APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 23, no. 2 (2022), 425-436
doi:10.4995/agt.2022.16940

Fredholm theory for demicompact linear relations

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Communicated by D. Werner

Abstract

We first attempt to determine conditions on a linear relation T such that μT becomes a demicompact linear relation for each $\mu \in [0,1)$ (see Theorems 2.4 and 2.5). Second, we display some results on Fredholm and upper semi-Fredholm linear relations involving a demicompact one (see Theorems 3.1 and 3.2). Finally, we provide some results in which a block matrix of linear relations becomes a demicompact block matrix of linear relations (see Theorems 4.2 and 4.3).

2020 MSC: *47A06.*

Keywords: *demicompact linear relations; Fredholm theory; block matrix.*

1. Introduction

Throughout this work, X, Y and Z are vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A mapping T, whose domain is a linear subspace

$$\mathcal{D}(T) := \{x \in X : Tx \neq \varnothing\}$$

of X, is called a linear relation (or a multivalued linear operator) if for all $x, z \in \mathcal{D}(T)$ and non-zero scalars α; we have

$$Tx + Tz = T(x + z)$$

$$\alpha Tx = T(\alpha x).$$

Evidently, the domain of linear relation is a linear subspace.

Received 25 December 2021 – Accepted 10 May 2022

In this notation, $\mathcal{LR}(X,Y)$ denotes the class of all linear relations on X into Y, if $X=Y$ simply denotes $\mathcal{LR}(X,X):=\mathcal{LR}(X)$. If T maps the points of its domain to singletons, then it is said to be a single valued linear operator (or simply an operator). The simplest naturally occurring example of a multivalued linear operator is the inverse T^{-1} of a linear map T from X to Y defined by the set of solutions

$$T^{-1}y := \{x \in X : Tx = y\}$$

for equation $Tx=y$. Each linear relation is identified only by its graph, $G(T)$, which is defined by

$$G(T) := \{(x,y) \in X \times Y : x \in \mathcal{D}(T) \ \ and \ \ y \in Tx\}.$$

The inverse of T is the linear relation, T^{-1} expressed by

$$G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}.$$

The subspace

$$\mathcal{N}(T) := \{x \in \mathcal{D}(T) \ \ such \ \ that \ \ (x,0) \in G(T)\}$$

is called the null space of T, and T is called injective if $\mathcal{N}(T)=\{0\}$, that is, if T^{-1} is a single valued linear operator.

$$T^{-1}(0) := \mathcal{N}(T).$$

The range of T is the subspace

$$\mathcal{R}(T) := \{y \in Y, \exists x \in \mathcal{D}(T) : (x,y) \in G(T)\}$$

and T is called surjective if $\mathcal{R}(T)=Y$. If T is injective and surjective, then we state that T is bijective. The quantities

$$\alpha(T) := dim\ (\mathcal{N}(T)) \ \ and \ \ \beta(T) := codim(\mathcal{R}(T)) = dim(Y/_{\mathcal{R}(T)})$$

are called the nullity (or the kernel index) and the deficiency of T, respectively. We also write $\overline{\beta}(T) := codim(\overline{\mathcal{R}(T)})$. The index of T is defined by $i(T) := \alpha(T)-\beta(T)$. If $\alpha(T)$ and $\beta(T)$ are infinite, then T is said to have no index. Let M be a subspace of X such that $M \cap \mathcal{D}(T) \neq \varnothing$ and let $T \in \mathcal{LR}(X,Y)$; then, the restriction $T_{|M}$, is the linear relation indicated by

$$G(T_{|M}) := \{(m,y) \in G(T) : m \in M\} = G(T) \cap (M \times Y).$$

For S, $T \in \mathcal{LR}(X,Y)$ and $R \in \mathcal{LR}(Y,Z)$, the sum $S+T$ and the product RS are the linear relations determined by

$$G(T+S) := \{(x, y+z) \in X \times Y : (x,y) \in G(T)\ and\ (x,z) \in G(S)\},\ and$$

$$G(RS) := \{(x,z) \in X \times Z : (x,y) \in G(S), (y,z) \in G(R)\ for\ some\ y \in Y\},$$

respectively and if $\lambda \in \mathbb{K}$, the λT is computed by

$$G(\lambda T) := \{(x, \lambda y) : (x,y) \in G(T)\}.$$

If $T \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{K}$, then the linear relation $\lambda - T$ is identified by

$$G(\lambda - T) := \{(x, y - \lambda x) : (x,y) \in G(T)\}.$$

Let $T \in \mathcal{LR}(X,Y)$. We write Q_T for the quotient map from Y into $Y/\overline{T(0)}$. Clearly, $Q_T T$ is an operator. For all $x \in \mathcal{D}(T)$, we define $\|Tx\| := \|Q_T Tx\|$, and the norm of T is defined by $\|T\| := \|Q_T T\|$. We note that $\|Tx\|$ and $\|T\|$ are not real norms. In fact, a non-zero relation can have a zero norm. T is said to be closed if its graph $G(T)$ is a closed subspace of $X \times Y$. The closure of T denoted by $\overline{T}$ is defined in terms of its graph $G(\overline{T}) := \overline{G(T)}$. We denote by $\mathcal{CR}(X,Y)$ the class of all the closed linear relations on X into Y, if $X = Y$ which simply denotes $\mathcal{CR}(X,X) := \mathcal{CR}(X)$. If $\overline{T}$ is an extension to T, we say that T is closable. Let $T \in \mathcal{LR}(X,Y)$. We say that T is continuous if for each neighbourhood V in $\mathcal{R}(T)$, the inverse image $T^{-1}(V)$ is a neighbourhood in $\mathcal{D}(T)$ equivalently if $\|T\| < \infty$; open if T^{-1} is continuous, bounded if $\mathcal{D}(T) = X$ and T is continuous, bounded below if it is injective and open and compact if $\overline{Q_T T(B_{\mathcal{D}(T)})}$ is compact in Y ($B_{\mathcal{D}(T)} := \{x \in \mathcal{D}(T) : \|x\| \leq 1\}$). We denote by $\mathcal{KR}(X,Y)$ the class of all the compact linear relations on X into Y, if $X = Y$ simply denotes $\mathcal{KR}(X,X) := \mathcal{KR}(X)$.

If X is a normed linear space, then $X^{'}$ will denote the dual space of X, i.e., the space of all the continuous linear functionals x' which are defined on X, with the norm

$$\|x'\| = \inf\{\lambda : |x'x| \leq \lambda\|x\| \ for\ all\ x \in X\}.$$

If $K \subset X$ and $L \subset X^{'}$, we shall adopt the following notations:

$$\begin{aligned} K^{\perp} &:= \{x' \in X^{'} : x' = 0 \ for\ all\ x \in K\}, \\ L^{\top} &:= \{x \in X : x' = 0 \ for\ all\ x' \in L\}. \end{aligned}$$

Clearly, $K^{\perp}$ and $L^{\top}$ are closed linear subspaces of $X^{'}$ and X, respectively. Let $T \in \mathcal{LR}(X,Y)$. The adjoint of T, which is T', is defined by

$$G(T') = G(-T^{-1})^{\perp} \subset Y^{'} \times X^{'}$$

where $\left\langle (y,x),(y',x') \right\rangle := \langle x,x'\rangle + \langle y,y'\rangle$. This means that

$$(y',x') \in G(T') \text{ if, and only if, } y'y - x'x = 0 \text{ for all } (x,y) \in G(T).$$

Similarly, we have $y'y = x'x$ for all $y \in Tx$, $x \in \mathcal{D}(T)$. Hence, $x' \in T'y$ if, and only if, $y'Tx = x'x$ for all $x \in \mathcal{D}(T)$.

Definition 1.1 ([7, Definition, V.1.1])**.** (i) A linear relation $T \in \mathcal{LR}(X,Y)$ is said to be upper semi-Fredholm and denoted by $T \in \mathcal{F}_+(X,Y)$, if there exists a finite codimensional subspace M of X for which $T_{|M}$ is injective and open.
(ii) A linear relation T is said to be lower semi-Fredholm and denoted by $T \in \mathcal{F}_-(X,Y)$, if its conjugate T' is upper semi-Fredholm.

If $X = Y$, this simply denotes $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ by respectively $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$.
For the case, when X and Y are Banach spaces, we extend the class of closed single valued Fredholm type operators provided earlier to include closed multivalued operators. Note that the definitions of $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ are,

respectively, consistent with

$$\Phi_+(X,Y) := \{T \in \mathcal{CR}(X,Y) : \ R(T) \ is\ closed,\ and\ \alpha(T) < \infty\},$$

$$\Phi_-(X,Y) := \{T \in \mathcal{CR}(X,Y) : \ R(T) \ is\ closed,\ and\ \beta(T) < \infty\}.$$

If $X = Y$, this simply denotes $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by respectively $\Phi_+(X)$ and $\Phi_-(X)$.

Lemma 1.2 ([1, Lemma 2.1]). *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ be a closed linear relation. Then,*

(i) $T \in \Phi_+(X,Y)$ if, and only if, $Q_T T \in \Phi_+(X, Y/T(0))$.

(ii) $T \in \Phi_-(X,Y)$ if, and only if, $Q_T T \in \Phi_-(X, Y/T(0))$.

Definition 1.3 ([9]). Let X be a Banach space. Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be $\inf\{d > 0$ such that D can be covered by a finite number of sets of a diameter less than or equal to d $\}$.

The following Proposition displays some properties of the Kuratowski measure of noncompactness which are frequently used.

Proposition 1.4 ([9]). *Let D and D' be two bounded subsets of X. Then, we have the following properties:*

(i) $\gamma(D) = 0$ if, and only if, D is relatively compact.

(ii) if $D \subseteq D'$, then $\gamma(D) \leq \gamma(D')$.

(iii) $\gamma(D + D') \leq \gamma(D) + \gamma(D')$.

(iv) For every $\alpha \in \mathbb{C}$, $\gamma(\alpha D) = |\alpha|\gamma(D)$.

The linear relations, which were introduced into a functional analysis by J. Von Neumann, were motivated by the need to consider adjoints of non-densely defined linear differential operators. These linear relations were widely investigated in a large number of papers (see, for example, [2], [3] and [5]).

The notion of demicompactess for linear operators (that is, single valued operators) was introduced into the functional analysis by W.V Petryshyn [10], to discuss fixed points. Since this notion has be come a hot area of research triggering significant scientific concern, several research papers such as [8, 10] invested it in their investigation. In 2012, W. Chaker, A. Jeribi and B. Krichen achieved some results on Fredholm and upper semi-Fredholm operators involving demicompact operators [6].

In what follows, we shall present two definitions set forward by A. Ammar, H. Daoud and A. Jeribi in 2017 [4], who extended the concept of demicompact and k-set-contraction of linear operators on multivalued linear operators and developed some pertinent properties.

Definition 1.5 ([4, Definition 3.1]). A linear relation $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$, such that $Q_{I-T}(I-T)x_n = Q_T(I-T)x_n \to x \in X/\overline{T(0)}$, there is a convergent subsequence of $Q_T x_n$.

Definition 1.6 ([4, Definition 4.1])**.** $T : \mathcal{D}(T) \subseteq X \to Y$ is a linear relation, while δ_1 and δ_2 are respectively Kuratowski measures of noncompactness in X/D and Y, where D is a closed subspace of $\mathcal{N}(T)$. Let $k \geq 0$, T is said to be $k - D$-set-contraction if, for any bounded subset B of $D(T)$, $Q_T T(B)$ is a bounded subset of $Y/\overline{T(0)}$ and

$$\delta_2(Q_T TB) \leq k\delta_1(Q_D B).$$

If $D = \{0\}$, then T is said to be $k-\{0\}$-set-contractive linear relation or simply k-set-contractive.

According to these definitions and referring to certain notations and some basic concepts of demicompact linear relations, we elaborate the following propositions.

Proposition 1.7. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a closed single-valued linear operator.*
(i) [6, Theorem 4] *If T is demicompact, then $I - T$ is an upper single-valued linear operator semi-Fredholm.*
(ii) [6, Theorem 5] *If μT is demicompact for each $\mu \in [0, 1]$, then $I - T$ is a single-valued linear operator Fredholm and $i(I - T) = 0$.*

The basic objective of this paper is to attempt to answer the following question "Under which conditions does the linear relation μT for each $\mu \in [0, 1)$ become a demicompact linear relation ?" Subsequently, we shall exhibit some results on Fredholm linear relations and upper semi-Fredholm demicompact linear relations. Thereafter, we shall display some results about a demicompact block matrix of linear relations.

The rest of the current paper is organized as follows. In section 2 which is entitled "Auxiliary results on demicompact linear relation", we provide conditions so that any linear relation becomes a demicompact linear relation and we present the results deriving from these demicompact relations (see Theorems 2.4 and 2.5). In section 3 which is entitled "Fredholm and upper semi-Fredholm linear relations", we investigate Fredholm linear relations as well as upper semi-Fredholm demicompact linear relations (see Theorems 3.1 and 3.2). Finally we exhibit some results in which a block matrix of linear relations becomes a demicompact block matrix of linear relations (see Theorems 4.2 and 4.3).

2. Auxiliary results on demicompact linear relations

In this Section, we try to answer the following question "Under which conditions does the linear relation μT for each $\mu \in [0, 1)$ become a demicompact linear relation ?" We then present some fundamental results about demicompact linear relations.

Lemma 2.1. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a linear relation. If $I - Q_T$ is compact, then T is demicompact if, and only if, $Q_T T$ is demicompact.*

Proof. We suppose that T is demicompact. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $x_n - Q_T T x_n \to y$. We have

$$x_n - Q_T T x_n = (I - Q_T)x_n + Q_T x_n - Q_T T x_n. \tag{2.1}$$

Based upon Eq. (2.1) and considering that $I - Q_T$ is compact and $\{x_n - Q_T T x_n\}$ is a convergent sequence, $\{Q_T x_n - Q_T T x_n\}$ has a convergent subsequence. When the latter is added to demicompact T, we get $\{Q_T x_n\}$, as a convergent subsequence. On the other side, we have

$$\begin{aligned} x_n &= x_n - Q_T x_n + Q_T x_n \\ &= (I - Q_T)x_n + Q_T x_n. \end{aligned}$$

Since $I - Q_T$ is compact and $\{Q_T x_n\}$ has a convergent subsequence, $\{x_n\}$ has a convergent subsequence. Conversely, we suppose that $Q_T T$ is demicompact. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $Q_T x_n - Q_T T x_n \to y$. We have

$$Q_T x_n - Q_T T x_n = -(I - Q_T)x_n + x_n - Q_T T x_n. \tag{2.2}$$

According to Eq. (2.2), and considering the fact that $I - Q_T$ is compact and $\{Q_T x_n - Q_T T x_n\}$ is a convergent sequence, we infer that $\{x_n - Q_T T x_n\}$ has a convergent subsequence. Bearing in mind the fact that $Q_T T$ is demicompact and $\{x_n - Q_T T x_n\}$ has a convergent subsequence, we obtain $\{x_n\}$ as a convergent subsequence. On the other side, we have

$$Q_T x_n = Q_T x_n - x_n + x_n = -(I - Q_T)x_n + x_n.$$

Besides, we have $I - Q_T$ which is compact and $\{x_n\}$ which has a convergent subsequence. Thus, $\{Q_T x_n\}$ has a convergent subsequence. □

Proposition 2.2. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a continuous linear relation. If T is a $k - \overline{T(0)}$-set-contraction, then μT is demicompact for each $\mu k < 1$.*

Proof. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $Q_{\mu T} x_n - Q_{\mu T} \mu T x_n \to y$. We have

$$Q_{\mu T} x_n = Q_{\mu T}(x_n - \mu T x_n) + Q_{\mu T} \mu T x_n. \tag{2.3}$$

Suppose that $\gamma(\{Q_{\mu T} x_n\}) \neq 0$. Therefore, using Eq. (2.3) and Proposition 1.4, we obtain

$$\begin{aligned} \gamma(\{Q_{\mu T} x_n\}) &\leq \gamma(\{Q_{\mu T}(x_n - \mu T x_n)\}) + \gamma(\{Q_{\mu T} \mu T x_n\}) \\ &\leq \mu k \gamma(\{Q_{\mu T} x_n\}) \\ &< \gamma(\{Q_{\mu T} x_n\}). \end{aligned}$$

However, the result is not accurate. It follows that $\gamma(\{Q_{\mu T} x_n\}) = 0$. Hence, $\{Q_{\mu T} x_n\}$ is relatively compact. □

An immediate consequence of Proposition 2.2 is the following Corollary:

Corollary 2.3. *Let $k \geq 0$ and $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a continuous linear relation. If T is a $k - \overline{T(0)}$-set-contraction, then $\frac{1}{1+k}T$ is demicompact.*

Theorem 2.4. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a linear relation. If $m \in \mathbb{N}^*$, $(Q_T \mu T)^m$ is compact for each $\mu \in [0, 1)$ and $I - Q_{\mu T}$ is compact, then μT is demicompact for each $\mu \in [0, 1)$.*

Proof. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that

$$y_n = Q_{\mu T} x_n - Q_{\mu T} \mu T x_n \to y.$$

Let's consider the various cases for m:
Case 1: For $m = 1$. Using Lemma 2.1, we notice that, μT is demicompact for each $\mu \in [0, 1)$.
Case 2: For $m \in \mathbb{N}^* \setminus \{1\}$, we have

$$\begin{aligned} \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^k Q_{\mu T} x_n &= \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^k y_n + \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^{k+1} x_n \\ Q_{\mu T} x_n + \sum_{k=1}^{m-1} (Q_{\mu T}\mu T)^k Q_{\mu T} x_n &= \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^k y_n + \sum_{k=0}^{m-2} (Q_{\mu T}\mu T)^{k+1} x_n \\ &\quad + (Q_{\mu T}\mu T)^m x_n \\ Q_{\mu T} x_n + \sum_{k=0}^{m-2} (Q_{\mu T}\mu T)^{k+1} Q_{\mu T} x_n &= \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^k y_n + \sum_{k=0}^{m-2} (Q_{\mu T}\mu T)^{k+1} x_n \\ &\quad + (Q_{\mu T}\mu T)^m x_n. \end{aligned}$$

Since $Q_T T Q_T$ and $(Q_T T)^n Q_T$ are single-valued linear operators for all $n \geq 1$, we get $\sum_{k=0}^{m-2} (Q_{\mu T}\mu T)^{k+1} Q_{\mu T} x_n$ which is single-valued. Therefore,

$$\begin{aligned} Q_{\mu T} x_n &= \sum_{k=0}^{m-1} (Q_{\mu T}\mu T)^k y_n + \sum_{k=0}^{m-2} (Q_{\mu T}\mu T)^{k+1} (I - Q_{\mu T}) x_n \\ &\quad + (Q_{\mu T}\mu T)^m x_n. \end{aligned}$$

As a matter of fact,

$$\begin{aligned} \gamma(\{Q_{\mu T} x_n\}) &\leq \sum_{k=0}^{m-1} \overline{\gamma}((Q_{\mu T}\mu T)^k) \gamma(\{y_n\}) + \overline{\gamma}((Q_{\mu T}\mu T)^m) \gamma(\{x_n\}) \\ &\quad + \sum_{k=0}^{m-2} \overline{\gamma}((Q_{\mu T}\mu T)^{k+1}) \overline{\gamma}(I - Q_{\mu T}) \gamma(\{x_n\}) \\ &= 0. \end{aligned}$$

We conclude that $\gamma(\{Q_{\mu T} x_n\}) = 0$. Hence, $\{Q_{\mu T} x_n\}$ is relatively compact. Thus, there is a convergent subsequence of $\{Q_{\mu T} x_n\}$. □

Theorem 2.5. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a linear relation and $k \geq 0$.*

(i) If $m \in \mathbb{N}^*$, $(Q_T T)^m$ and $I - Q_{\mu T}$ are compact, then $\frac{1}{1+k} T$ is demicompact.

(ii) If $(Q_T\mu T)^m$ is compact for each $\mu \in [0,1)$ and $m > 0$, then μT is demicompact for each $\mu \in [0,1)$.

(iii) If $m \in \mathbb{N}^*$, $\overline{\gamma}((Q_T\mu T)^m) \leq k$ and $I - Q_{\mu T}$ is compact, then μT is demicompact for each $0 \leq \mu^m k < 1$.

(iv) If $m \in \mathbb{N}^*$, $\overline{\gamma}(T^m) \leq k$ and $I - Q_{\frac{1}{1+k}T}$ is compact, then $\frac{1}{1+k}T$ is demicompact.

Proof. (i) An immediate consequence of Theorem 2.4 for $\mu = \frac{1}{1+k}$.

(ii) Likewise, based on the preceding proof of Theorem 2.4, we obtain

$$\begin{aligned} Q_{\mu T}x_n &= \sum_{k=0}^{m-1}(Q_{\mu T}\mu T)^k y_n + \sum_{k=0}^{m-2}(Q_{\mu T}\mu T)^{k+1}(I - Q_{\mu T})x_n \\ &\quad +(Q_{\mu T}\mu T)^m x_n. \end{aligned}$$

As a matter of fact,

$$\begin{aligned} \gamma(\{Q_{\mu T}x_n\}) &\leq \sum_{k=0}^{m-1}\overline{\gamma}((Q_{\mu T}\mu T)^k)\gamma(\{y_n\}) + \overline{\gamma}((Q_{\mu T}\mu T)^m)\gamma(\{x_n\}) \\ &\quad +\sum_{k=0}^{m-2}\overline{\gamma}((Q_{\mu T}\mu T)^{k+1})\overline{\gamma}(I - Q_{\mu T})\gamma(\{x_n\}) \\ &= 0. \end{aligned}$$

We conclude that $\gamma(\{Q_{\mu T}x_n\}) = 0$. Hence, $\{Q_{\mu T}x_n\}$ is relatively compact. Thus, there is a convergent subsequence of $\{Q_{\mu T}x_n\}$.

(iii) Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $y_n = Q_{\mu T}x_n - Q_{\mu T}\mu Tx_n \to y$. Suppose that $\gamma(\{Q_{\mu T}x_n\}) \neq 0$. We have

$$\begin{aligned} \gamma(\{Q_{\mu T}x_n\}) &\leq \sum_{k=0}^{m-1}\overline{\gamma}((Q_{\mu T}\mu T)^k)\gamma(\{y_n\}) + \overline{\gamma}((Q_{\mu T}\mu T)^m)\gamma(\{x_n\}) \\ &\quad +\sum_{k=1}^{m-1}\overline{\gamma}((Q_{\mu T}\mu T)^k)\overline{\gamma}(I - Q_{\mu T})\gamma(\{x_n\}) \\ &\leq \mu^m\overline{\gamma}((Q_{\mu T}T)^m)\gamma(\{x_n\}) \\ &\leq \mu^m\overline{\gamma}((Q_T T)^m)\gamma(\{x_n\}) \\ &\leq \mu^m k\gamma(\{x_n\}) \\ &< \gamma(\{x_n\}). \end{aligned}$$

However, this result is not accurate. It follows that $\gamma(\{Q_{\mu T}x_n\}) = 0$. Hence, $\{Q_{\mu T}x_n\}$ is relatively compact.

(iv) An immediate consequence of (iii) for $\mu = \frac{1}{1+k}$ resides in the fact that, we have $\frac{k}{(1+k)^m} < 1$ for each $k \geq 0$. □

3. Fredholm and upper semi-Fredholm demicompact linear relations

In this Section, we set forward some results on Fredholm and upper semi-Fredholm linear relations involving demicompact linear relations. In particular, in both Theorems stated below, we extend Proposition 1.7, to linear relations:

Theorem 3.1. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a closed linear relation. If T is demicompact and $I - Q_T$ is compact, then $I - T$ is an upper semi-Fredholm relation.*

Proof. Let T be a demicompact and $I-Q_T$ be a compact linear relation. Using Lemma 2.1, we infer that $Q_T T$ is demicompact. Based on the latter and using Proposition 1.7 (i), we obtain $I-Q_T T$ which is an upper semi-Fredholm single valued linear operator. On the other side,

$$\begin{aligned} Q_{I-T}(I-T) &= Q_T(I-T) \\ &= Q_T I - Q_T T + I - I \\ &= -(I-Q_T)I + I - Q_T T. \end{aligned}$$

Since $I-Q_T$ is compact and $I-Q_T T$ is an upper single valued linear operator semi-Fredholm, we notice that $Q_{I-T}(I-T)$ is an upper single valued linear operator semi-Fredholm. Using Lemma 1.2, we obtain $I-T$ which is an upper semi-Fredholm relation. □

Theorem 3.2. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a closed linear relation. If μT is demicompact and $I - Q_T$ is compact, then $I - T$ is a Fredholm relation and $i(I-T) = 0$.*

Proof. Let T be a demicompact linear relation and $I-Q_T$ be a compact operator. Applying Lemma 2.1, we get $Q_T T$ which is a demicompact linear relation. Using Proposition 1.7 (ii) and demicompact $Q_T T$, we obtain $I-Q_T T$, which is a single valued linear operator Fredholm and $i(I-Q_T T) = 0$. On the other side,

$$\begin{aligned} Q_{I-T}(I-T) &= Q_T(I-T) \\ &= Q_T I - Q_T T + I - I \\ &= -(I-Q_T)I + I - Q_T T. \end{aligned}$$

Moreover, we have $I-Q_T$ which is compact and $I-Q_T T$ which is a single valued linear operator Fredholm and $i(I-Q_T T) = 0$. Therefore, $Q_{I-T}(I-T)$ is a Fredholm operator of index zero. Using Lemma 1.2, we notice that $I-T$ is a Fredholm relation and $i(I-T) = 0$. □

Proposition 3.3. *Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a continuous linear relation. If T is demicompact, $k - \overline{T(0)}$ is a set-contraction and $I - Q_T$ is compact, then $I - T$ is a Fredholm relation and $i(I-T) = 0$.*

Proof. Since T is demicompact, $k-\overline{T(0)}$ is a set-contraction and $I-Q_T$ is compact, grounded on Corollary 2.3, we deduce that $\frac{1}{1+k}T$ is demicompact. Using Theorem 3.2, we obtain $I-T$ which is Fredholm relation and $i(I-T)=0$. □

4. Demicompact block matrix of linear relations

In this section, a block matrix of linear relations $\mathcal{L}$ is identified. Afterwards, some results, where this block matrix of linear relations $\mathcal{L}$ becomes a demicompact block matrix of linear relations, are displayed.
In the Banach space $X\oplus Y$, we consider the linear relation $\mathcal{L}$ provided by the block matrix of linear relations

$$\mathcal{L}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.1}$$

where $A:\mathcal{D}(A)\subseteq X\longrightarrow X$, $B:\mathcal{D}(B)\subseteq Y\longrightarrow X$, $C:\mathcal{D}(C)\subseteq X\longrightarrow Y$ and $D:\mathcal{D}(D)\subseteq Y\longrightarrow Y$ are linear relations with their natural domain

$$\mathcal{D}(\mathcal{L}):=\Big(\mathcal{D}(A)\cap\mathcal{D}(C)\Big)\oplus\Big(\mathcal{D}(B)\cap\mathcal{D}(D)\Big).$$

The graph of $\mathcal{L}$ is defined by
$G(\mathcal{L}):=\left\{\Big((x_1,x_2),(y_1,y_2)\Big):(x_1,x_2)\in\mathcal{D}(\mathcal{L}),\ y_1\in Ax_1+Bx_2 \text{ and } y_2\in Cx_1+Dx_2\right\}$.

Lemma 4.1 ([5, Remark 2.3])**.** *Let $\mathcal{L}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix of linear relations where $A:\mathcal{D}(A)\subseteq X\longrightarrow X$, $B:\mathcal{D}(B)\subseteq Y\longrightarrow X$, $C:\mathcal{D}(C)\subseteq X\longrightarrow Y$ and $D:\mathcal{D}(D)\subseteq Y\longrightarrow Y$. If $B(0)\subset A(0)$ and $C(0)\subset D(0)$, then*

$$Q_{\mathcal{L}}\mathcal{L}=\begin{pmatrix} Q_AA & Q_AB \\ Q_DC & Q_DD \end{pmatrix}.$$

Theorem 4.2. *Let $A:\mathcal{D}(A)\subseteq X\longrightarrow X$ and $D:\mathcal{D}(D)\subseteq Y\longrightarrow Y$ be two demicompact linear relations. Then, $\mathcal{M}=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is a demicompact linear relation.*

Proof. Let $\{t_n\}=\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ be a bounded sequence of $\mathcal{D}(\mathcal{M})$ such that $Q_{\mathcal{M}}t_n-Q_{\mathcal{M}}\mathcal{M}t_n$ is convergent. We have

$$\begin{aligned} Q_{\mathcal{M}}t_n-Q_{\mathcal{M}}\mathcal{M}t_n &= \begin{pmatrix} Q_A & 0 \\ 0 & Q_D \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}-\begin{pmatrix} Q_AA & 0 \\ 0 & Q_DD \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ &= \begin{pmatrix} Q_Ax_n-Q_AAx_n \\ Q_Dy_n-Q_DDy_n \end{pmatrix}. \end{aligned}$$

Since $\{x_n\}$ is a bounded sequence of $\mathcal{D}(A)$, $Q_Ax_n-Q_AAx_n$ are convergent and A is a demicompact linear relation; then $\{Q_Ax_n\}$ has a convergent subsequence. Similarly, we get $\{Q_Dy_n\}$ which has a convergent subsequence. Hence, $\{Q_{\mathcal{M}}t_n\}$ has a convergent subsequence.

□

Theorem 4.3. *Let $A : \mathcal{D}(A) \subseteq X \longrightarrow X$ and $D : \mathcal{D}(D) \subseteq Y \longrightarrow Y$ be two demicompact linear relations and let $B : \mathcal{D}(B) \subseteq Y \longrightarrow X$ and $C : \mathcal{D}(C) \subseteq X \longrightarrow Y$ be two linear relations.*
If $Q_A(I-B)$ and $Q_D(I-C)$ are compact and $B(0) \subseteq A(0)$ and $C(0) \subseteq D(0)$, then $\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a demicompact linear relation.

Proof. Let $\{t_n\} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ be a bounded sequence of $\mathcal{D}(\mathcal{L})$ such that $Q_{\mathcal{L}}t_n - Q_{\mathcal{L}}\mathcal{L}t_n$ is convergent. Using Lemma 4.1, we obtain

$$\begin{aligned} Q_{\mathcal{L}}t_n - Q_{\mathcal{L}}\mathcal{L}t_n &= \begin{pmatrix} Q_A & Q_A \\ Q_D & Q_D \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} Q_A A & Q_A B \\ Q_D C & Q_D D \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ &= \begin{pmatrix} Q_A x_n + Q_A y_n - Q_A A x_n - Q_A B y_n \\ Q_D x_n + Q_D y_n - Q_D C x_n - Q_D D y_n \end{pmatrix} \\ &= \begin{pmatrix} Q_A x_n - Q_A A x_n + Q_A(I-B)y_n \\ Q_D y_n - Q_D D y_n + Q_D(I-C)x_n \end{pmatrix}. \end{aligned}$$

We have $\{x_n\}$ which is a bounded sequence and $Q_A(I-B)$ which is compact. Then, $\{Q_A(I-B)y_n\}$ is bounded. Since $\{Q_A x_n - Q_A A x_n + Q_A(I-B)y_n\}$ is convergent and $\{Q_A(I-B)y_n\}$ is bounded, $\{Q_A x_n - Q_A A x_n\}$ is convergent. Subsequently, using the fact that A is a demicompact linear relation, $\{Q_A x_n\}$, therefore, has a convergent subsequence. Similarly, we get $\{Q_D y_n\}$ which has a convergent subsequence. As a matter of fact, $\{Q_{\mathcal{L}} t_n\}$ has a convergent subsequence.

□

An immediate consequence of Theorem 4.3 is the following Corollary:

Corollary 4.4. *Let $A : \mathcal{D}(A) \subseteq X \longrightarrow X$ and $D : \mathcal{D}(D) \subseteq Y \longrightarrow Y$ be two demicompact linear relations and let $B : \mathcal{D}(B) \subseteq Y \longrightarrow X$ and $C : \mathcal{D}(C) \subseteq X \longrightarrow Y$ be two linear relations.*
Thus, the block matrix $\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a demicompact linear relation, if one of the following conditions holds:

- **a.:** *$Q_A(I-B)$ and $Q_C(I-D)$ are compact and $B(0) \subseteq A(0)$ and $D(0) \subseteq C(0)$.*
- **b.:** *$Q_B(I-A)$ and $Q_D(I-C)$ are compact and $A(0) \subseteq B(0)$ and $C(0) \subseteq D(0)$.*
- **c.:** *$Q_B(I-A)$ and $Q_C(I-D)$ are compact and $A(0) \subseteq B(0)$ and $D(0) \subseteq C(0)$.*

Acknowledgements. *The authors wish to express their gratitude to the referee for his valuable comments which helped to improve the quality of this paper.*

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