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The largest topological ring of functions endowed with the m-topology

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Abstract

The purpose of this article is to identify the largest subring of the ring of all real valued functions on a Tychonoff space X, which forms a topological ring endowed with the m-topology.

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1. Introduction

For a Tychonoff (completely regular and Hausdorff) space X, $\mathbb{R}^X$ represents the ring of all real valued functions defined on X. Two of its important subrings are $C(X)$, the set of all continuous functions in $\mathbb{R}^X$, and $C^*(X)$, the set of all bounded and continuous members of $\mathbb{R}^X$. The algebraic properties of the rings $C(X)$ and $C^*(X)$ vis-à-vis topological properties of X have been studied extensively in the literature (see, [3]). Besides studying the algebraic properties of $C(X)$, one can also define many interesting topologies on $C(X)$. Consequently, one may study the interaction between various algebraic structures of $C(X)$ with the corresponding topology on it. Two commonly studied topologies on $C(X)$ and $C^*(X)$ are the u-topology (uniform topology) and the m-topology. Both of these topologies can be defined on $\mathbb{R}^X$ (definitions are given in the next section).

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Though the convergence concerning u-topology, popularly known as the uniform convergence, has been known for centuries, the m-topology was introduced by E. Hewitt ([5]) in 1948. In many aspects, the u-topology is the most relevant for studying $C^*(X)$, while the m-topology is appropriate to study the ring $C(X)$ (see, Theorem 1 and Theorem 3 in [5]). The m-topology has been studied in detail in [1, 2, 4, 6, 7, 8, 11, 12]. For more about u-topology and m-topology, we refer readers to the recent research monograph [13].

It is known that $C^*(X)$ equipped with the u-topology forms a topological ring while $C(X)$ need not. However, $C(X)$ equipped with the m-topology is a topological ring. But $\mathbb{R}^X$ with any of these topologies does not form a topological ring in general. There are several interesting subrings that are intermediate between $C^*(X)$ and $\mathbb{R}^X$, such as the ring of all Baire one functions and the ring of all locally bounded functions. The main results of the paper (Theorems 2.8 and 2.11) help to recognize such subrings, which are topological rings endowed with u-topology and m-topology. It can be shown that $C^*(X)$ is the largest subring of $C(X)$, which is a topological ring under the u-topology. It follows that a subring $S(X)$ of $C(X)$ is a topological ring for the u-topology if and only if $S(X) \subseteq C^*(X)$. This article aims to formulate the largest subrings of $\mathbb{R}^X$, which form topological rings when equipped with u-topology and m-topology, respectively.

2. Main Results

Throughout this article, X is assumed to be a Tychonoff space (though we may specify that it has some additional properties). We first recall the definitions of u-topology and m-topology.

Definition 2.1. The u-topology or uniform topology (denoted by τ^u) is determined on $\mathbb{R}^X$ by taking all sets of the form

$$B_u(f,\epsilon) = \{g \in \mathbb{R}^X : |f(x) - g(x)| < \epsilon, \forall x \in X\}; \quad (\epsilon > 0 \text{ is constant})$$

as a base for the neighborhood system at $f \in \mathbb{R}^X$.

Definition 2.2. The m-topology (denoted by τ^m) is determined on $\mathbb{R}^X$ by taking all sets of the form

$$B_m(f,\eta) = \{g \in \mathbb{R}^X : |f(x) - g(x)| < \eta(x), \forall x \in X\}; \quad (\eta \in U_+(X))$$

as a base for the neighborhood system at $f \in \mathbb{R}^X$. Here $U_+(X)$ represents the set of all positive units in $C(X)$.

Clearly, τ^m is finer than τ^u. These topologies coincide if and only if X is pseudocompact.

The u-topology can also be determined on $\mathbb{R}^X$ by using the positive units of $C^*(X)$ in a way given in the following proposition.

Proposition 2.3. *Let τ be the topology on $\mathbb{R}^X$ determined by taking all sets of the form*

$$B(f,\gamma) = \{g \in \mathbb{R}^X : |f(x) - g(x)| < \gamma(x), \forall x \in X\}; \quad (\gamma \in U_+^*(X))$$

as a base for the neighborhood system at f for each $f \in \mathbb{R}^X$, where $U_+^*(X)$ denotes the set of all positive units in $C^*(X)$. Then $\tau = \tau^u$ on $\mathbb{R}^X$.

There is a considerable difference between the units of the rings $C(X)$ and $C^*(X)$. A function $f \in C(X)$ is a positive unit of $C(X)$ if and only if $f(x) > 0$. But a function $f \in C^*(X)$ is a positive unit of $C^*(X)$ if and only if $f(x) > 0$ and $\frac{1}{f} \in C^*(X)$. Equivalently, $f \in C^*(X)$ is a positive unit of $C^*(X)$ if and only if $\inf\{f(x) : x \in X\} > 0$. So it is clear that every positive unit of $C^*(X)$ is also a positive unit of $C(X)$, but the converse need not be true.

To identify the largest subrings of $\mathbb{R}^X$, which are topological rings endowed with τ^u and τ^m, we define two families $B(X)$ and $\mathcal{D}(X)$ of functions in the following manner.

$$B(X) = \{f \in \mathbb{R}^X : \exists\ \psi \in U_+^*(X),\ |f(x)| < \psi(x) \text{ for all } x \in X\},$$
$$\mathcal{D}(X) = \{f \in \mathbb{R}^X : \exists\ \phi \in U_+(X),\ |f(x)| < \phi(x) \text{ for all } x \in X\}.$$

It is not hard to see that $B(X)$ and $\mathcal{D}(X)$ are subrings of $\mathbb{R}^X$, and $B(X)$ is the same as the family of all bounded functions in $\mathbb{R}^X$. Clearly, $C(X) \subseteq \mathcal{D}(X)$ and $B(X) \subseteq \mathcal{D}(X)$. Also $\mathcal{D}(X) = B(X)$ if and only if X is pseudocompact.

We now relate $\mathcal{D}(X)$ with two important subrings of $\mathbb{R}^X$, namely $B_1(X)$ and $LB(X)$ which denote respectively, the ring of all Baire one functions and the ring of all locally bounded functions. Recall that a function $f : X \to \mathbb{R}$ is called Baire one if f is the pointwise limit of a sequence of continuous functions from X to $\mathbb{R}$, and f is called locally bounded if it is bounded on some neighborhood of each point of X.

Clearly, $\mathcal{D}(X) \subseteq LB(X)$. In general, this containment is strict. To see an example of a space X for which $\mathcal{D}(X) \neq LB(X)$, we need the following definition and discussion.

Definition 2.4 (Horne, [9])**.** A space X is called a *cb-space* if for each $h \in LB(X)$, there exists $f \in C(X)$ such that $|h| \leq f$.

The next proposition follows immediately from the Definition 2.4.

Proposition 2.5. *$\mathcal{D}(X) = LB(X)$ if and only if X is a cb-space.*

It is known that a *cb*-space is countably paracompact, and a normal space is a *cb*-space if and only if it is countably paracompact ([10]). Consequently, for a non countably paracompact space X, $\mathcal{D}(X) \neq LB(X)$.

The following example proves that $B_1(X)$ and $\mathcal{D}(X)$ may not be comparable.

Example 2.6. Let $X = \mathbb{R}$ with the usual topology. Define the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in (-\infty, 0]$ and $f(x) = \frac{1}{x}$ for all $x \in (0, \infty)$. For each $n \in \mathbb{N}$, define a function $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0], \\ n^2 x & \text{if } x \in (0, 1/n), \\ \frac{1}{x} & \text{if } x \in [1/n, \infty). \end{cases}$$

Clearly, each f_n is continuous and the sequence (f_n) converges pointwise to f. Therefore, $f \in B_1(X)$. But $f \notin \mathcal{D}(X)$. Now define another function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = 0$ for every rational number x and $h(x) = 1$ for every irrational number x. We can easily see that $h \in \mathcal{D}(X) \setminus B_1(X)$.

Using Proposition 2.5 and Example 2.6, by Theorem 2.8, we can conclude that $LB(X)$ and $B_1(X)$ need not form topological rings endowed with the m-topology.

Theorem 2.7. *$\mathcal{D}(X)$ endowed with τ^m is a topological ring.*

Proof. The continuity of the map $(f, g) \to f + g$ is easy to check. We only prove that the map $(f, g) \to fg$ is continuous. Let $f, g \in \mathcal{D}(X)$. So there exist $\phi_f, \phi_g \in U_+(X)$ such that $|f(x)| < \phi_f(x)$ and $|g(x)| < \phi_g(x)$ for all $x \in X$. Let $B_m(fg, \eta)$ be any basic neighborhood of fg in $(\mathcal{D}(X), \tau^m)$ for some $\eta \in U_+(X)$. Consider the basic neighborhoods $B_m(f, \eta_1)$ and $B_m(g, \eta_2)$ of f and g respectively for $\eta_1 = \frac{\eta}{2(1+\phi_g)}$ and $\eta_2 = \frac{\eta}{2(\phi_f+\eta_1+1)}$. It is enough to show that for any $h_1 \in B_m(f, \eta_1)$ and $h_2 \in B_m(g, \eta_2)$, we have $h_1h_2 \in B_m(fg, \eta)$. It follows as

$$\begin{aligned} |(fg)(x) - (h_1h_2)(x)| &\le |g(x)||f(x) - h_1(x)| + |h_1(x)||g(x) - h_2(x)| \\ &< \phi_g(x)\eta_1(x) + |h_1(x)|\eta_2(x) \qquad \text{for all } x \in X \\ &< \eta(x) \qquad \text{for all } x \in X. \end{aligned}$$

□

Our next theorem establish the fact that $\mathcal{D}(X)$ is the largest subring of $\mathbb{R}^X$ which is a topological ring endowed with the m-topology.

Theorem 2.8. *Let $S(X)$ be a subring of $\mathbb{R}^X$. Then the following conditions are equivalent:*

(a) *$S(X)$ endowed with τ^m is a topological ring;*
(b) *$S(X) \subseteq \mathcal{D}(X)$.*

Proof. $(a) \Rightarrow (b)$. Suppose $S(X) \nsubseteq \mathcal{D}(X)$. Let $f \in S(X) \setminus \mathcal{D}(X)$. We show that pointwise multiplication $(f, g) \to fg$ is not continuous at point $(0_X, f)$, where 0_X is the constant function zero on X. Consider the basic neighborhood $B_m(0_X, 1)$ of the function $0_X f = 0_X$ in $(S(X), \tau^m)$. Since $f \notin \mathcal{D}(X)$, for every $\eta \in U_+(X)$ there exists a point $x_\eta \in X$ such that $|f(x_\eta)| \ge \dfrac{2}{\eta(x_\eta)}$, that is, $\left|\dfrac{\eta(x_\eta)}{2} f(x_\eta)\right| \ge 1$. Therefore for any $\eta, \mu \in U_+(X)$, we have $\dfrac{\eta}{2} \in B_m(0_X, \eta)$ and $f \in B_m(f, \mu)$ but $\dfrac{\eta}{2} f \notin B_m(0_X, 1)$.

$(b) \Rightarrow (a)$. It follows from Theorem 2.7. □

Corollary 2.9. *$LB(X)$ equipped with τ^m is a topological ring if and only if X is a cb-space.*

Corollary 2.10. *For a first countable space X, the following conditions are equivalent:*

(a) *X is discrete;*
(b) *$\mathbb{R}^X = \mathcal{D}(X)$;*
(c) *$\mathbb{R}^X$ endowed with τ^m is a topological ring;*
(d) *$\mathbb{R}^X = LB(X)$.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate and the implications $(b) \Rightarrow (c) \Rightarrow (d)$ follow from Theorem 2.8.

$(d) \Rightarrow (a)$. Suppose there is a non-isolated point $x_0 \in X$. Since X is first countable, there exists a sequence (x_n) of distinct points in $X \setminus \{x_0\}$ which converges to x_0. Define a function $f : X \to \mathbb{R}$ such that $f(x_n) = n$ for every $n \in \mathbb{N}$ and $f(x) = 0$ for all $x \in X \setminus \{x_n : n \in \mathbb{N}\}$. It is easy to see that $f \notin LB(X)$. We arrive at a contradiction. □

It should be noted that X being first countable is used only to prove the implication $(d) \Rightarrow (a)$. It may be interesting to know whether Corollary 2.10 is true for any Tychonoff space X.

Theorem 2.11. *$B(X)$ is the largest subring of $\mathbb{R}^X$, which is a topological ring endowed with τ^u.*

Proof. It can be proved in a manner similar to Theorems 2.7 and 2.8. □

Corollary 2.12. *$\mathbb{R}^X$ equipped with τ^u is a topological ring if and only if X is finite.*

Corollary 2.13. *Every subring of $\mathbb{R}^X$ which forms a topological ring under τ^u is also a topological ring under τ^m.*

Corollary 2.14. *For a space X, the following conditions are equivalent:*

(a) *$LB(X)$ endowed with τ^u is a topological ring;*
(b) *$B(X) = LB(X)$;*
(c) *X is a pseudocompact cb-space;*
(d) *X is countably compact.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Theorem 2.11 and $(c) \Leftrightarrow (d)$ follows from Theorem 9 of [10].

$(b) \Leftrightarrow (c)$. It follows from Proposition 2.5 and the fact that $\mathcal{D}(X) = B(X)$ if and only if X is pseudocompact. □

We conclude this article with the following question.

Question 2.15. *For what spaces X, the ring $B_1(X)$ endowed with τ^m or τ^u forms a topological ring?*

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