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Dynamics of induced mappings on symmetric products, some answers

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Abstract

Let X be a metric continuum and $n \in \mathbb{N}$. Let $F_n(X)$ be the hyperspace of nonempty subsets of X with at most n points. If $1 \leq m < n$, we consider the quotient space $F^n_m(X) = F_n(X)/F_m(X)$. Given a mapping $f\colon X \to X$, we consider the induced mappings $f_n\colon F_n(X) \to F_n(X)$ and $f^n_m\colon F^n_m(X) \to F^n_m(X)$. In this paper we study relations among the dynamics of the mappings f, f_n and f^n_m and we answer some questions, by F. Barragán, A. Santiago-Santos and J. Tenorio, related to the properties: minimality, irreducibility, strong transitivity and turbulence.

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1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a nonempty compact metric space X and integers $1 \leq m < n$ we consider the following hyperspaces of X:

$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\}$,

$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}$,

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and the quotient space $F_m^n(X) = F_n(X)/F_m(X)$.

The hyperspace 2^X is considered with the Hausdorff metric [13, Theorem 2.2]. Given subsets $U_1, \ldots, U_k$ of X, let

$$\langle U_1, \ldots, U_k \rangle = \{A \in F_n(X) : A \subset U_1 \cup \cdots \cup U_k \text{ and } A \cap U_i \neq \varnothing \text{ for each } i \in \{1, \ldots, k\}\}.$$

Then the family of sets of the form $\langle U_1, \ldots, U_k \rangle$, where the sets U_i are open subsets of X, is a base of the topology in $F_n(X)$ [13, Theorem 3.1]. The hyperspace $F_n(X)$ is called the *n^{th}-symmetric product of X*. We denote the quotient mapping by $q_m \colon F_n(X) \to F_m^n(X)$ (or q_m^n, if necessary) and we denote by F_X^m the element in $F_m^n(X)$ such that $q_m(F_m(X)) = \{F_X^m\}$. A *mapping* is a continuous function. Given a mapping $f \colon X \to X$, the *induced mapping* $2^f \colon 2^X \to 2^X$ is defined by $2^f(A) = f(A)$ (the image of A under f). The induced mapping $f_n \colon F_n(X) \to F_n(X)$ (also denoted in some papers by $F_n(f)$) is the restriction of 2^f to $F_n(X)$. The induced mapping $f_m^n \colon F_m^n(X) \to F_m^n(X)$ (also denoted by $SF_m^n(f)$) is the mapping that makes commutative the following diagram [8, Theorem 4.3, Chapter VI].

$$\begin{array}{ccc} F_n(X) & \xrightarrow{f_n} & F_n(X) \\ {\scriptstyle q_m}\downarrow & & \downarrow{\scriptstyle q_m} \\ F_m^n(X) & \xrightarrow[f_m^n]{} & F_m^n(X) \end{array}$$

A *dynamical system* is a pair (X, f), where X is a non-degenerate compact metric space and $f \colon X \to X$ is a mapping. Given a point $p \in X$, *the orbit of p under f* is the set $\mathrm{orb}(p, f) = \{f^k(p) \in X : k \in \mathbb{N} \cup \{0\}\}$. A dynamical system (X, f) induces the dynamical systems $(2^X, 2^f)$, $(F_n(X), f_n)$ and $(F_n^m(X), f_m^n)$.

H. Hosokawa [12] was the first author that studied induced mappings to hyperspaces. Since then, this topic has been widely studied. The most common problem studied in this area is the following. Given a class of mappings $\mathcal{M}$, determine whether one of the following statements implies another:

(a) $f \in \mathcal{M}$,
(b) $2^f \in \mathcal{M}$,
(c) $f_n \in \mathcal{M}$,
(d) $f_m^n \in \mathcal{M}$.

Of course, this problem has also been considered for other hyperspaces. Dynamical properties of induced mappings on symmetric products have been considered in [2], [3], [4], [5], [6], [9], [10], [11] and [14]. In particular, in [4] and [5], the properties of being: exact, mixing, weakly mixing, transitive, totally transitive, strongly transitive, chaotic, minimal, irreducible, feebly open and turbulent were studied.

The aim of this paper is to solve most of the problems posed by F. Barragán, A. Santiago-Santos and J. Tenorio in [4] and [5], related to the properties: minimality, irreducibility, strong transitivity and turbulence.

Throughout this paper the word space means a non-degenerate compact metric space.

We are aware that some of our proofs can be copied to obtain results with less restrictions either on the spaces or in the functions, however we consider that, point out the more general setting under each result holds, is worthless and breaks the continuity of the paper.

2. Minimality

Let X be a space. A mapping $f\colon X \to X$ is *minimal* [1, p.7] if there is no nonempty proper closed subset M of X which is invariant under f (invariance of M means that $f(M) \subset M$); equivalently, if the orbit of every point of X is dense in X. The mapping f is *totally minimal* if f^s is minimal for each $s \in \mathbb{N}$.

Given $n \in \mathbb{N}$, in this section we consider the following statements.

(1) f is minimal,
(2) f_n is minimal, and
(3) f_1^n is minimal.

In [4, Theorem 4.18], it was proved that (2) implies (3), (3) implies (1), (2) implies (1), (1) does not imply (2) and (1) does not imply (3), for the case that X is a continuum. Moreover, in [4, Question 4.2] it was asked whether (3) implies (2). The following theorem solves this question and even when it has a very simple proof, it shows that the question and several results on minimal induced mappings are irrelevant.

Theorem 2.1. *Let X be a space, $f\colon X \to X$ a mapping and $1 \leq m < n$. Then:*

(a) $f_n(F_1(X)) \subset F_1(X)$,
(b) $f_m^n(F_X^m) = F_X^m$,
(c) for each $A \in F_m(X)$, $orb(A, f_n) \subset F_m(X)$. *Thus,* $orb(A, f_n)$ *is not dense in* $F_n(X)$ *and* f_n *is not minimal, and*
(d) $orb(F_X^m, f_m^n) = \{F_X^m\}$. *Thus,* $orb(F_X^m, f_m^n)$ *is not dense in* $F_m^n(X)$ *and* f_m^n *is not minimal.*

Proof. Take a point $p \in X$. Then $f_n(\{p\}) = f(\{p\}) = \{f(p)\} \in F_1(X)$. Moreover, $f_m^n(F_X^m) = f_m^n(q_m(\{p\})) = q_m(f_n(\{p\})) = q_m(\{f(p)\}) = F_X^m$. This proves (a), (b) and (d). The proof of (c) is similar. □

Theorem 2.1 (b) implies that the mappings f_n and f_m^n are never minimal or totally minimal. Then proved results in which minimality or total minimality of f_n or f_m^n is either assumed or concluded become irrelevant or partially irrelevant, such is the case of the following results by Barragán, Santiago-Santos and Tenorio: Theorem 4.18, Corollary 4.19, Corollary 4.20 and Corollary 4.21 of [4]; Corollary 5.13 and Corollary 5.17 of [6].

3. Irreducibility

Let X be a space. A mapping $f\colon X \to X$ is *irreducible* [1, p.171] if the only closed subset A of X for which $f(A) = X$ is $A = X$;

Given $n \in \mathbb{N}$, in this section we consider the following statements.

(1) f is irreducible,
(2) f_n is irreducible,
(3) f_1^n is irreducible, and
(4) f_m^n is irreducible.

Using [4, Theorem 5.1], in [5, Theorem 4.1] it was shown that each one of the statements (2), (3) and (4) implies (1). The authors of [4] and [5] supposed that the spaces are continua, however, it is easy see that the proofs for these results are valid for infinite compact metric spaces without isolated points. The rest of the implications among (1), (2), (3) and (4) are left as questions in [4, Questions 5.5] and [5, Question 4.2]. The purpose of this section is to complete the proof that, in fact, all the statements (1)-(4) are equivalent.

Theorem 3.1. *Let X be a space without isolated points, $f\colon X \to X$ a mapping and $1 \leq m < n$. If f is irreducible, then f_n is irreducible.*

Proof. Suppose that f is irreducible.

Claim 1. If U is a nonempty open subset of X, then there exists $p \in U$ such that $f(p) \notin f(X \setminus U)$.

In order to prove Claim 1, let $A = X \setminus U$. Then A is a proper closed subset of X. Since f is irreducible, f is onto. Thus there exist $q \in X$ such that $q \notin f(A)$ and $p \in X$ such that $f(p) = q$. Observe that $p \in U$. This finishes the proof of Claim 1.

Claim 2. If $\mathcal{U}$ is a nonempty open subset of $F_n(X)$, then there exists $B \in \mathcal{U}$ such that $B \in F_n(X) \setminus F_{n-1}(X)$ and $f(B) \notin f_n(F_n(X) \setminus \mathcal{U})$.

We prove Claim 2. Since X does not have isolated points, $F_n(X) \setminus F_{n-1}(X)$ is dense in $F_n(X)$. Then there exists $D = \{p_1, \dots, p_n\} \in (F_n(X) \setminus F_{n-1}(X)) \cap \mathcal{U}$. Then there exist pairwise disjoint open subsets $U_1, \dots, U_n$ of X such that for each $i \in \{1, \dots, n\}$, $p_i \in U_i$ and $D \in \langle U_1 \dots, U_n \rangle \subset \mathcal{U}$. By Claim 1, for each $i \in \{1, \dots, n\}$, there exists $u_i \in U_i$ such that $f(u_i) \notin f(X \setminus U_i)$.

Define $B = \{u_1, \dots, u_n\}$. Clearly, $B \in F_n(X) \setminus F_{n-1}(X)$. Suppose that there exists $E \in F_n(X) \setminus \mathcal{U}$ such that $f(E) = f(B)$. Given $i \in \{1, \dots, n\}$, let $e_i \in E$ be such that $f(e_i) = f(u_i)$. By the choice of u_i, $e_i \in U_i$. Thus $E \in \langle U_1 \dots, U_n \rangle \subset \mathcal{U}$, a contradiction. This proves that $f(B) \notin f_n(F_n(X) \setminus \mathcal{U})$. This finishes the proof of Claim 2.

We are ready to prove that f_n is irreducible. Let $\mathcal{A}$ be a proper closed subset of $F_n(X)$ and $\mathcal{U} = F_n(X) \setminus \mathcal{A}$. By Claim 2, there exists $B \in \mathcal{U}$ such that $B \in F_n(X) \setminus F_{n-1}(X)$ and $f(B) \notin f_n(\mathcal{A})$. Therefore $f_n(\mathcal{A}) \neq F_n(X)$ and f_n is irreducible. □

Theorem 3.2. *Let X be an space without isolated points, $f\colon X \to X$ a mapping and $1 \leq m < n$. If f_n is irreducible, then f_m^n is irreducible.*

Proof. Suppose that f_m^n is not irreducible. We will prove that f_n is not irreducible. Then there exists a proper closed subset $\mathcal{A}$ of $F_m^n(X)$ such that $f_m^n(\mathcal{A}) = F_m^n(X)$. Let $\mathcal{B} = q_m^{-1}(\mathcal{A} \cup \{F_X^m\})$. Then $\mathcal{B}$ is a closed subset of $F_n(X)$.

We check that $\mathcal{B} \neq F_n(X)$. Set $\mathcal{U} = F_m^n(X) \setminus \mathcal{A}$. Then $\mathcal{U}$ is a nonempty open subset of $F_m^n(X)$. This implies that $q_m^{-1}(\mathcal{U})$ is a nonempty open subset of $F_n(X)$. Since X does not have isolated points, $F_n(X) \setminus F_{n-1}(X)$ is dense in $F_n(X)$. So, there exists $G \in (F_n(X) \setminus F_{n-1}(X)) \cap q_m^{-1}(\mathcal{U})$. Thus $q_m(G) \in \mathcal{U} \setminus \{F_X^m\} = F_m^n(X) \setminus (\mathcal{A} \cup \{F_X^m\})$. Hence $G \notin \mathcal{B}$. Therefore $\mathcal{B} \neq F_n(X)$.

Now, we prove that $f_n(\mathcal{B}) = F_n(X)$. Since f_n is surjective, we have that f is surjective [4, Theorem 3.2]. Take $E \in F_n(X)$. In the case that $E = \{q_1, \ldots, q_k\} \in F_m(X)$, with $k \leq m$. Since f is surjective, for each $i \in \{1, \ldots, k\}$ there exists $p_i \in X$ such that $f(p_i) = q_i$. Thus $\{p_1, \ldots, p_k\} \in F_m(X) = q_m^{-1}(F_X^m) \subset \mathcal{B}$. Therefore $E = f_n(\{p_1, \ldots, p_k\}) \in f_n(\mathcal{B})$. Now we suppose that $E \notin F_m(X)$. Let $A \in \mathcal{A}$ be such that $f_m^n(A) = q_m(E)$. Let $B \in F_n(X)$ be such that $A = q_m(B)$. Then $B \in \mathcal{B}$. Since $q_m(E) = f_m^n(A) = f_m^n(q_m(B)) = q_m(f_n(B))$ and $E \notin F_m(X)$, we have that $\{E\} = q_m^{-1}(q_m(E)) = f_n(B)$. Therefore $E \in f_n(\mathcal{B})$. We have shown that $f_n(\mathcal{B}) = F_n(X)$. Therefore f_n is not irreducible. Therefore, we have shown that if f_m^n is not irreducible, then f_n is not irreducible. □

Corollary 3.3. *Let X be a space without isolated points, $1 \leq m < n$ and $f\colon X \to X$ a mapping. Then the following statements are equivalent.*

(1) f is irreducible,
(2) f_n is irreducible, and
(3) f_m^n is irreducible.

4. Strong transitivity

Let X be a space. A mapping $f\colon X \to X$ is *strongly transitive* [15, p.369] if for each nonempty open subset U of X, there exists $r \in \mathbb{N}$ such that $\bigcup_{i=0}^{r} f^i(U) = X$.

Given $1 \leq m < n$, in this section we consider the following statements.

(1) f is strongly transitive,
(2) f_n is strongly transitive,
(3) f_1^n is strongly transitive, and
(4) f_m^n is strongly transitive.

Using [4, Theorem 4.13], in [5, Theorem 3.17] it was shown that (2) implies (1), (2) implies (3), (2) implies (4), (3) implies (1), (4) implies (1), (1) does not imply (2), (1) does not imply (3) and (1) does not imply (4). The authors of [4] and [5] supposed that the spaces are continua, however it is easy to see that the proofs for these results are valid for non-degenerate compact metric spaces without isolated points.

The questions whether the rest of the implications hold are contained in [4, Question 4.1] and [5, Question 3.18]. With the following theorem we show that all these implications hold.

Theorem 4.1. *Let X be a space without isolated points, $f\colon X \to X$ a mapping and $1 \leq m < n$. If f_m^n is strongly transitive, then f_n is is strongly transitive.*

Proof. Let $\mathcal{U}$ be a nonempty open subset of $F_n(X)$.

Fix an element $A = \{a_1, \ldots, a_k\} \in \mathcal{U}$, where $k \leq n$ and the cardinality of A is k. Let $W_1', \ldots W_k'$ be pairwise disjoint open subsets of X such that for each $i \in \{1, \ldots, k\}$, $a_i \in W_i'$ and $\mathcal{W}' = \langle W_1', \ldots, W_k' \rangle \subset \mathcal{U}$. For each $i \in \{1, \ldots, k\}$, choose an open subset W_i of X such that $a_i \in W_i \subset \mathrm{cl}_X(W_i) \subset W_i'$. Let $\mathcal{W} = \langle W_1, \ldots, W_k \rangle$. Since X does not have isolated points, $F_n(X) \setminus F_m(X)$ is dense in $F_n(X)$ and the set $\mathcal{V} = \mathcal{W} \setminus F_m(X)$ is a nonempty open subset of $F_n(X)$. Observe that $q_m(\mathcal{V})$ is a nonempty open subset of $F_m^n(X)$. By hypothesis there exists $r \in \mathbb{N}$ such that $\bigcup_{i=0}^{r} (f_m^n)^i(q_m(\mathcal{V})) = F_m^n(X)$.

We claim that $\bigcup_{i=0}^{r} (f_n)^i(\mathcal{W}') = F_n(X)$. Take an element $B \in F_n(X)$. Let $\{B_s\}_{s=1}^{\infty}$ be a sequence in $F_n(X) \setminus F_{n-1}(X)$ such that $\lim_{s \to \infty} B_s = B$. Given $s \in \mathbb{N}$, there exist $D_s \in \mathcal{V}$ and $i_s \in \{0, 1, \ldots, r\}$ such that $(f_m^n)^{i_s}(q_m(D_s)) = q_m(B_s)$. This implies that $q_m(f^{i_s}(D_s)) = q_m(B_s)$.

Since $F_n(X)$ is compact, we may suppose that the sequence $\{D_s\}_{s=1}^{\infty}$ converges to an element $D \in F_n(X)$ and there exists $j \in \{0, 1, \ldots, r\}$ such that for each $s \in \mathbb{N}$, $i_s = j$.

Given $s \in \mathbb{N}$, $q_m(f^j(D_s)) = q_m(B_s)$. Since $B_s \notin F_m(X)$, we obtain that $f^j(D_s) = B_s$. By the continuity of f^j, $f^j(D) = B$. Since $D_s \in \mathcal{V} \subset \mathcal{W} \subset \mathrm{cl}_{F_n(X)}(\mathcal{W})$, we conclude that $D \in \mathrm{cl}_{F_n(X)}(\mathcal{W}) \subset \langle \mathrm{cl}_X(W_1), \ldots, \mathrm{cl}_X(W_k) \rangle \subset \langle W_1', \ldots, W_k' \rangle = \mathcal{W}'$. Therefore $B \in (f_n)^j(\mathcal{W}')$. This finishes the proof that $\bigcup_{i=0}^{r} (f_n)^i(\mathcal{W}') = F_n(X)$, so, $\bigcup_{i=0}^{r} (f_n)^i(\mathcal{U}) = F_n(X)$ and completes the proof of the theorem. □

5. Turbulence

Let X be a space. A mapping $f\colon X \to X$ is *turbulent* [7, p.588] if there are compact non-degenerate subsets K and L of X such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$.

Given $1 \leq m < n$, in this section we consider the following statements.

(1) f is turbulent,
(2) f_n is turbulent,
(3) f_1^n is turbulent, and
(4) f_m^n is turbulent.

Using [4, Theorem 5.6] in [5, Theorem 4.5] it follows that (1) implies (2), (3) and (4). In [5, Questions 4.6], it was asked whether one of the rest of the possible implications holds, when X is a continuum.

The following example shows that (2) and (3) does not imply (1), when X is a compact metric space.

Problem 5.1. *Does one of the statements (2), (3) or (4) implies another for a compact metric space?*

Example 5.2. There exist a non-degenerate compact metric space X and a mapping $f\colon X \to X$ such that f_2 and f_1^2 are turbulent but f is not turbulent.

Define $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. For each $m \in \mathbb{N}$, let $a_m = \frac{1}{3m-2}$, $b_m = \frac{1}{3m-1}$ and $c_m = \frac{1}{3m}$. Then

$$X = \{0\} \cup \{a_m : m \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{c_m : m \in \mathbb{N}\}.$$

Define $f\colon X \to X$ by

$$f(p) = \begin{cases} 0, & \text{if } p = 0, \\ c_k, & \text{if } p = a_{2k-1}, \\ b_k, & \text{if } p = a_{2k}, \\ a_k, & \text{if } p \in \{b_{2k}, c_{2k}, b_{2k-1}, c_{2k-1}\}. \end{cases}$$

Clearly, f is an onto mapping.

Suppose to the contrary that f is turbulent. Then there are compact non-degenerate subsets K and L of X such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$.

If there exists $k \geq 2$ such that $c_k \in K \cup L$, since $f^{-1}(c_k) = \{a_{2k-1}\}$, we have that $a_{2k-1} \in K \cap L$. Since $f^{-1}(a_{2k-1}) = \{b_{4k-2}, c_{4k-2}, b_{4k-3}, c_{4k-3}\}$, there is $p \in \{b_{4k-2}, c_{4k-2}, b_{4k-3}, c_{4k-3}\} \cap K$ such that $f(p) = a_{2k-1}$. Since $f^{-1}(p) = \{a_i\}$ for some $i > 4k-3 > 2k-1$, we have that $a_i \in K \cap L$. Thus $\{a_i, a_{2k-1}\} \subset K \cap L$, a contradiction. Thus $(K \cup L) \cap \{c_k : k \geq 2\} = \varnothing$. Similarly, $(K \cup L) \cap \{b_k : k \geq 2\} = \varnothing$. Therefore $K \cup L \subset \{a_k : k \in \mathbb{N}\} \cup \{b_1, c_1\} \cup \{0\}$.

If there exists $k \geq 2$ such that $a_k \in K \cup L$, then there exists $k' > 2$ such that $\{b_{k'}, c_{k'}\} \cup (K \cup L) \neq \varnothing$. This contradicts what we proved in the previous paragraph. Thus $K \cup L \subset \{a_1, b_1, c_1\} \cup \{0\}$. Since $(\{a_1, b_1, c_1\} \cup \{0\}) \cap f^{-1}(b_1) = \varnothing$, we have that $b_1 \notin K \cup L$. Hence $K \cup L \subset \{a_1, c_1\} \cup \{0\}$. Since $(\{a_1, c_1\} \cup \{0\}) \cap f^{-1}(a_1) = \{c_1\}$ and $(\{a_1, c_1\} \cup \{0\}) \cap f^{-1}(c_1) = \{a_1\}$, we obtain that if $\{a_1, c_1\} \cap (K \cup L) \neq \varnothing$, then $\{a_1, c_1\} \subset K \cap L$, a contradiction. This proves that $K \cup L \subset \{0\}$, a contradiction. This completes the proof that f is not turbulent.

Now, we check that f_2 is turbulent. Define

$$\mathcal{K} = \{\{a_m, b_m\} \in F_2(X) : m \in \mathbb{N}\} \cup \{\{0\}\}, \text{ and}$$
$$\mathcal{L} = \{\{a_m, c_m\} \in F_2(X) : m \in \mathbb{N}\} \cup \{\{0\}\}.$$

Then $\mathcal{K}$ and $\mathcal{L}$ are compact non-degenerate subsets of $F_2(X)$ and $\mathcal{K} \cap \mathcal{L} = \{\{0\}\}$.

Given $m \in \mathbb{N}$, $\{a_m, b_m\} = \{f(c_{2m}), f(a_{2m})\} = f_2(\{c_{2m}, a_{2m}\}) \in f_2(\mathcal{L})$. Moreover, $\{a_m, b_m\} = \{f(b_{2m}), f(a_{2m})\} = f_2(\{b_{2m}, a_{2m}\}) \in f_2(\mathcal{K})$. Since $\{0\} = \{f(0)\} = f(\{0\}) = f_2(\{0\}) \in f_2(\mathcal{K}) \cap f_2(\mathcal{L})$. We have shown that $\mathcal{K} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$. Similarly, $\mathcal{L} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$. Therefore, f_2 is turbulent.

Using $\mathcal{K}_0 = q_1(\mathcal{K})$ and $\mathcal{L}_0 = q_1(\mathcal{L})$, it can be proved that f_1^2 is turbulent.

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