldsymbol{x}_1) * \ldots * F(\boldsymbol{x}_{n+1}) \\ &= 1 * \ldots * 1 = 1 \end{aligned}$$

for some $n \in \mathbb{N} \cup \{0\}$. Since $a \neq b$ this contradicts (FQM2').

(3) $\Rightarrow$ (4). This is a direct consequence of Corollary 3.35.

(4) $\Rightarrow$ (1). This is similar to (4)$\Rightarrow$ (1) of Theorem 4.15 and the only difference is with proving condition (FQM2'). Let $\{(X, M_i, *) : i \in I\}$ be a family of fuzzy quasi-metric spaces. Let $x, y \in X$ such that $F \circ \boldsymbol{M}(x, y, t) = F \circ \boldsymbol{M}(y, x, t) = 1$ for all $t > 0$. By Corollary 3.35 $F$ is $*$-supmultiplicative. Hence we have that $1 = 1 * 1 = F(\boldsymbol{M}(x, y, t)) * F(\boldsymbol{M}(y, x, t)) \leq F(\boldsymbol{M}(x, y, t) *^I \boldsymbol{M}(y, x, t))$ for all $t > 0$, so $F((M_i(x, y, t))_{i\in I} * (M_i(y, x, t))_{i\in I}) = 1$. Define $\boldsymbol{x}_n = (M_i(x, y, \frac{1}{n}))_{i\in I} *^I (M_i(y, x, \frac{1}{n}))_{i\in I}$. Then $\{\boldsymbol{x}_n : n \in \mathbb{N}\} \subseteq F^{-1}(1)$. By assumption there exists $i \in I$ such that $(\boldsymbol{x}_n)_i = 1$ for all $n \in \mathbb{N}$, i. e. $M_i(x, y, \frac{1}{n}) * M_i(x, y, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Hence $M_i(x, y, \frac{1}{n}) = M_i(x, y, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Since $M_i(x, y, \cdot)$ and $M_i(y, x, \cdot)$ are increasing we deduce that $M_i(x, y, t) = M_i(y, x, t) = 1$ for all $t > 0$. Consequently, $x = y$ since $(M_i, *)$ is a fuzzy quasi-metric. $\square$

From the previous theorems we can obtain the following result which can be easily proved independently (see Remark 4.7).

**Corollary 4.20.** *Let $F : [0, 1]^I \to [0, 1]$ be a function and $*$ be a t-norm. If $F$ is a ($*$-)fuzzy (quasi-)metric aggregation function on products then it is a ($*$-)fuzzy (quasi-)metric aggregation function on sets.*

*Moreover, if $|I| = 1$ then the converse is also true.*

As already observed, the converse of the previous corollary is not true in general (see Example 4.8).

In case that we treat with ($*$-)fuzzy (quasi-)pseudometric aggregation functions, there is no difference if they are considered on products or on sets as the next result shows.

**Theorem 4.21.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy quasi-pseudometric aggregation function on products;*
*(2) $F$ is a ($*$-)fuzzy pseudometric aggregation function on products;*
*(3) $F$ is a ($*$-)fuzzy quasi-pseudometric aggregation function on sets;*
*(4) $F$ is a ($*$-)fuzzy pseudometric aggregation function on sets;*
*(5) $F$ is an aggregation function, ($*$-)supmultiplicative and left-continuous;*
*(6) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, $F$ is left-continuous and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* (1)$\Rightarrow$ (2), (1)$\Rightarrow$ (3), (2)$\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are obvious. (4) $\Rightarrow$ (5) is similar to the proof of (2) $\Rightarrow$ (3) in Theorem 4.19 but in this case we cannot prove that the core of $F$ is countably included in a unitary face. (5) $\Rightarrow$ (6) is a consequence of Corollary 3.35. (6) $\Rightarrow$ (1) follows as implication (4)$\Rightarrow$(1) of Theorem 4.15 but since in this case we only need to prove (FQM2), it is not necessary that $F$ has trivial core. □

Next we discuss the case in which $F$ only aggregates stationary fuzzy (quasi-)metrics, in the sense of Definition 4.6.

**Theorem 4.22.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)stationary fuzzy quasi-metric aggregation function on products;*
*(2) $F$ is an aggregation function, ($*$-)supmultiplicative and $F$ has trivial core;*
*(3) $F(\mathbf{0}) = 0$, $F$ has trivial core and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* (1) $\Rightarrow$ (2). That $F(\mathbf{0}) = 0$ and $F^{-1}(1) = \{\mathbf{1}\}$ follows as in Theorem 4.15.

Let $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$. Given a t-norm $*$, let us consider the family of stationary fuzzy quasi-metric spaces $\{([0,1], M_i, *) : i \in I\}$ where $(M_i, *) = (M^{\overset{*}{\to}}, *)$ is the stationary fuzzy quasi-metric of Example 2.13. Then $(F \circ \widetilde{\boldsymbol{M}}, *)$ is also a fuzzy quasi-metric on $[0,1]^I$ so for any $t, s > 0$ we have that

$$F \circ \widetilde{\boldsymbol{M}}(\mathbf{1}, \boldsymbol{a}, t) * F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{b}, s) \leq F \circ \widetilde{\boldsymbol{M}}(\mathbf{1}, \boldsymbol{b}, t+s)$$
$$F((M^{\overset{*}{\to}}(1, \boldsymbol{a}_i, t))_{i\in I}) * F((M^{\overset{*}{\to}}(\boldsymbol{a}_i, \boldsymbol{b}_i, s))_{i\in I}) \leq F((M^{\overset{*}{\to}}(1, \boldsymbol{b}_i, t+s))_{i\in I})$$
$$F((1 \overset{*}{\to} \boldsymbol{a}_i)_{i\in I}) * F((\boldsymbol{a}_i \overset{*}{\to} \boldsymbol{b}_i)_{i\in I}) \leq F((1 \overset{*}{\to} \boldsymbol{b}_i)_{i\in I})$$
$$F(\boldsymbol{a}) = F(\boldsymbol{a}) * F(\mathbf{1}) \leq F(\boldsymbol{b}).$$

Whence we obtain that $F$ is isotone.

Next we show that $F$ is ($*$-)supmultiplicative. If $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^I$ are arbitrary, then

$$F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{1}, \boldsymbol{a}, t) * F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{a}, \boldsymbol{a} *^I \boldsymbol{b}, s) \leq F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{1}, \boldsymbol{a} *^I \boldsymbol{b}, t+s)$$
$$F((1 \xrightarrow{*} \boldsymbol{a}_i)_{i \in I}) * F((\boldsymbol{a}_i \xrightarrow{*} \boldsymbol{a}_i * \boldsymbol{b}_i)_{i \in I}) \leq F((1 \xrightarrow{*} \boldsymbol{a}_i * \boldsymbol{b}_i)_{i \in I})$$
$$F(\boldsymbol{a}) * F((\boldsymbol{a}_i \xrightarrow{*} \boldsymbol{a}_i * \boldsymbol{b}_i)_{i \in I}) \leq F(\boldsymbol{a} *^I \boldsymbol{b})$$
$$F(\boldsymbol{a}) * F(\boldsymbol{b}) \leq F(\boldsymbol{a}) * F((\boldsymbol{a}_i \xrightarrow{*} \boldsymbol{a}_i * \boldsymbol{b}_i)_{i \in I}) \leq F(\boldsymbol{a} *^I \boldsymbol{b})$$

where in the last implication we have used that $F$ is isotone and $\boldsymbol{b}_i \leq (\boldsymbol{a}_i \to \boldsymbol{a}_i * \boldsymbol{b}_i)$ for all $i \in I$. Hence $F$ is ($*$-)supmultiplicative.

(2) $\Rightarrow$ (3). This is a direct consequence of Corollary 3.35.

(3) $\Rightarrow$ (1). This is similar to the proof provided in Theorem 4.15 for (4) $\Rightarrow$ (1), except that in this case left-continuity is trivial. $\square$

It seems natural to wonder what is the relationship between ($*$-)stationary fuzzy quasi-metric aggregation function on products and ($*$-)fuzzy quasi-metric aggregation function on products. In view of Theorems 4.19 and 4.22, it is clear that every ($*$-)fuzzy quasi-metric aggregation function on products is always a ($*$-)stationary fuzzy quasi-metric aggregation function on products. Next example brings to light that the converse is not true.

**Example 4.23.** Let us consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

It is obvious that $f$ is an aggregation function with trivial core and $f$ preserves asymmetric triangular triplets. Consequently, by Theorem 4.22, $f$ is a stationary fuzzy quasi-metric aggregation function on products. Nevertheless, it is not left-continuous. Indeed the sequence $(1 - \frac{1}{n})_{n \in \mathbb{N}}$ converges to 1 but $(f(1 - \frac{1}{n}))_{n \in \mathbb{N}} = (0)_{n \in \mathbb{N}}$ does not converge to $f(1) = 1$. Hence, by Theorem 4.15, $f$ is not a fuzzy quasi-metric aggregation function on products.

Next we address the case in which $F$ only aggregates, in the sense of Definition 4.6, stationary fuzzy metrics.

**Theorem 4.24.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)stationary fuzzy metric aggregation function on products;*
*(2) $F(\boldsymbol{0}) = 0$, $F$ has trivial core and preserves ($*$-)triangular triplets.*

*Proof.* (1) $\Rightarrow$ (2). That $F(\boldsymbol{0}) = 0$ and $F^{-1}(1) = \{\boldsymbol{1}\}$ follows as in Theorem 4.15.

Let $*$ be a t-norm and $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ be an $*$-triangular triplet. Let us consider $I_1 = \{i \in I : \boldsymbol{a}_i \neq 1, \boldsymbol{b}_i \neq 1, \boldsymbol{c}_i \neq 1\}$, $I_2 = \{i \in I : \boldsymbol{a}_i = 1, \boldsymbol{b}_i = \boldsymbol{c}_i \neq 1\}$, $I_3 = \{i \in I : \boldsymbol{b}_i = 1, \boldsymbol{a}_i = \boldsymbol{c}_i \neq 1\}$, $I_4 = \{i \in I : \boldsymbol{c}_i = 1, \boldsymbol{a}_i = \boldsymbol{b}_i \neq 1\}$ and $I_5 = \{i \in I : \boldsymbol{a}_i = \boldsymbol{b}_i = \boldsymbol{c}_i = 1\}$. Notice that $I = \cup_{k=1}^5 I_k$ (see Remark 3.14). Let us consider a set with three distinct elements $X = \{x_1, x_2, x_3\}$ and let us define

$$X_i = \begin{cases} X & \text{if } i \in I_1 \\ \{x_2, x_3\} & \text{if } i \in I_2 \\ \{x_1, x_2\} & \text{if } i \in I_3 \cup I_4 \\ \{x_1\} & \text{if } i \in I_5 \end{cases}.$$

Given $i \in I$, we also define $M_i' : X \times X \times [0,+\infty) \to [0,1]$ as follows

$$M_i'(x,y,t) = M_i'(y,x,t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } x = y, t > 0 \\ \boldsymbol{a}_i & \text{if } x = x_1, y = x_2, t > 0 \\ \boldsymbol{b}_i & \text{if } x = x_2, y = x_3, t > 0 \\ \boldsymbol{c}_i & \text{if } x = x_1, y = x_3, t > 0 \end{cases}.$$

Given $i \in I$ define $M_i$ as the restriction of $M_i'$ to $X_i$. Then $\{(X_i, M_i, *) : i \in I\}$ is a family of stationary fuzzy metric spaces. By assumption, $(\prod_{i\in I} X_i, F \circ \widetilde{\boldsymbol{M}}, *)$ is also a stationary fuzzy metric space. Let us define $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \prod_{i\in I} X_i$ as

$$\boldsymbol{x}_i = \begin{cases} x_1 & \text{if } i \in I_1 \cup I_3 \cup I_4 \cup I_5 \\ x_2 & \text{if } i \in I_2 \end{cases}, \qquad \boldsymbol{y}_i = \begin{cases} x_2 & \text{if } i \in I_1 \cup I_2 \cup I_3 \cup I_4 \\ x_1 & \text{if } i \in I_5 \end{cases}$$

$$\boldsymbol{z}_i = \begin{cases} x_3 & \text{if } i \in I_1 \cup I_2 \\ x_2 & \text{if } i \in I_3 \\ x_1 & \text{if } i \in I_4 \cup I_5 \end{cases}.$$

For any $t, s > 0$, it is straightforward to check that $M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t) = a_i$, $M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s) = b_i$ and $M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s) = c_i$ for all $i \in I$. Hence

$$\begin{aligned} F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{y}, t) * F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{y}, \boldsymbol{z}, s) &\le F \circ \widetilde{\boldsymbol{M}}(\boldsymbol{x}, \boldsymbol{z}, t+s) \\ F((M_i(\boldsymbol{x}_i, \boldsymbol{y}_i, t))_{i\in I}) * F((M_i(\boldsymbol{y}_i, \boldsymbol{z}_i, s))_{i\in I}) &\le F((M_i(\boldsymbol{x}_i, \boldsymbol{z}_i, t+s))_{i\in I}) \\ F(\boldsymbol{a}) * F(\boldsymbol{b}) &\le F(\boldsymbol{c}). \end{aligned}$$

Since $(F \circ \widetilde{\boldsymbol{M}}, *)$ is stationary, you can prove in a similar way that $F(\boldsymbol{b}) * F(\boldsymbol{c}) \le F(\boldsymbol{a})$ and $F(\boldsymbol{c}) * F(\boldsymbol{a}) \le F(\boldsymbol{b})$ only by permuting $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$. Consequently $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an $*$-triangular triplet.

(2) $\Rightarrow$ (1). This is straightforward. □

Observe that every ($*$-)stationary fuzzy quasi-metric aggregation function on products is always a ($*$-)stationary fuzzy metric aggregation function on products. However, the next example illustrates that ($*$-)stationary fuzzy metric aggregation functions on products are not necessarily isotone and, thus, they are not necessarily ($*$-)stationary fuzzy quasi-metric aggregation functions on products.

**Example 4.25.** Let $*$ be a t-norm different from $\wedge$. Since $*$ is not idempotent there exists $\alpha \in ]0,1[$ such that $\alpha * \alpha < \alpha$. Moreover, if $*$ has zero divisors, i.e. there exist $a, b \in ]0,1[$ such that $a * b = 0$ but $a \ne 0$ and $b \ne 0$, then we can choose $\alpha \in ]0,1[$ verifying $\alpha * \alpha = 0$. In any case, consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \alpha & \text{if } 0 < x < \frac{1}{2} \\ \alpha * \alpha & \text{if } \frac{1}{2} \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

Let us check that $f$ preserves $*$-triangular triplets. Let $(a, b, c)$ be an arbitrary $*$-triangular triplet with all its elements different from 1 and 0. Then the triplet $(f(a), f(b), f(c))$ is some of the following triplets or a permutation of them: $(\alpha *$

$\alpha, \alpha, \alpha)$, $(\alpha * \alpha, \alpha * \alpha, \alpha)$, $(\alpha * \alpha, \alpha * \alpha, \alpha * \alpha)$, $(\alpha, \alpha, \alpha)$. Notice that all these triplets are $*$-triangular.

On the other hand, suppose that some of the elements of the $*$-triangular triplet $(a, b, c)$ is 1, let's say $a = 1$. Then $b = c$ so $(f(1), f(b), f(c)) = (1, f(b), f(b))$ also is an $*$-triangular triplet.

Finally, let us suppose that some of $a, b, c$ is 0, for example $a = 0$. Then $b * c = 0$. We distinguish two cases:

- $b = 0$ or $c = 0$. Without loss of generality suppose that $b = 0$. Then $(f(a), f(b), f(c)) = (0, 0, f(c))$ which is also an $*$-triangular triplet;
- $b \neq 0$ and $c \neq 0$. $*$ has zero divisors so $\alpha * \alpha = 0$. Furthermore, since $b * c = 0$ then $b \neq 1, c \neq 1$, so it is clear that $f(b) * f(c) \leq \alpha * \alpha = 0 = f(0)$. Hence $(0, f(b), f(c))$ is an $*$-triangular triplet.

Consequently, $f$ preserves $*$-triangular triplets.

Since $f(0) = 0$ and $f^{-1}(1) = \{1\}$ then, by Theorem 4.24, $f$ is an $*$-stationary fuzzy metric aggregation function on products. However, $f$ is not isotone. Hence Theorem 4.22 guarantees that $f$ is not an $*$-stationary fuzzy quasi-metric aggregation function on products.

Next we give a characterization of ($*$-)stationary fuzzy quasi-metric aggregation functions on sets.

**Theorem 4.26.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)stationary fuzzy quasi-metric aggregation function on sets;*
*(2) $F$ is an aggregation function, ($*$-)supmultiplicative and its core is pairwise included in a unitary face;*
*(3) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, the core of $F$ is pairwise included in a unitary face and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* (1) $\Rightarrow$ (3). Let $\{(X, M_i, *) : i \in I\}$ be an arbitrary family of stationary fuzzy quasi-metric spaces. Since $(F \circ \boldsymbol{M}, *)$ is a fuzzy quasi-metric on $X$ then for each $x \in X$ and each $t > 0$ we have that $1 = F \circ \boldsymbol{M}(x, x, t) = F((M_i(x, x, t))_{i \in I}) = F(\mathbf{1})$. Furthermore, $0 = F \circ \boldsymbol{M}(x, x, 0) = F(\mathbf{0})$.

Now let us prove that the core of $F$ is pairwise included in a unitary face. Suppose that $F(\boldsymbol{a}) = F(\boldsymbol{b}) = 1$ but $I_{\boldsymbol{a}} \cap I_{\boldsymbol{b}} = \varnothing$. Then consider a set $X = \{x_0, y_0\}$ with two different elements and the family of stationary fuzzy quasi-metric spaces $\{(X, M_i, *) : i \in I\}$ where $M_i$ is defined as

$$M_i(x, y, t) = \begin{cases} 1 & \text{if } x = y,\ t > 0 \\ \boldsymbol{a}_i & \text{if } x = x_0,\ y = y_0,\ t > 0 \\ \boldsymbol{b}_i & \text{if } x = y_0,\ y = x_0,\ t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Notice that we cannot find $i \in I$ such that $\boldsymbol{a}_i = \boldsymbol{b}_i = 1$ so $(M_i, *)$ verifies the condition (FQM2') for all $i \in I$. By assumption, $(F \circ \boldsymbol{M}, *)$ is a fuzzy quasi-metric on $X$. Nevertheless,

$$F \circ \boldsymbol{M}(x_0, y_0, t) = F((M_i(x_0, y_0, t))_{i \in I}) = F(\boldsymbol{a}) = 1$$

and

$$F \circ \boldsymbol{M}(y_0, x_0, t) = F((M_i(y_0, x_0, t))_{i \in I}) = F(\boldsymbol{b}) = 1$$

for all $t > 0$ so, since $(F \circ \boldsymbol{M}, *)$ verifies (FQM2'), we have that $x_0 = y_0$ which is a contradiction. Hence, $I_{\boldsymbol{a}} \cap I_{\boldsymbol{b}} \neq \varnothing$, i. e. the core of $F$ is pairwise included in a unitary face.

Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ be an asymmetric $(*\text{-})$triangular triplet. Let us consider $X = \{x_1, x_2, x_3\}$ a set with three different elements and $*$ a t-norm. For each $i \in I$, the stationary fuzzy quasi-metric $(M_i, *)$ on $X$ is defined by

$$M_i(x,y,t) = \begin{cases} 1 & \text{if } x = y, t > 0 \\ \boldsymbol{a}_i & \text{if } x = x_1, y = x_2, t > 0 \\ \boldsymbol{b}_i & \text{if } x = x_2, y = x_3, t > 0 \\ \boldsymbol{c}_i & \text{if } x = x_1, y = x_3, t > 0 \\ 0 & \text{otherwise} \end{cases}.$$

By assumption, $(F \circ \boldsymbol{M}, *)$ is a stationary fuzzy quasi-metric on $X$. Therefore, for any $t, s > 0$ we have that

$$\begin{aligned} F \circ \boldsymbol{M}(x_1, x_2, t) * F \circ \boldsymbol{M}(x_2, x_3, s) &\leq F \circ \boldsymbol{M}(x_1, x_3, t+s) \\ F((M_i(x_1, x_2, t))_{i \in I}) * F((M_i(x_2, x_3, s))_{i \in I}) &\leq F((M_i(x_1, x_3, t+s))_{i \in I}) \\ F(\boldsymbol{a}) * F(\boldsymbol{b}) &\leq F(\boldsymbol{c}) \end{aligned}$$

so $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is an asymmetric $(*\text{-})$triangular triplet.

(3) $\Rightarrow$ (1). Let $\{(X, M_i, *) : i \in I\}$ be a family of stationary fuzzy quasi-metric spaces. We must show that $(F \circ \boldsymbol{M}, *)$ is a stationary fuzzy quasi-metric on $X$. It is obvious that $F \circ \boldsymbol{M}(x, y, 0) = 0$ and $F \circ \boldsymbol{M}(x, x, t) = 1$ for all $x, y \in X$ and for all $t > 0$ since $F(\boldsymbol{1}) = 1$ and $F(\boldsymbol{0}) = 0$. Let $x, y \in X$ such that $F \circ \boldsymbol{M}(x, y, t) = F \circ \boldsymbol{M}(y, x, t) = 1$ for all $t > 0$. Since the core of $F$ is pairwise included in a unitary face we can find $i \in I$ such that $M_i(x, y, t) = M_i(y, x, t) = 1$ for all $t > 0$ (since $M_i$ is stationary). Then $x = y$.

Let us consider now $x, y, z \in X$ and $t, s > 0$. Since $(M_i, *)$ is a fuzzy quasi-metric then $M_i(x, y, t) * M_i(y, z, s) \leq M_i(x, z, t+s)$. So $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an asymmetric $*$-triangular triplet where $\boldsymbol{a} = (M_i(x, y, t))_{i \in I}$, $\boldsymbol{b} = (M_i(y, z, s))_{i \in I}$ and $\boldsymbol{c} = (M_i(x, z, t+s))_{i \in I}$. By assumption $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c})) = (F((M_i(x, y, t))_{i \in I}), F((M_i(y, z, s))_{i \in I}), F((M_i(x, z, t+s))_{i \in I}))$ is also an asymmetric $(*\text{-})$triangular triplet. Hence

$$F \circ \boldsymbol{M}(x, y, t) * F \circ \boldsymbol{M}(y, z, s) \leq F \circ \boldsymbol{M}(x, z, t+s).$$

It is clear that $F \circ \boldsymbol{M}(x, y, \cdot)$ is left-continuous for all $x, y \in X$ since $F \circ \boldsymbol{M}(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is constant. Moreover $(F \circ \boldsymbol{M}, *)$ is stationary since $(M_i, *)$ is stationary for all $i \in I$.

(2) $\Leftrightarrow$ (3). It is a direct consequence of Corollary 3.35. $\square$

**Remark 4.27.** Note that if $F$ satisfies the conditions of Theorem 4.26 then, by Proposition 4.12, the core of $F$ is also finitely included in a unitary face since $F$ is $(*\text{-})$supmultiplicative.

In the following we give a characterization of $(*\text{-})$stationary fuzzy metric aggregation functions on sets.

**Theorem 4.28.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a $(*\text{-})$stationary fuzzy metric aggregation function on sets;*

*(2)* $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$, *the core of* $F$ *is included in the unitary boundary and if* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ *is a* $(*\text{-})$*triangular triplet on* $[0,1]^I$ *such that its elements verify that all their coordinates are not equal to 1, then* $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ *is also a* $(*\text{-})$*triangular triplet.*

*Proof.* (1) $\Rightarrow$ (2). The facts that $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$ can be proved as in Theorem 4.26. That the core of $F$ is included in the unitary boundary can be proved by an easy modification of the proof that the core of $F$ is pairwise included in a unitary face in Theorem 4.26.

Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ be a $(*\text{-})$triangular triplet on $[0,1]^I$ such that none of the elements of the triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ has a coordinate equal to 1. Consider a set $X = \{x_1, x_2, x_3\}$ with three different elements and the stationary fuzzy metrics $(M_i, *)$ on $X$ given by

$$M_i(x, y, t) = M_i(y, x, t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } x = y, t > 0 \\ \boldsymbol{a}_i & \text{if } x = x_1, y = x_2, t > 0 \\ \boldsymbol{b}_i & \text{if } x = x_2, y = x_3, t > 0 \\ \boldsymbol{c}_i & \text{if } x = x_1, y = x_3, t > 0 \end{cases}.$$

By assumption $(F \circ \boldsymbol{M}, *)$ is a stationary fuzzy metric on $X$. Therefore for any $t, s > 0$

$$\begin{aligned} F \circ \boldsymbol{M}(x_1, x_2, t) * F \circ \boldsymbol{M}(x_2, x_3, s) &\leq F \circ \boldsymbol{M}(x_1, x_3, t + s) \\ F((M_i(x_1, x_2, t))_{i \in I}) * F((M_i(x_2, x_3, s))_{i \in I}) &\leq F((M_i(x_1, x_3, t + s))_{i \in I}) \\ F(\boldsymbol{a}) * F(\boldsymbol{b}) &\leq F(\boldsymbol{c}). \end{aligned}$$

If we permute $x_1, x_2, x_3$ we deduce that $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is a $(*\text{-})$triangular triplet.

(2) $\Rightarrow$ (1). Let $\{(X, M_i, *) : i \in I\}$ be a family of stationary fuzzy metric spaces. Everything is similar to the proof of (3) $\Rightarrow$ (1) in Theorem 4.26 except for proving (FQM3). Now let us consider $x, y, z \in X$ and $t, s > 0$. Since $(M_i, *)$ is a fuzzy metric for all $i \in I$ then it satisfies (FQM3). So $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is an $*$-triangular triplet where $\boldsymbol{a} = (M_i(x, y, t))_{i \in I}$, $\boldsymbol{b} = (M_i(y, z, s))_{i \in I}$ and $\boldsymbol{c} = (M_i(x, z, t + s))_{i \in I}$. If none of the elements of the triplet $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ has a coordinate equal to 1, then by hypothesis $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c})) = (F \circ \boldsymbol{M}(x, y, t), F \circ \boldsymbol{M}(y, z, s), F \circ \boldsymbol{M}(x, z, t + s))$ is also an $*$-triangular triplet. It follows that

$$F \circ \boldsymbol{M}(x, y, t) * F \circ \boldsymbol{M}(y, z, s) \leq F \circ \boldsymbol{M}(x, z, t + s).$$

If some of the coordinates is equal to 1, let us suppose without loss of generality that we can find $i \in I$ such that $\boldsymbol{a}_i = 1$, i. e. $M_i(x, y, t) = 1$. Since $(M_i, *)$ is stationary then it follows by (FQM2') that $x = y$. Therefore $\boldsymbol{a} = \mathbf{1}$ and $\boldsymbol{b} = \boldsymbol{c}$. Hence $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c})) = (1, F(\boldsymbol{b}), F(\boldsymbol{b}))$ is also an $*$-triangular triplet. Consequently, we conclude that $F \circ \boldsymbol{M}$ satisfies (FQM3). $\square$

Clearly every $(*\text{-})$stationary fuzzy quasi-metric aggregation function on sets is also a $(*\text{-})$stationary fuzzy metric aggregation function on sets. However, the converse is not true as shows the example below. Indeed, we provide an $*$-stationary fuzzy metric aggregation function on sets which does not preserve every $*$-triangular triplet.

**Example 4.29.** Let $*$ be a t-norm different from $\wedge$. Since $*$ is not idempotent there exists $\alpha \in ]0, 1[$ such that $\alpha * \alpha < \alpha$. Moreover, if $*$ has zero divisors, i.e. there exist

$a, b \in ]0,1[$ such that $a * b = 0$ but $a \neq 0$ and $b \neq 0$, then we can choose $\alpha \in ]0,1[$ verifying $\alpha * \alpha = 0$. In any case, let us consider the function $F : [0,1]^2 \to [0,1]$ given by

$$F(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ \alpha & \text{otherwise} \end{cases}.$$

Let us check that $F$ is an $*$-stationary fuzzy metric aggregation function on sets. It is obvious that $F(0,0) = 0$, $F(1,1) = 1$ and the core of $F$ is included in the unitary boundary.

Now, let us consider an $*$-triangular triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in $[0,1]^2$ such that all the coordinates of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are different from 1. Let us check that $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is also an $*$-triangular triplet.

If all the coordinates of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are also different from 0 then $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c})) = (\alpha, \alpha, \alpha)$ which obviously is an $*$-triangular triplet. So let us suppose that some coordinate of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is 0, for example, $\mathbf{a}_1 = 0$. Since $(\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1)$ is an $*$-triangular triplet then $\mathbf{b}_1 * \mathbf{c}_1 = 0$. If $\mathbf{b}_1 = 0$ or $\mathbf{c}_1 = 0$ then $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is $(0,0,\alpha)$, $(0,0,0)$ or a permutation of these triplets which are $*$-triangular. If $\mathbf{b}_1 \neq 0$ and $\mathbf{c}_1 \neq 0$ then $*$ has zero divisors so $\alpha * \alpha = 0$. Since $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is $(0,0,0), (0,0,\alpha), (0,\alpha,\alpha)$ or a permutation of these triplets, we conclude that $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is $*$-triangular in any case.

Therefore, by Theorem 4.28, $F$ is an $*$-stationary fuzzy metric aggregation function on sets.

However, $F$ does not preserve every $*$-triangular triplet. For example, if $\mathbf{a} = (1, \frac{1}{2})$, $\mathbf{b} = (\frac{1}{2}, 1)$ and $\mathbf{c} = (\frac{1}{2}, \frac{1}{2})$ then $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an $*$-triangular triplet. Nonetheless $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c})) = (1, 1, \alpha)$ is not $*$-triangular because $1 * 1 = 1 \nleq \alpha$.

We also notice that the core of $F$ is not pairwise included in a unitary face and, thus, by Theorem 4.26 it is not an ($*$-)stationary fuzzy quasi-metric aggregation function on sets.

Example 4.25 also shows that there exist $*$-stationary fuzzy metric aggregation functions on sets which are not isotone.

Observe that in the light of Theorems 4.22 and 4.26, ($*$-)stationary fuzzy quasi-metric aggregation functions on products and ($*$-)stationary fuzzy quasi-metric aggregation functions on sets are different. Nonetheless, functions which aggregate ($*$-)stationary fuzzy quasi-pseudometrics on products are the same that functions which aggregate ($*$-)stationary fuzzy quasi-pseudometrics on sets as we next prove.

**Theorem 4.30.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)stationary fuzzy quasi-pseudometric aggregation function on products;*
*(2) $F$ is a ($*$-)stationary fuzzy quasi-pseudometric aggregation function on sets;*
*(3) $F$ is an aggregation function and ($*$-)supmultiplicative;*
*(4) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$ and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* It is obvious that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) follows as the implication (1) $\Rightarrow$ (3) in Theorem 4.26 (and using Corollary 3.35), except that we cannot obtain that the core of $F$ is pairwise included in a unitary face. We only can prove that $F(\mathbf{1}) = 1$.

(3) $\Rightarrow$ (4) follows from Corollary 3.35.

(4) $\Rightarrow$ (1) is similar to the proof (3) $\Rightarrow$ (1) of Theorem 4.22 except that in this case we only need the hypothesis $F(\mathbf{1}) = 1$ instead of $F$ has trivial core. $\square$

Stationary fuzzy metric aggregation functions are different depending if they are considered on products or on sets (see Theorems 4.24 and 4.28). In case that we treat with stationary fuzzy pseudometrics, there is no difference as the next result shows.

**Theorem 4.31.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)stationary fuzzy pseudometric aggregation function on products;*
*(2) $F$ is a ($*$-)stationary fuzzy pseudometric aggregation function on sets;*
*(3) $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$ and $F$ preserves ($*$-)triangular triplets.*

*Proof.* (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3) is similar to (1)$\Rightarrow$ (2) in Theorem 4.28 but under the current hypothesis it is not true in general that the core of $F$ is included in the unitary boundary. Nevertheless, notice that in this case, if we consider an arbitrary $*$-triangular triplet (with coordinates that can be 1) the fuzzy sets $M_i$ constructed in that implication are in this case fuzzy pseudometrics.

(3) $\Rightarrow$ (1) is like implication (2) $\Rightarrow$ (1) in Theorem 4.24 except that in this case we only need the hypothesis $F(\mathbf{1}) = 1$ instead of $F$ has trivial core. $\square$

Finally, as a consequence of the results that we have already obtained for functions which aggregate different types of stationary fuzzy quasi-pseudometrics, we can characterize functions which aggregate well-known types of fuzzy binary relations. As previously observed in Remark 3.3, there exist equivalences between subfamilies of stationary fuzzy quasi-pseudometrics and subfamilies of fuzzy binary relations. These relationships are summarized in the following:

| | | |
|---|---|---|
| stationary fuzzy quasi-pseudometric | $\simeq$ | fuzzy preorder |
| stationary fuzzy quasi-metric | $\simeq$ | fuzzy partial order |
| stationary fuzzy pseudometric | $\simeq$ | indistinguishability operator |
| stationary fuzzy metric | $\simeq$ | equality |

**Corollary 4.32.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy preorder aggregation function on products;*
*(2) $F$ is a ($*$-)fuzzy preorder aggregation function on sets;*
*(3) $F(\mathbf{1}) = 1$ and $F$ is ($*$-)supmultiplicative and isotone;*
*(4) $F(\mathbf{1}) = 1$ and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* We can prove this result proceeding as in Theorem 4.30 except that in this case it is not necessary that $F(\mathbf{0}) = 0$. $\square$

**Corollary 4.33.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy partial order aggregation function on products;*
*(2) $F$ has trivial core and $F$ is ($*$-)supmultiplicative and isotone;*
*(3) $F$ has trivial core and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* Everything is similar to the proof of Theorem 4.22 but in this case the condition $F(\mathbf{0}) = 0$ can be avoided. □

**Corollary 4.34.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)fuzzy partial order aggregation function on sets;*
*(2) $F(\mathbf{1}) = 1$, the core of $F$ is pairwise included in a unitary face and $F$ is ($*$-)supmultiplicative and isotone;*
*(3) $F(\mathbf{1}) = 1$, the core of $F$ is pairwise included in a unitary face and $F$ preserves asymmetric ($*$-)triangular triplets.*

*Proof.* It can be proved as Theorem 4.26 but in this case the condition $F(\mathbf{0}) = 0$ is unnecessary. □

**Corollary 4.35** (see Theorem 3.25)**.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)indistinguishability operator aggregation function on products;*
*(2) $F$ is a ($*$-)indistinguishability operator aggregation function on sets;*
*(3) $F(\mathbf{1}) = 1$ and $F$ preserves ($*$-)triangular triplets.*

*Proof.* We can prove this result proceeding as in Theorem 4.31 except that in this case it is not necessary that $F(\mathbf{0}) = 0$. □

**Corollary 4.36.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)equality aggregation function on products;*
*(2) $F$ has trivial core and $F$ preserves ($*$-)triangular triplets.*

*Proof.* We can prove this result proceeding as in Theorem 4.24 except that in this case it is not necessary the condition $F(\mathbf{0}) = 0$. □

**Corollary 4.37.** *Let $F : [0,1]^I \to [0,1]$ be a function and $*$ be a t-norm. The following statements are equivalent:*

*(1) $F$ is a ($*$-)equality aggregation function on sets;*
*(2) $F(\mathbf{1}) = 1$, the core of $F$ is included in the unitary boundary and if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a ($*$-)triangular triplet on $[0,1]^I$ such that its elements verify that all their coordinates are not equal to 1, then $(F(\boldsymbol{a}), F(\boldsymbol{b}), F(\boldsymbol{c}))$ is also a ($*$-)triangular triplet.*

*Proof.* Everything is similar to the proof of Theorem 4.28 but in this case the condition $F(\mathbf{0}) = 0$ is unnecessary. □

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## References


[1] J. C. R. Alcantud, A. Z. Khameneh and A. Kilicman, *Aggregation of infinite chains of intuitionistic fuzzy sets and their application to choices with temporal intuitionistic fuzzy information*, Inform. Sci. **514** (2020), 106–117.

[2] M. Baczyński and B. Jayaram, *Fuzzy implications*, Studies in Fuzziness and Soft Computing, vol. 231, Springer, 2008.

[3] G. Beliakov, H. Bustince and T. Calvo, *A practical guide to averaging functions*, Studies in Fuzziness and Soft Computing, Springer, 2016.

[4] G. Beliakov, A. Pradera and T. Calvo, *Aggregation functions: a guide for practitioners*, Studies in Fuzziness and Soft Computing, Springer, 2007.

[5] R. Belohlavek and V. Vychodil, *Codd's relational model from the point of view of fuzzy logic*, J. Logic Comput. **21** (2011), no. 5, 851–862.

[6] U. Bodenhofer and J. Küng, *Enriching vague queries by fuzzy orderings*, Proceedings 2nd. Int. Conf. in Fuzzy Logic and Technology (EUSFLAT 2001), 2001, pp. 360–364.

[7] D. Boixader and J. Recasens, *On the relationship between fuzzy subgroups and indistinguishability operators*, Fuzzy Sets Syst. **373** (2019), 149–163.

[8] J. Borsík and J. Doboš, *On a product of metric spaces*, Math. Slovaca **31** (1981), no. 1, 193–205

[9] B. De Baets and R. Mesiar, *$T$-partitions*, Fuzzy Sets Syst. **97** (1998), 211–223.

[10] B. De Baets and R. Mesiar, *Pseudo-metrics and $T$-equivalences*, J. Fuzzy Math. 5 (1997), 471–481.

[11] B. De Baets and R. Mesiar, *Metrics and $T$-equalities*, J. Math. Anal. Appl. **267** (2002) 531–547.

[12] M. Delić, L. Nedović and E. Pap, *Extended power-based aggregation of distance functions and application in image segmentation*, Inform. Sci. **494** (2019), 155–173.

[13] J. Drewniak and U. Dudziak, *Aggregation in classes of fuzzy relations*, Stud. Math. **5** (2006), 33–43.

[14] J. Drewniak and U. Dudziak, *Preservation of properties of fuzzy relations during aggregation processes*, Kybernetika **43** (2007), 115–132.

[15] U. Dudziak, *Preservation of $t$-norm and $t$-conorm based properties of fuzzy relations during aggregation process*, Proc. of 8th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2013), 2013, pp. 376–383.

[16] P. Eklund, J. Gutiérrez-García, U. Höhle and J. Kortelainen, *Semigroups in complete lattices. Quantales, modules and related topics*, Springer, 2018.

[17] J. Fodor and M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Kluwer, 1994.

[18] P. Fuster-Parra, J. Martín, J.-J. Miñana and O. Valero, *A study on the relationship between relaxed metrics and indistinguishability operators*, Soft Computing **23** (2019), 6785–6795.

[19] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst. **64** (1994), no. 3, 395–399.

[20] G. Gerla, *Representation theorems for fuzzy orders and quasi-metrics*, Soft Computing **8** (2004), 571–58.

[21] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation functions*, Encyclopedia of Mathematics and its Applications, vol. 127, Cambridge University Press, 2009.

[22] S. Gottwald, *On $t$-norms which are related to distances of fuzzy sets*, BUSEFAL **50** (1992), 25–30.

[23] V. Gregori, S. Morillas and A. Sapena, *Examples of fuzzy metrics and applications*, Fuzzy Sets Syst. **170** (2011), 95–111.

[24] V. Gregori and S. Romaguera, *On fuzzy quasi-metric spaces*, Appl. Gen. Topol. **5** (2004), no. 1, 129–136.

[25] U. Höhle, *Fuzzy equalities and indistinguishability*, Proc. of EUFIT'93, Vol. 1, 1993 pp. 358–363.

[26] E. P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Kluwer, 2000.
[27] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[28] F. Klawonn and R. Kruse, *From fuzzy sets to indistinguishability and back*, Proc. of the First ICSC International Symposium on Fuzzy Logic (ISFL'95), 1995, pp. A57–A59.
[29] H. Lai and D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets Syst. **157** (2006), 1865–1885.
[30] S. Massanet and O. Valero, *On the aggregation of metric structures: the extended quasi-metric case*, Int. J. Comput. Int. Sys. **6** (2013), 115–126.
[31] G. Mayor and J. Recasens, *Preserving T-transitivity*, Artificial Intelligence Reserach and Development, vol. 288, 2016, pp. 79–87.
[32] G. Mayor and O. Valero, *Aggregation of asymmetric distances in Computer Science*, Inform. Sci. **180** (2010), 803–812.
[33] G. Mayor and O. Valero, *Metric aggregation functions revised*, Eur. J. Combin. **80** (2019), 390–400.
[34] R. Mesiar and A. Kolesárová, *Construction of fuzzy implication functions based on $F$-chains*, in: New trends in aggregation theory, vol. 981, eds. R. Halaš and M. Gagolewski and R. Mesiar, Springer, pp. 315–326.
[35] R. Mesiar and E. Pap, *Aggregation of infinite sequences*, Inform. Sci. 178 (2008), 3557-3564.
[36] J.-J. Miñana and O. Valero, *On indistinguishability operators, fuzzy metrics and modular metrics*, Axioms **2017**, 6, 34.
[37] J.-J. Miñana and O. Valero, *A duality relationship between fuzzy metrics and metrics*, Int. J. Gen. Syst. **47** (2018), no. 6, 593–612.
[38] J.-J. Miñana and O. Valero, *Characterizing quasi-metric aggregation functions*, Int. J. Gen. Syst. **48** (2019), no. 8, 890–909.
[39] A. Pradera, E. Trillas and E. Castiñeria, *On the aggregation of some classes of fuzzy relations*, in: Technologies for constructing intelligent systems, B. Bouchon-Meunier, J. Gutierrez, L. Magdalena and R. Yager (eds), Springer-Verlag, pp. 125–147, 2002.
[40] A. Pradera and E. Trillas, *A note of pseudometrics aggregation*, Int. J. Gen. Syst. **31** (2002), no. 1, 41–51.
[41] F. E. Petry and P. Bosc, *Fuzzy databases: principles and applications*, Kluwer Academic Publishers, 1996.
[42] J. Recasens, *Indistinguishability operators. Modelling fuzzy equalities and fuzzy equivalence relations*, Springer-Verlag Berlin Heidelberg, 2010.
[43] J. Rodríguez-López, S. Romaguera and J. M. Sánchez-Álvarez, *The Hausdorff fuzzy quasi-metric*, Fuzzy Sets Syst. **161** (2010), no. 8, 1078–1096.
[44] J. Doboš, *Metric preserving functions*, Štroffek, Košice, 1998.
[45] S. Ovchinnikov, *Representation of transitive fuzzy relations*, Aspects of Vageness, H. Skala, S. Termini and E. Trillas (eds), Springer-Verlag, pp. 105–118, 1984.
[46] S. Saminger, R. Mesiar and U. Bodenhofer, *Domination of aggregation operators and preservation of transitivity*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **10** (2002), 11–35.
[47] A. Sapena, *A contribution to the study of fuzzy metric spaces*, Appl. Gen. Topol. **2** (2001), no. 1, 63–75.
[48] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 314–334.
[49] L. Valverde, *On the structure of F-indistinguishability operators*, Fuzzy Set. Syst. 17 (1985), 313–328.
[50] L. Zadeh, *Similarity relations and fuzzy orderings*, Inform. Sci. **1** (1971), 177–200.

(T. Pedraza) Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
*Email address*: `tapedraz@mat.upv.es`

(J. Rodríguez-López) Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
*Email address*: `jrlopez@mat.upv.es`

(O. Valero) Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, Ctra. de Valldemossa km. 7.5, 07122 Palma, Spain., Institut d'Investigació Sanitària Illes Balears (IdISBa), Ctra. de Valldemossa, 79, 07120 Palma, Spain.

*Email address*: o.valero@uib.es