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Fixed point theorems for a new class of nonexpansive mappings

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Abstract

We consider a new class of nonlinear mappings that generalizes two well-known classes of nonexpansive type mappings and extends some other classes of mappings. We present some existence and convergence results for this class of mappings. Some illustrative examples presented herein show the generality of the obtained results.

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1. Introduction

Let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$ of a Banach space $(\mathcal{X}, \|.\|)$. A self-mapping $\Psi : \mathcal{K} \to \mathcal{K}$ is 1-Lipschitz or nonexpansive if

$$\|\Psi(\sigma) - \Psi(\upsilon)\| \leq \|\sigma - \upsilon\|$$

for all $\sigma, \upsilon \in \mathcal{K}$. A fixed point σ of the mapping Ψ is the point at which the mapping is invariant, that is, $\Psi(\sigma) = \sigma$. In 1965, Browder [6, 7], Göhde [9] and Kirk [10] initiated the existence theory for fixed points of nonexpansive mapping, independently (cf. [8]). In general nonexpansive mapping are uniformly continuous on their domains. To generalize, extend and accommodate discontinuous nonexpansive type mappings, many authors considered various

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classes of mappings [18, 11, 19, 2, 3, 14, 1, 8] for more details, see [15]. In 2008, Suzuki [18] considered a more general class of nonexpansive mappings (also known as Suzuki type generalized nonexpansive mapping) and presented some interesting results for these mappings:

Definition 1.1 ([18])**.** Assume that $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$. A mapping $\Psi : \mathcal{K} \to \mathcal{K}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\sigma - \Psi(\sigma)\| \leq \|\sigma - \upsilon\| \text{ implies } \|\Psi(\sigma) - \Psi(\upsilon)\| \leq \|\sigma - \upsilon\|$$

for all $\sigma, \upsilon \in \mathcal{K}$.

In 2011, Aoyama and Kohsaka [3] introduced another class of nonexpansive type mappings (called as α-nonexpansive mappings). This class of mappings generalizes several classes of mappings including λ-hybrid and nonspreading mappings. For more details one may refer to [11, 19, 2].

Definition 1.2. Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{X}$ and $\Psi : \mathcal{K} \to \mathcal{K}$ a self-mapping. Then Ψ is an α-nonexpansive if there exists an $\alpha < 1$ such that

$$\begin{aligned} \|\Psi(\sigma) - \Psi(\upsilon)\|^2 \leq \quad & \alpha\|\Psi(\sigma) - \upsilon\|^2 + \alpha\|\Psi(\upsilon) - \sigma\|^2 \\ & +(1-2\alpha)\|\sigma - \upsilon\|^2 \end{aligned} \tag{1.1}$$

for all $\sigma, \upsilon \in \mathcal{K}$.

Remark 1.3. Even though, the class of α-nonexpansive mappings was considered in [3] for any real number $\alpha < 1$, but Ariza-Ruiz *et al.* [4] pointed out that for $\alpha < 0$, this concept is trivial (see also [17]).

We note that α-nonexpansive and mappings satisfying the condition (C) are independent, and need not be continuous on their domains of definitions, unlike nonexpansive mappings. A couple of examples below illustrate these facts.

Example 1.4. Let $\mathcal{K} = [0, 5] \subset \mathbb{R}$ with the usual norm on $\mathbb{R}$. Assume that $\Psi : \mathcal{K} \to \mathcal{K}$ is a self-mapping defined as:

$$\Psi(\sigma) = \begin{cases} 1 - \sigma, & \text{if } \sigma \in [0, 1] \\ 0, & \text{if } \sigma \in (1, 5) \\ 1, & \text{if } \sigma = 5. \end{cases}$$

If $\sigma < \upsilon$ and $(\sigma, \upsilon) \in ([0, 5] \times [0, 5]) \backslash ((4, 5) \times \{5\})$, then it can be easily seen that $\|\Psi(\sigma) - \Psi(\upsilon)\| \leq \|\sigma - \upsilon\|$ holds. If $\sigma \in (4, 5)$ and $\upsilon = 5$, then

$$\frac{1}{2}\|\sigma - \Psi(\sigma)\| = \frac{\sigma}{2} > 1 > \|\sigma - \upsilon\| \text{ and } \frac{1}{2}\|\upsilon - \Psi(\upsilon)\| = 2 > \|\sigma - \upsilon\|.$$

Hence Ψ satisfies condition (C).

Contrarily, at $\sigma = 0$ and $\upsilon = 1$

$$\alpha\|\Psi(\sigma) - \upsilon\|^2 + \alpha\|\Psi(\upsilon) - \sigma\|^2 + (1 - 2\alpha)\|\sigma - \upsilon\|^2 = 1 - 2\alpha \leq 1 = \|\Psi(\sigma) - \Psi(\upsilon)\|^2$$

holds only if $\alpha = 0$. But for $\alpha = 0$, and $\sigma = \frac{9}{2}$, $\upsilon = 5$, we get

$$\|\Psi(\sigma) - \Psi(\upsilon)\| = 1 > \frac{1}{2} = \|\sigma - \upsilon\|.$$

Therefore, Ψ is not an α-nonexpansive mapping for any $\alpha \in [0, 1)$.

Example 1.5. [14]. Let $\mathcal{K} = [0, 4] \subset \mathbb{R}$ endowed with the usual norm. Define $\Psi : \mathcal{K} \to \mathcal{K}$ as follows:

$$\Psi(\sigma) = \begin{cases} 0, & \text{if } \sigma \neq 4, \\ 2, & \text{if } \sigma = 4. \end{cases}$$

Then it can be easily verified that Ψ is α-nonexpansive mapping for $\alpha \geq \frac{1}{2}$. However Ψ is not a mpping satisfying the condition (C) for $\sigma \in (2, 3]$ and $\upsilon = 4$.

In [14], we introduced the following class of mappings:

Definition 1.6. Suppose $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$, and $\Psi : \mathcal{K} \to \mathcal{K}$ a self-mapping. Then Ψ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ such that

$$\begin{aligned} \frac{1}{2}\|\sigma - \Psi(\sigma)\| &\leq \|\sigma - \upsilon\| \text{ implies} \\ (1.2)\|\Psi(\sigma) - \Psi(\upsilon)\| &\leq \alpha\|\Psi(\sigma) - \upsilon\| + \alpha\|\Psi(\upsilon) - \sigma\| + (1 - 2\alpha)\|\sigma - \upsilon\| \end{aligned}$$

for all $\sigma, \upsilon \in \mathcal{K}$.

The implication in inequality (1.2) is more restrictive than in (1.1), and therefore the above mapping does not contain α-nonexpansive mapping, properly. The present paper deals with this problem. Indeed, we consider a class of mappings which properly contains the class of α-nonexpansive mappings. To show the generality of the class of mappings considered herein, we present some illustrative examples. We also obtain the Demi-closedness principle in Banach spaces. Further, we employ a three step iterative method to approximate the fixed point of mapping considered herein.

2. Preliminaries

Now onwards, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N}$ the set of natural numbers.

Definition 2.1. Assume that $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$. A self-mapping $\Psi : \mathcal{K} \to \mathcal{K}$ is a quasinonexpansive mapping if

$$\|\Psi(\sigma) - w^\dagger\| \leq \|\sigma - w^\dagger\|$$

for all $\sigma \in \mathcal{K}$ and $w^\dagger \in F(\Psi)$.

A Banach space $\mathcal{X}$ is said to be *uniformly convex,* for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that the following holds: for each $\sigma, \upsilon \in \mathcal{X}$

$$\left.\begin{aligned} \|\sigma\| &\leq 1 \\ \|\upsilon\| &\leq 1 \\ \|\sigma - \upsilon\| &\geq \varepsilon \end{aligned}\right\} \Rightarrow \left\|\frac{\sigma + \upsilon}{2}\right\| \leq 1 - \delta.$$

Definition 2.2 ([16])**.** Let $\mathcal{X}$ be a normed space and $\mathcal{K}$ nonempty subset of $\mathcal{X}$. A mapping $\Psi : \mathcal{K} \to \mathcal{K}$ is said to satisfy Condition (I) if there exists a function $f : [0, \infty) \to [0, \infty)$ with the following properties:

- f is nondecreasing;
- $f(r) > 0$ for all $r \in (0, \infty)$ and $f(0) = 0$;
- $\|\sigma - \Psi(\sigma)\| \geq f(d(\sigma, F(\Psi)))$ for all $\sigma \in \mathcal{K}$,

where $d(x, F(\Psi))$ denotes distance of x from $F(\Psi)$.

A Banach $\mathcal{X}$ satisfies the Opial conditions [13] if for each weakly convergent sequence $\{\sigma_n\} \subset \mathcal{X}$ having weak limit σ, we have

$$\liminf_{n\to\infty} \|\sigma_n - \sigma\| < \liminf_{n\to\infty} \|\sigma_n - \upsilon\|$$

for all $\upsilon \in \mathcal{X}$, $\sigma \neq \upsilon$. It can be easily seen that on passing through appropriate subsequences, the lower limit can be replaced with upper limits in Opial property. The sequence $\{\sigma_n\}$ is *an approximate fixed point sequence* for Ψ (in short, a.f.p.s.) if $\lim_{n\to\infty} \|\sigma_n - \Psi(\sigma_n)\| = 0$.

3. C-α NONEXPANSIVE MAPPING

We introduce the following notion of C-α nonexpansive mapping

Definition 3.1. Suppose $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$ and $\Psi : \mathcal{K} \to \mathcal{K}$ a self-mapping. We say Ψ is a C-α nonexpansive mapping if

$$\begin{aligned} \frac{1}{2}\|\sigma - \Psi(\sigma)\| &\leq \|\sigma - \upsilon\| \text{ implies } \|\Psi(\sigma) - \Psi(\upsilon)\|^2 \\ &\leq \alpha\|\Psi(\sigma) - \upsilon\|^2 + \alpha\|\Psi(\upsilon) - \sigma\|^2 + (1 - 2\alpha)\|\sigma - \upsilon\|^2 \end{aligned} \tag{3.1}$$

for all $\sigma, \upsilon \in \mathcal{K}$, where $\alpha \in [0, 1)$.

We discuss some fundamental properties of C-α nonexpansive mapping.

Proposition 3.2. *Let $\Psi : \mathcal{K} \to \mathcal{K}$ be a mapping satisfying the condition (C). Then Ψ is a C-α nonexpansive mapping.*

In the next example we show that the reverse implication is not true, in general.

Example 3.3. Let $(\ell^2, \|.\|_2)$ be the Banach space of square-summable sequences endowed with its standard norm. Assume that $\{e_n\}$ is the canonical basis of ℓ^2. Define

$$\mathcal{K} := \overline{\text{conv}}\{e_1, e_2\} = \{\mu e_1 + (1 - \mu)e_2 : \mu \in [0, 1]\},$$

where $\overline{\text{conv}}\{e_1, e_2\}$ denotes the convex closure of $\{e_1, e_2\}$. Now, define $\Psi : \mathcal{K} \to \mathcal{K}$ as follows:

$$\Psi(\mu e_1 + (1 - \mu)e_2) = \begin{cases} e_1, & \text{if } \mu = 0, \\ \dfrac{(e_1 + e_2)}{2}, & \text{otherwise.} \end{cases}$$

Then Ψ is C-$\frac{1}{3}$ nonexpansive mapping. Indeed, if $\sigma := e_2$ and $\upsilon := \mu e_1 + (1-\mu)e_2$, $\mu \in (0,1]$, we have

$$\|\Psi(\sigma)-\Psi(\upsilon)\|_2 = \left\|e_1 - \left(\frac{e_1+e_2}{2}\right)\right\|_2 = \left\|\frac{e_1-e_2}{2}\right\|_2 = \frac{\sqrt{2}}{2},$$

$$\|\Psi(\upsilon)-\sigma\|_2 = \left\|\frac{e_1+e_2}{2} - e_2\right\|_2 = \left\|\frac{e_1-e_2}{2}\right\|_2 = \frac{\sqrt{2}}{2},$$

$$\|\Psi(\sigma)-\upsilon\|_2 = \|e_1 - (\mu e_1 + (1-\mu)e_2)\|_2 = (1-\mu)\sqrt{2}$$

$$\|\sigma-\upsilon\|_2 = \|e_2 - (\mu e_1 + (1-\mu)e_2)\|_2 = \mu\sqrt{2}.$$

Therefore, for $\alpha = \frac{1}{3}$

$$\begin{aligned}\frac{1}{3}\|\Psi(\sigma)-\upsilon\|_2 + \frac{1}{3}\|\Psi(\upsilon)-\sigma\|_2 + \left(1-\frac{2}{3}\right)\|\sigma-\upsilon\|_2 &= \frac{1}{3}(1-\mu)\sqrt{2} + \frac{1}{3}\frac{\sqrt{2}}{2} + \frac{1}{3}\mu\sqrt{2} \\ &= \frac{\sqrt{2}}{2} = \|\Psi(\sigma)-\Psi(\upsilon)\|_2.\end{aligned}$$

By the convexity of function $t \mapsto t^2$, we obtain

$$(\|\Psi(\sigma)-\Psi(\upsilon)\|_2)^2 \leq \frac{1}{3}(\|\Psi(\sigma)-\upsilon\|_2)^2 + \frac{1}{3}(\|\Psi(\upsilon)-\sigma\|_2)^2 + \left(1-\frac{2}{3}\right)(\|\sigma-\upsilon\|_2)^2.$$

Contrarily, if $\sigma := e_2$ and $\upsilon := \frac{1}{3}e_1 + \frac{2}{3}e_2$, then

$$\begin{aligned}\frac{1}{2}\|\upsilon-\Psi(\upsilon)\|_2 = \frac{1}{2}\left\|\frac{1}{3}e_1 + \frac{2}{3}e_2 - \frac{(e_1+e_2)}{2}\right\|_2 &= \frac{1}{12}\|e_1-e_2\|_2 \\ &= \frac{1}{12}\sqrt{2} \leq \frac{1}{3}\sqrt{2} = \|\sigma-\upsilon\|_2\end{aligned}$$

and $\|\Psi(\sigma)-\Psi(\upsilon)\|_2 = \frac{\sqrt{2}}{2} > \frac{1}{3}\sqrt{2} = \|\sigma-\upsilon\|_2$. Therefore, Ψ does not satisfy the criterion of condition (C). Note that $\dfrac{(e_1+e_2)}{2}$ is a fixed point of Ψ.

A generalized α-nonexpansive mapping is C-α nonexpansive mapping but the reverse implication is not true (see Example 3.5 below).

Proposition 3.4. *Assume that $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$ and $\Psi : \mathcal{K} \to \mathcal{K}$ a generalized α-nonexpansive mapping for all $\alpha \in [0, \frac{1}{2}]$. Then Ψ is C-α nonexpansive mapping for $\alpha \in [0, \frac{1}{2}]$.*

Proof. Let $\sigma, \upsilon \in \mathcal{K}$ and $\alpha \in [0, \frac{1}{2}]$. Note that $1-2\alpha \geq 0$. Since Ψ a generalized α-nonexpansive mapping, by implication in (1.2), we have

$$\|\Psi(\sigma)-\Psi(\upsilon)\| \leq \alpha\|\Psi(\sigma)-\upsilon\| + \alpha\|\Psi(\upsilon)-\sigma\| + (1-2\alpha)\|\sigma-\upsilon\|.$$

Considering the convexity of function $t \mapsto t^2$, we conclude that

$$\|\Psi(\sigma)-\Psi(\upsilon)\|^2 \leq \|\Psi(\sigma)-\upsilon\|^2 + \|\Psi(\upsilon)-\sigma\|^2 + (1-2\alpha)\|\sigma-\upsilon\|^2.$$

That is, Ψ is C-α nonexpansive mapping for all $\alpha \in [0, \frac{1}{2}]$. $\square$

Example 3.5. Let $\mathcal{K} = [0,3] \subset \mathbb{R}$ endowed with the usual norm in $\mathbb{R}$. Define a mapping $\Psi : \mathcal{K} \to \mathcal{K}$ as follows:

$$\Psi(\sigma) = \begin{cases} \frac{\sigma}{2}, & \text{if } \sigma \neq 3, \\ \frac{5}{2}, & \text{otherwise.} \end{cases}$$

Then Ψ is C-α nonexpansive mapping for $\alpha = \frac{3}{4}$. Indeed, if $\sigma, \upsilon \neq 3$, then

$$\begin{aligned} & \frac{3}{4}|\Psi(\sigma) - \upsilon|^2 + \frac{3}{4}|\Psi(\upsilon) - \sigma|^2 + \left(1 - 2 \times \frac{3}{4}\right)|\sigma - \upsilon|^2 \\ = \;& \frac{3}{4}\left|\frac{\sigma}{2} - \upsilon\right|^2 + \frac{3}{4}\left|\frac{\upsilon}{2} - \sigma\right|^2 - \frac{1}{2}|\sigma - \upsilon|^2 \\ = \;& \frac{7}{16}\sigma^2 + \frac{7}{16}\upsilon^2 - \frac{1}{2}\sigma\upsilon \\ = \;& \left(\frac{1}{4}\sigma^2 + \frac{1}{4}\upsilon^2 - \frac{1}{2}\sigma\upsilon\right) + \frac{3}{16}\sigma^2 + \frac{3}{16}\upsilon^2 \\ \geq \;& \frac{1}{4}\sigma^2 + \frac{1}{4}\upsilon^2 - \frac{1}{2}\sigma\upsilon \\ = \;& \left|\frac{\sigma}{2} - \frac{\upsilon}{2}\right|^2 = |\Psi(\sigma) - \Psi(\upsilon)|^2. \end{aligned}$$

Again if $\sigma = 3$ and $\upsilon \neq 3$, then

$$\begin{aligned} & \frac{3}{4}|\Psi(\sigma) - \upsilon|^2 + \frac{3}{4}|\Psi(\upsilon) - \sigma|^2 + \left(1 - 2 \times \frac{3}{4}\right)|\sigma - \upsilon|^2 \\ = \;& \frac{3}{4}\left|\frac{5}{2} - \upsilon\right|^2 + \frac{3}{4}\left|\frac{\upsilon}{2} - 3\right|^2 - \frac{1}{2}|3 - \upsilon|^2 \\ = \;& \frac{7}{16}\upsilon^2 - \frac{12}{4}\upsilon + \frac{111}{16} \\ = \;& \left(\frac{1}{4}\upsilon^2 - \frac{10}{4}\upsilon + \frac{100}{16}\right) + \frac{3}{16}\upsilon^2 - \frac{1}{2}\upsilon + \frac{11}{16}. \end{aligned}$$

Since $\frac{3}{16}\upsilon^2 - \frac{1}{2}\upsilon + \frac{11}{16} \geq 0$ for all $\upsilon \in [0,3]$, we have

$$\begin{aligned} & \frac{3}{4}|\sigma - \Psi(\upsilon)|^2 + \frac{3}{4}|\Psi(\sigma) - \upsilon|^2 + \left(1 - 2 \times \frac{3}{4}\right)|\sigma - \upsilon|^2 \\ \geq \;& \frac{1}{4}\upsilon^2 - \frac{10}{4}\upsilon + \frac{100}{16} \\ = \;& \left|\frac{5}{2} - \frac{\upsilon}{2}\right|^2 = |\Psi(\sigma) - \Psi(\upsilon)|^2. \end{aligned}$$

Contrarily at $\sigma = 3$ and $\upsilon = 2$, we get

$$\frac{1}{2}|\sigma - \Psi(\sigma) = \frac{1}{2}\left|3 - \frac{5}{2}\right| = \frac{1}{4} \leq 1 = |3 - 2| = |\sigma - \upsilon|$$

and

$$\begin{aligned} & \alpha|\Psi(3)-2|+\alpha|\Psi(2)-3|+(1-2\alpha)|3-2|= \\ = \; & \alpha\left|\frac{5}{2}-2\right|+\alpha|1-3|+1-2\alpha \\ = \; & \frac{1}{2}\alpha+2\alpha+1-2\alpha=1+\frac{1}{2}\alpha \\ < \; & \frac{3}{2}=\left|\frac{5}{2}-\frac{2}{2}\right|=|\Psi(\sigma)-\Psi(\upsilon)|. \end{aligned}$$

Hence Ψ is not a generalized α-nonexpansive mapping for any value of $\alpha \in [0,1)$.

Proposition 3.6. *Every α-nonexpansive is C-α nonexpansive mapping, but the converse is not true.*

Example 3.7. Let $(\ell^\infty, \|.\|_\infty)$ be the Banach space of all bounded real sequences endowed with the supremum norm. Assume that $\{e_n\}$ is the canonical basis of ℓ^∞. Define

$$\mathcal{K} := \{\mu e_1 : \mu \in [0,1]\}$$

Define $\Psi : \mathcal{K} \to \mathcal{K}$ as follows:

$$\Psi(\mu e_1) = \begin{cases} 0, & \text{if } \mu \neq 1, \\ \dfrac{e_1}{3}, & \text{if } \mu = 1. \end{cases}$$

Then Ψ is a C-$\frac{1}{10}$ nonexpansive mapping. Indeed, if $\sigma = \mu_1 e_1$, $\upsilon = \mu_2 e_1$, where $\mu_1, \mu_2 \in [0,1)$ then

$$\|\Psi(\sigma)-\Psi(\upsilon)\|_\infty^2 = 0 \leq \frac{1}{10}\|\Psi(\sigma)-\upsilon\|_\infty^2+\frac{1}{10}\|\Psi(\upsilon)-\sigma\|_\infty^2+\left(1-2\times\frac{1}{10}\right)\|\sigma-\upsilon\|_\infty^2.$$

Again if, $\sigma = \mu_1 e_1$, where $\mu_1 \in [0, \frac{2}{3}]$ and $\upsilon = e_1$, then

$$\begin{aligned} & \frac{1}{10}\|\Psi(\sigma)-\upsilon\|_\infty^2+\frac{1}{10}\|\Psi(\upsilon)-\sigma\|_\infty^2+\left(1-2\times\frac{1}{10}\right)\|\sigma-\upsilon\|_\infty^2 \\ = \; & \frac{1}{10}\|e_1\|_\infty^2+\frac{1}{10}\left\|\frac{e_1}{3}-\sigma\right\|_\infty^2+\frac{4}{5}\|e_1-\sigma\|_\infty^2 \\ \geq \; & \frac{1}{10}+\frac{4}{5}\times\frac{1}{9}=\frac{17}{90}>\frac{1}{9}=\|\Psi(\sigma)-\Psi(\upsilon)\|_\infty^2. \end{aligned}$$

If $\sigma = \mu_1 e_1$, where $\mu_1 \in \left(\frac{2}{3}, 1\right)$ and $\upsilon = e_1$, then

$\frac{1}{2}\|\sigma-\Psi(\sigma)\|_\infty = \frac{1}{2}\|\sigma\|_\infty > \|e_1-\sigma\|_\infty$ and $\frac{1}{2}\|\upsilon-\Psi(\upsilon)\|_\infty = \frac{1}{2}\left\|e_1-\frac{e_1}{3}\right\|_\infty = \frac{1}{3} > \|e_1-\sigma\|_\infty$.

On the other hand, at $\sigma = \frac{9}{10}e_1$ and $\upsilon = e_1$,

$$\begin{aligned} & \frac{1}{10}\left\|\Psi\left(\frac{9}{10}e_1\right)-e_1\right\|_\infty^2+\frac{1}{10}\left\|\Psi(e_1)-\frac{9}{10}e_1\right\|_\infty^2+\left(1-2\times\frac{1}{10}\right)\left\|\frac{9}{10}e_1-e_1\right\|_\infty^2 \\ = \; & \frac{1}{10}+\frac{289}{9000}+\frac{8}{1000}=\frac{397}{9000}<\frac{1}{9}=\left\|\Psi\left(\frac{9}{10}e_1\right)-\Psi(e_1)\right\|_\infty^2. \end{aligned}$$

Thus Ψ is not $\frac{1}{10}$-nonexpansive mapping.

Proposition 3.8. *Suppose that $\mathcal{K}$ is a nonempty subset a Banach space $\mathcal{X}$ and $\Psi : \mathcal{K} \to \mathcal{K}$ a C-α nonexpansive mapping and has a fixed point. Then Ψ is quasinonexpansive.*

Proof. It follows from the proof of [14, Proposition 3.5]. □

Lemma 3.9. *Suppose that $\mathcal{K}$ is a nonempty subset a Banach space $\mathcal{X}$ and $\Psi : \mathcal{K} \to \mathcal{K}$ a C-α nonexpansive mapping. Then $F(\Psi)$ is closed. In addition, if $\mathcal{K}$ is convex and $\mathcal{X}$ is strictly convex, then $F(\Psi)$ is convex.*

Proof. The proof is much similar to proof [14, Lemma 3.6] □

4. Main results

Proposition 4.1 (Demiclosedness principle)**.** *Assume that $\mathcal{K}$ is a nonempty subset of a Banach space $\mathcal{X}$ which has the Opial property and $\Psi : \mathcal{K} \to \mathcal{K}$ be a C-α nonexpansive mapping. If $\{\sigma_n\}$ converges weakly to a point σ and $\lim_{n\to\infty} \|\Psi(\sigma_n) - \sigma_n\| = 0$ then $\Psi(\sigma) = \sigma$. That is, $I - \Psi$ is demiclosed at zero, where I is the identity mapping on $\mathcal{X}$.*

Proof. Since the sequence $\{\sigma_n\}$ is weakly convergent and $\lim_{n\to\infty} \|\Psi(\sigma_n) - \sigma_n\| = 0$, both sequences $\{\sigma_n\}$ and $\{\Psi(\sigma_n)\}$ are bounded. First we assume that $\limsup_{n\to\infty} \|\sigma_n - \sigma\| = 0$. Now by the triangle inequality, we get

$$\begin{aligned}\|\sigma - \Psi(\sigma)\| &\le \limsup_{n\to\infty} \|\sigma_n - \sigma\| + \limsup_{n\to\infty} \|\sigma_n - \Psi(\sigma)\| \\ &= \limsup_{n\to\infty} \|\sigma_n - \Psi(\sigma)\|.\end{aligned}$$

Indeed, by Opial property

$$\|\sigma - \Psi(\sigma)\| \le \limsup_{n\to\infty} \|\sigma_n - \Psi(\sigma)\| < \limsup_{n\to\infty} \|\sigma_n - \sigma\| = 0.$$

Thus $\Psi(\sigma) = \sigma$.

If we assume that $\limsup_{n\to\infty} \|\sigma_n - \sigma\| = r > 0$. Since $\lim_{n\to\infty} \|\Psi(\sigma_n) - \sigma_n\| = 0$, for large enough n, there exists a $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}\|\sigma_n - \Psi(\sigma_n)\| \le \|\sigma_n - \sigma\| \text{ for all } n \ge n_0.$$

By (4.5), we have

$$\|\Psi(\sigma_n) - \Psi(\sigma)\|^2 \le \alpha\|\Psi(\sigma_n) - \sigma\|^2 + \alpha\|\Psi(\sigma) - \sigma_n\|^2 + (1-2\alpha)\|\sigma_n - \sigma\|^2. \tag{4.1}$$

Now by the triangle inequality and (4.1), we have

$$\begin{aligned}
\|\sigma_n-\Psi(\sigma)\|^2 &\leq (\|\sigma_n-\Psi(\sigma_n)\|+\|\Psi(\sigma_n)-\Psi(\sigma)\|)^2\\
&\leq \|\sigma_n-\Psi(\sigma_n)\|^2+\|\Psi(\sigma_n)-\Psi(\sigma)\|^2+2\|\sigma_n-\Psi(\sigma_n)\|\|\Psi(\sigma_n)-\Psi(\sigma)\|\\
&\leq \|\sigma_n-\Psi(\sigma_n)\|^2+\alpha\|\Psi(\sigma_n)-\sigma\|^2+\alpha\|\Psi(\sigma)-\sigma_n\|^2\\
&\quad +(1-2\alpha)\|\sigma_n-\sigma\|^2+2\|\sigma_n-\Psi(\sigma_n)\|\|\Psi(\sigma_n)-\Psi(\sigma)\|\\
&\leq \|\sigma_n-\Psi(\sigma_n)\|^2+\alpha(\|\Psi(\sigma_n)-\sigma_n\|+\|\sigma_n-\sigma\|)^2+\alpha\|\Psi(\sigma)-\sigma_n\|^2\\
&\quad +(1-2\alpha)\|\sigma_n-\sigma\|^2+2\|\sigma_n-\Psi(\sigma_n)\|\|\Psi(\sigma_n)-\Psi(\sigma)\|\\
&\leq \|\sigma_n-\Psi(\sigma_n)\|^2+\alpha\|\Psi(\sigma_n)-\sigma_n\|^2+\alpha\|\sigma_n-\sigma\|^2\\
&\quad +2\alpha\|\Psi(\sigma_n)-\sigma_n\|\|\sigma_n-\sigma\|+\alpha\|\Psi(\sigma)-\sigma_n\|^2\\
&\quad +(1-2\alpha)\|\sigma_n-\sigma\|^2+2\|\sigma_n-\Psi(\sigma_n)\|\|\Psi(\sigma_n)-\Psi(\sigma)\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\sigma_n-\Psi(\sigma)\|^2 &\leq \frac{(1+\alpha)}{(1-\alpha)}\|\sigma_n-\Psi(\sigma_n)\|^2+\frac{2}{(1-\alpha)}(\alpha\|\sigma_n-\sigma\|+\|\Psi(\sigma_n)-\Psi(\sigma)\|)\\
&\quad \|\Psi(\sigma_n)-\sigma_n\|+\|\sigma_n-\sigma\|^2.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\|\sigma_n-\Psi(\sigma)\|^2\leq\limsup_{n\to\infty}\|\sigma_n-\sigma\|^2$$

as an application of Opial property we conclude that $\Psi(\sigma)=\sigma$. □

Theorem 4.2. *Suppose $\mathcal{X}$ is a Banach space having the Opial property. Assume that $\mathcal{K}$ is a nonempty subset of $\mathcal{X}$ and $\Psi:\mathcal{K}\to\mathcal{K}$ a C-α nonexpansive mapping such that Ψ admits an a.f.p.s.. Then Ψ has a fixed point.*

Proof. Demiclosedness principle implies the conclusion. □

Noor [12] considered the following iterative process:

$$\begin{cases}\sigma_1\in\mathcal{K}\\ \sigma_{n+1}=(1-\zeta_n)\sigma_n+\zeta_n\Psi(\upsilon_n)\\ \upsilon_n=(1-\gamma_n)\sigma_n+\gamma_n\Psi(w_n)\\ w_n=(1-\delta_n)\sigma_n+\delta_n\Psi(\sigma_n),\quad n\in\mathbb{N},\end{cases}\tag{4.2}$$

where $\{\zeta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$.

Lemma 4.3 ([20, p.484])**.** *Assume that $0<a\leq l_n\leq b<1$ for all $n\in\mathbb{N}$ and $\mathcal{X}$ is a uniformly convex Banach space. Suppose $\{\sigma_n\}$ and $\{\upsilon_n\}$ are sequences such that $\limsup_{n\to\infty}\|\sigma_n\|\leq r, \limsup_{n\to\infty}\|\upsilon_n\|\leq r$ and $\lim_{n\to\infty}\|l_n\sigma_n+(1-l_n)\upsilon_n\|=r$ hold for some $r\geq 0$. Then $\lim_{n\to\infty}\|\sigma_n-\upsilon_n\|=0$.*

Lemma 4.4. *Suppose $\mathcal{K}$ is a nonempty closed convex subset of a Banach space $\mathcal{X}$. Let $\Psi:\mathcal{K}\to\mathcal{K}$ be a C-α nonexpansive mapping. Let $\{\sigma_n\}$ be a sequence defined by (4.2). If $F(\Psi)\neq\varnothing$, then the following postulation hold:*

- **(1):** $\max\{\|\sigma_{n+1}-w^\dagger\|,\|\upsilon_n-w^\dagger\|,\|w_n-w^\dagger\|\}\leq\|\sigma_n-w^\dagger\|$ *for all $n\in\mathbb{N}$ and $w^\dagger\in F(\Psi)$;*

(2): $\lim_{n\to\infty} \|\sigma_n - w^\dagger\|$ *exists;*

(3): $\lim_{n\to\infty} d(\sigma_n, F(\Psi))$ *exists, where* $d(\sigma, F(\Psi))$ *denotes the distance from* σ *to* $F(\Psi)$.

Proof. In view (4.2) and Proposition 3.8, we get

$$\begin{aligned} \|w_n - w^\dagger\| &= \|(1-\delta_n)\sigma_n + \delta_n\Psi(\sigma_n) - w^\dagger\| \\ &\leq (1-\delta_n)\|\sigma_n - w^\dagger\| + \delta_n\|\Psi(\sigma_n) - w^\dagger\| \\ &\leq (1-\delta_n)\|\sigma_n - w^\dagger\| + \delta_n\|\sigma_n - w^\dagger\| \\ &= \|\sigma_n - w^\dagger\|. \end{aligned} \tag{4.3}$$

By (4.2), (4.3) and Proposition 3.8, we have

$$\begin{aligned} \|\upsilon_n - w^\dagger\| &= \|(1-\gamma_n)\sigma_n + \gamma_n\Psi(w_n) - w^\dagger\| \\ &\leq (1-\gamma_n)\|\sigma_n - w^\dagger\| + \gamma_n\|\Psi(w_n) - w^\dagger\| \\ &\leq (1-\gamma_n)\|\sigma_n - w^\dagger\| + \gamma_n\|w_n - w^\dagger\| \\ &\leq (1-\gamma_n)\|\sigma_n - w^\dagger\| + \gamma_n\|\sigma_n - w^\dagger\| \\ &= \|\sigma_n - w^\dagger\|. \end{aligned} \tag{4.4}$$

Using (4.2), (4.4) and Proposition 3.8, we get

$$\begin{aligned} \|\sigma_{n+1} - w^\dagger\| &= \|(1-\zeta_n)\sigma_n + \zeta_n\Psi(\upsilon_n) - w^\dagger\| \\ &\leq (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\Psi(\upsilon_n) - w^\dagger\| \\ &\leq (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\upsilon_n - w^\dagger\| \\ &\leq (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\sigma_n - w^\dagger\| \\ &= \|\sigma_n - w^\dagger\|. \end{aligned} \tag{4.5}$$

Combining (4.3), (4.4) and (4.5) proves (1). Also by (4.5) the sequence $\{\|\sigma_n - w^\dagger\|\}$ is bounded and hence monotone decreasing. Therefore $\lim_{n\to\infty}\|\sigma_n - w^\dagger\|$ exists and proves (2). Now, since $\|\sigma_{n+1} - w^\dagger\| \leq \|\sigma_n - w^\dagger\|$ for each $w^\dagger \in F(\Psi)$ and for all $n \in \mathbb{N}$, $d(\sigma_{n+1}, F(\Psi)) \leq d(\sigma_n, F(\Psi))$ for all $n \in \mathbb{N}$. Thus $\{d(\sigma_n, F(\Psi))\}$ is a bounded sequence and monotone decreasing. Hence, $\lim_{n\to\infty} d(\sigma_n, F(\Psi))$ exists. ☐

Theorem 4.5. *Let* $\mathcal{K}$, $\{\sigma_n\}$ *and* Ψ *be same as in Lemma 4.4. If* $F(\Psi) \neq \varnothing$ *and* $\mathcal{X}$ *is a uniformly convex Banach space. Then* $\lim_{n\to\infty}\|\Psi(\sigma_n) - \sigma_n\| = 0$.

Proof. Let $w^\dagger \in F(\Psi)$. Then from Lemma 4.4, $\{\sigma_n\}$ is bounded and $\lim_{n\to\infty}\|\sigma_n - w^\dagger\|$ exists. Call it r. That is

$$\lim_{n\to\infty}\|\sigma_n - w^\dagger\| = r. \tag{4.6}$$

In view of (4.6) and Proposition 3.8 it implies that

$$\limsup_{n\to\infty}\|\Psi(\sigma_n) - w^\dagger\| \leq r. \tag{4.7}$$

By (4.3) and (4.6), we get

$$\limsup_{n\to\infty} \|w_n - w^\dagger\| \le \lim_{n\to\infty} \|\sigma_n - w^\dagger\| = r. \tag{4.8}$$

From (4.4) and (4.6), we get

$$\limsup_{n\to\infty} \|\upsilon_n - w^\dagger\| \le r. \tag{4.9}$$

In view of (4.9) and Proposition 3.8 it follows that

$$\limsup_{n\to\infty} \|\Psi(\upsilon_n) - w^\dagger\| \le r. \tag{4.10}$$

Similarly,

$$\limsup_{n\to\infty} \|\Psi(w_n) - w^\dagger\| \le r. \tag{4.11}$$

By (4.2), (4.4) and Proposition 3.8, we have

$$\begin{aligned}
\|\sigma_{n+1} - w^\dagger\| &= \|(1-\zeta_n)\sigma_n + \zeta_n\Psi(\upsilon_n) - w^\dagger\| \\
&\le (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\Psi(\upsilon_n) - w^\dagger\| \\
&\le (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\upsilon_n - w^\dagger\| \\
&\le (1-\zeta_n)\|\sigma_n - w^\dagger\| + \zeta_n\|\sigma_n - w^\dagger\| \\
&= \|\sigma_n - w^\dagger\|,
\end{aligned}$$

or

$$\|\sigma_{n+1} - w^\dagger\| \le \|(1-\zeta_n)\sigma_n + \zeta_n\Psi(\upsilon_n) - w^\dagger)\| \le \|\sigma_n - w^\dagger\|,$$

it implies that

$$r \le \lim_{n\to\infty} \|(1-\zeta_n)\sigma_n + \zeta_n\Psi(\upsilon_n) - w^\dagger)\| \le r.$$

Then,

$$\lim_{n\to\infty} \|(1-\zeta_n)\sigma_n + \zeta_n\Psi(\upsilon_n) - w^\dagger)\| = r. \tag{4.12}$$

By (4.10), (4.11), (4.12) and Lemma 4.3, we get that

$$\lim_{n\to\infty} \|\sigma_n - \Psi(\upsilon_n)\| = 0. \tag{4.13}$$

In view of the triangle inequality and Proposition 3.8, we obtain

$$\begin{aligned}
\|\sigma_n - w^\dagger\| &\le \|\sigma_n - \Psi(\upsilon_n)\| + \|\Psi(\upsilon_n) - w^\dagger\| \\
&\le \|\sigma_n - \Psi(\upsilon_n)\| + \|\upsilon_n - w^\dagger\|
\end{aligned}$$

By (4.13), we have

$$\begin{aligned}
\lim_{n\to\infty} \|\sigma_n - w^\dagger\| &\le \lim_{n\to\infty} \|\sigma_n - \Psi(\upsilon_n)\| + \liminf_{n\to\infty} \|\upsilon_n - w^\dagger\| \\
&\le \liminf_{n\to\infty} \|\upsilon_n - w^\dagger\|.
\end{aligned}$$

It follows that

$$r \le \liminf_{n\to\infty} \|\upsilon_n - w^\dagger\|. \tag{4.14}$$

Using (4.2), (4.9) and (4.14), we get

$$r = \lim_{n\to\infty} \|\upsilon_n - w^\dagger\| = \|(1-\gamma_n)(\sigma_n - w^\dagger) + \gamma_n(\Psi(w_n) - w^\dagger)\|. \tag{4.15}$$

In view of Lemma 4.3, and (4.6), (4.11), (4.15), we have

$$\lim_{n\to\infty} \|\sigma_n - \Psi(w_n)\| = 0. \tag{4.16}$$

By triangle inequality and Proposition 3.8, we have

$$\begin{aligned} \|\sigma_n - w^\dagger\| &\leq \|\sigma_n - \Psi(w_n)\| + \|\Psi(w_n) - w^\dagger\| \\ &\leq \|\sigma_n - \Psi(w_n)\| + \|w_n - w^\dagger\|, \end{aligned}$$

using (4.16) it follows that

$$r \leq \liminf_{n\to\infty} \|w_n - w^\dagger\|. \tag{4.17}$$

Combining (4.8) and (4.17) together we get

$$\lim_{n\to\infty} \|w_n - w^\dagger\| = 0. \tag{4.18}$$

By (4.2) and Proposition 3.8, we have

$$\begin{aligned} \|w_n - w^\dagger\| &= \|(1-\delta_n)\sigma_n + \delta_n\Psi(\sigma_n) - w^\dagger\| \\ &\leq (1-\delta_n)\|\sigma_n - w^\dagger\| + \delta_n\|\Psi(\sigma_n) - w^\dagger\| \\ &\leq (1-\delta_n)\|\sigma_n - w^\dagger\| + \delta_n\|\sigma_n - w^\dagger\| \\ &= \|\sigma_n - w^\dagger\|. \end{aligned}$$

This implies that

$$r \leq \lim_{n\to\infty} \|(1-\delta_n)(\sigma_n - w^\dagger) + \delta_n(\Psi(\sigma_n) - w^\dagger)\| \leq r.$$

Therefore, we get

$$\lim_{n\to\infty} \|(1-\delta_n)(\sigma_n - w^\dagger) + \delta_n(\Psi(\sigma_n) - w^\dagger)\| = r. \tag{4.19}$$

In view of Lemma 4.3 and (4.6), (4.7), (4.19), it follows that $\lim_{n\to\infty} \|\Psi(\sigma_n) - \sigma_n\| = 0$. □

Theorem 4.6. *Let $\mathcal{X}$ be a uniformly convex Banach space having the Opial's property, $\mathcal{K}$, Ψ and $\{\sigma_n\}$ same as in Theorem 4.5. If $F(\Psi) \neq \varnothing$ then $\{\sigma_n\}$ weakly converges to a fixed point of Ψ.*

Proof. This can be completed following [14, Theorem 5.8]. □

Theorem 4.7. *Suppose that $\mathcal{K}$, $\mathcal{X}$, $\{\sigma_n\}$ and Ψ are same as in Lemma 4.4. Let $F(\Psi) \neq \varnothing$ and $\liminf_{n\to\infty} d(\sigma_n, F(\Psi)) = 0$. Then the sequence $\{\sigma_n\}$ strongly converges to a fixed point of Ψ.*

Proof. This can be completed following [14, Theorem 5.9]. □

Theorem 4.8. *Assume that $\mathbb{K}$ is a subset of a uniformly convex Banach space$\mathcal{X}$. Let Ψ and $\{\sigma_n\}$ are same as in Theorem 4.5 and. Let Ψ satisfy condition (I) with $F(\Psi) \neq \varnothing$. Then the sequence $\{\sigma_n\}$ strongly converges to a fixed point of Ψ.*

Proof. This can be completed following [14, Theorem 5.10]. $\square$

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