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The D_π-property on products of π-decomposable groups

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Abstract The aim of this paper is to prove the following result: Let π be a set of odd primes. If the group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, then G has a unique conjugacy class of Hall π-subgroups, and any π-subgroup is contained in a Hall π-subgroup (i.e. G satisfies property D_π).

Keywords Finite groups · Product of subgroups · π-structure · Simple groups

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1 Introduction

All groups considered in this paper are assumed to be finite. A well-known result in the framework of finite factorized groups is the classical theorem by O. Kegel and H. Wielandt which asserts the solubility of a group which is the product of two nilpotent subgroups. This theorem has been the motivation for a great number of results in the literature on factorized groups. Particularly some of them consider the situation when either one or both factors are π-decomposable, for a set of primes π. A group X is said to be π-*decomposable* for a set of primes π, if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup X_π and a π'-subgroup $X_{\pi'}$, where π' stands for the complement of π in the set of all prime numbers. For any group X and any set of primes σ, we use X_σ to denote a Hall σ-subgroup of X, if it exists.

In this line, Y. G. Berkovich [5] proved that the Kegel and Wielandt's result remains true for a group $G = AB$ which is the product of subgroups A and B such that one of the factors, say A, is 2-decomposable, the other factor B is nilpotent of odd order, and A and B have coprime orders. If the subgroup B is metanilpotent instead of nilpotent, but preserving all the remaining conditions, P. J. Rowley [27] proved that the group G is σ-separable, for the set σ of all odd primes dividing the order of A. Then Z. Arad and D. Chillag [3] showed that this conclusion remains true without any restriction on the nilpotent length of B. Previously L. S. Kazarin [13] had obtained under the same hypotheses that $O_{2'}(A) \le O_{2'}(G)$.

A significant extension of these results was obtained in [17] by proving that $O_\pi(A) \le O_\pi(G)$ whenever $G = AB$, A is a π-decomposable subgroup of G for any set of odd primes π, and B is a π-subgroup of G; equivalently, $O_\pi(A)B$ is a Hall π-subgroup of G (see [17, Theorem 1, Lemma 1]). Under the additional hypothesis that A and B have coprime orders, such as considered in the mentioned previous results, it is easily derived the σ-separability of G for the set σ of all odd prime divisors of the order of A.

In fact the results in [17] make up the starting point of a longtime development carried out in [17–21], where the existence of Hall π-subgroups have been considered as a preliminary step to finding conditions of π-separability for products of π-decomposable subgroups. The goal was to prove the following theorem, which was first stated as a conjecture in [18]:

Theorem 1 ([21, Main Theorem]) *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi$ is a Hall π-subgroup of G.*

This theorem, whose proof uses deeply the classification of finite simple groups (CFSG), substantially extends all the above mentioned results. In particular, we have achieved some non-simplicity and π-separability criteria for certain products of π-decomposable groups, which contribute some new extensions of the theorem of Kegel and Wielandt (see also [19]). Some examples in [17, 18] show that the analogous result to Theorem 1 does not hold in general if the set of primes π contains the prime 2, although some related positive results were obtained in this case in [18] when the factors are both soluble.

It is well-known that, given a set of primes π, any π-separable group, has a unique conjugacy class of Hall π-subgroups, and that every π-subgroup is contained in a Hall π-subgroup (dominance). So it is worthwhile emphasizing that the above development is closely related to a classic but ongoing problem in the theory of finite groups: the search for conditions which guarantee the existence, conjugacy, and dominance of Hall subgroups in a finite group, for a given set of primes π. To be more accurate, we will say that a group G *satisfies the property*:

E_π if G has at least one Hall π-subgroup;
C_π if G satisfies E_π and any two Hall π-subgroups of G are conjugate in G;
D_π if G satisfies C_π and every π-subgroup of G is contained in some Hall π-subgroup of G.

Such a group is also called an E_π*-group*, C_π*-group*, and D_π*-group*, respectively.

After the seminal work of P. Hall (cf. [11]), where the above terminology was introduced, numerous researchers have addressed the mentioned problem. Specially significant is a theorem of H. Wielandt [30] which states that any group possessing nilpotent Hall π-subgroups is a D_π-group (see Lemma 3 below). In particular, several authors have investigated in a series of papers the Hall subgroups of the finite simple groups. The classification of the simple groups satisfying the properties E_π, C_π, or D_π, has been completed by E. Vdovin and D. Revin. We refer to the expository article [29] for a detailed account on this topic.

For products of π-decomposable groups, Theorem 1 provides that, when π is a set of odd primes, such factorized groups satisfy the property E_π. A natural question arises then: What can be said about the properties C_π and D_π for such products of groups? For any set of odd primes, F. Gross proved in [10, Theorem A] that any group satisfying the E_π-property also satisfies the C_π one. So, having in mind Theorem 1, it remains to analyze the dominance property. This will be the principal aim of the present paper, which we will attain by proving the following theorem:

Main Theorem *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then G satisfies the property D_π.*

In fact, our study on the D_π-property was initially motivated by the development carried out in [22] on trifactorized groups and π-decomposability, where the dominance property appeared as a relevant tool. *Trifactorized* groups, that is, groups of the form $G = AB = AC = BC$, where A, B, and C are subgroups of G, play a key role within the study of factorized groups. For instance, the so-called *factorizer* of a normal subgroup in a factorized group turns out to be trifactorized (see [1, Lemma 1.1.4]). Specifically, for a subgroup N of a group $G = AB$ which is the product of subgroups A and B, the *factorizer* of N in G, denoted $X(N)$, is the intersection of all factorized subgroups of G containing N. Recall that a subgroup S of $G = AB$ is *factorized* if $S = (S \cap A)(S \cap B)$ and $A \cap B \leq S$. In this setting, in [22] we proved the following result, which can be considered as a particular significant case of our Main Theorem:

Theorem 2 ([22, Theorem 3.2]) *Let π be a set of odd primes. Let A, B, C be subgroups of a group G, and let $G = AB = AC = BC$ be a trifactorized group, where $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups, and C is a D_π-group. Then G is a D_π-group.*

The notation is standard and is taken mainly from [8]. We also refer to this book for the basic terminology and results about classes of groups. If X, Y are subgroups of a group G, we use the notation $X^Y = \langle x^y \mid x \in X, y \in Y \rangle$; in particular, X^G is the normal closure of X in G. Also, if n is an integer and p a prime number, n_p will denote the largest power of p dividing n, and $\pi(n)$ the set of prime divisors of n; for a group G, we set $\pi(G) = \pi(|G|)$. Finally, we use the notation of [25] for the simple groups, especially for the unitary groups and the orthogonal groups.

The layout of the paper is as follows. In Section 2 we provide some necessary preliminaries. In Section 3 we first describe the structure of a minimal counterexample to our Main Theorem (cf. Proposition 1), which is shown to be an almost simple group. Next we will prove the Main Theorem by carrying out a case-by-case analysis of the simple groups occurring as the socle of the minimal counterexample, leading to a final contradiction. This proof relies deeply on the classification of the maximal factorizations of simple groups and their automorphism groups given by Liebeck, Praeger and Saxl in [25]. Also the information from [29] about simple groups satisfying property D_π will be needed in some cases.

2 Preliminary results

The next result is a reformulation of a useful one due to Kegel, and later on improved by Wielandt, which appears in [1, Lemma 2.5.1] (see also [18, Lemma 2]).

Lemma 1 *Let the group $G = AB$ be the product of the subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^gB_0 = B_0A_0^g$ for all $g \in G$.*

Moreover, if A_0 and B_0 are π-groups for a set of primes π, and $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.

We will use, without further reference, the following fact on Hall subgroups of factorized groups, which is applicable to π-separable groups (see [1, Lemma 1.3.2]).

Lemma 2 *Let $G = AB$ be the product of the subgroups A and B. Assume that A and B have Hall π-subgroups and that G is a D_π-group for a set of primes π. Then there exist Hall π-subgroups A_π of A and B_π of B such that $A_\pi B_\pi$ is a Hall π-subgroup of G.*

The next well known result due to Wielandt will be relevant in our development.

Lemma 3 ([30, Satz]) *If a group G has a nilpotent Hall π-subgroup, for a set of primes π, then G is a D_π-group.*

We need specifically the following result, whose proof uses CFSG.

Lemma 4 ([26, Theorem 7.7]) *Let G be a group, A a normal subgroup of G, and π be a set of primes. Then G is a D_π-group if and only if A and G/A are D_π-groups.*

The following results on simple groups of Lie type will be essential for the proof of our main theorem. We introduce first some additional terminology and notation. Let n be a positive integer and p be a prime number. A prime r is said to be *primitive with respect to the pair* (p, n) if r divides $p^n - 1$ but r does not divide $p^k - 1$ for every integer k such that $1 \leq k < n$. It was proved by Zsigmondy [31] that such a primitive prime r exists unless either $n = 2$ and p is a Mersenne prime or $(p, n) = (2, 6)$. For such a prime r, it holds $r - 1 \equiv 0 \pmod{n}$ and, in particular, $r \geq n + 1$.

In the sequel, for $q = p^e$, $e \geq 1$, we will denote by q_n any primitive prime divisor of $p^{en} - 1$, i.e. primitive with respect to the pair (p, ne).

We collect in the next Lemma 5 some information on centralizers of elements and maximal tori in classical simple groups of Lie type, as it appears in [21, Lemma 14]). Unitary groups $U_n(q)$ are not included in Table 1 since our strategy to deal with such groups is essentially different, as in many cases they do not admit a factorization (cf. Lemma 11). For additional information on maximal tori used along the paper we refer to [28, Lemma 1.2] (see also [6, Propositions 7-10]).

Lemma 5 ([21, Lemma 14]) *For $N = G(q)$ a classical simple group of Lie type of characteristic p and $q = p^e$, and $N \trianglelefteq G \leq Aut(N)$, there exist primes r, $s \in \pi(N) \setminus \pi(G/N)$ and maximal tori T_1 and T_2 of N as stated in Table 1.*

Moreover, except for the case denoted $(\star)$ in Table 1, for any element $a \in N$ of order r and any element $b \in N$ of order s, up to conjugacy in N we have that $C_N(a) \leq T_1$ and $C_N(b) \leq T_2$, and T_1, T_2 are abelian p'-groups.

On the other hand, there is neither a field automorphism nor a graph-field automorphism of N centralizing elements of N of order r or s (except for the triality automorphism in the case $P\Omega_8^+(q)$).

N	r	s	$\lvert T_1\rvert$	$\lvert T_2\rvert$	*Conditions*
$L_n(q)$, $n \geq 4$	q_n	q_{n-1}	$\frac{q^n-1}{(n,q-1)(q-1)}$	$\frac{q^{n-1}-1}{(n,q-1)}$	$(n,q) \neq (6,2)$, $(n-1,q) \neq (6,2)$
$PSp_{2n}(q)$	q_{2n}	$q_{2(n-1)}$	$\frac{q^n+1}{(2,q-1)}$	$\frac{(q^{n-1}+1)(q+1)}{(2,q-1)}$	n even $(\star)$, $(n,q) \neq (4,2)$
$P\Omega_{2n+1}(q)$, q odd, $n \geq 3$	q_{2n}	q_n	$\frac{q^n+1}{(2,q-1)}$	$\frac{q^n-1}{(2,q-1)}$	n odd, $(n,q) \neq (3,2)$
$P\Omega_{2n}^-(q)$, $n \geq 4$	q_{2n}	$q_{2(n-1)}$	$\frac{q^n+1}{(4,q^n+1)}$	$\frac{(q^{n-1}+1)(q-1)}{(4,q^n+1)}$	$(n,q) \neq (4,2)$
$P\Omega_{2n}^+(q)$	$q_{2(n-1)}$	q_{n-1}	$\frac{(q^{n-1}+1)(q+1)}{(4,q^n-1)}$	$\frac{(q^{n-1}-1)(q-1)}{(4,q^n-1)}$	n even, $(n,q) \neq (4,2)$
$n \geq 4$	$q_{2(n-1)}$	q_n	$\frac{(q^{n-1}+1)(q+1)}{(4,q^n-1)}$	$\frac{q^n-1}{(4,q^n-1)}$	n odd

Table 1

Lemma 6 ([21, Lemma 5]) *Let $N = G(q)$ be a classical simple group of Lie type over the field $GF(q)$ of characteristic p. Then $|Out(N)|_p \leq q$ and equality holds only when $q \in \{2, 3, 4\}$. Moreover, if $q = 3$, the only case in which $|Out(N)|_p = q$ is possible is when $N = P\Omega_8^+(q)$. In particular, $|Out(N)|_p < q^2$ for any classical simple group of Lie type.*

3 The minimal counterexample

We describe next the structure of a minimal counterexample to our Main Theorem.

Proposition 1 *Let π be a set of odd primes. Assume that the group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and G is a counterexample of minimal order to the assertion that G satisfies property D_π.*

Then G has a unique minimal normal subgroup N, which is a non-abelian simple group, so that $N \trianglelefteq G \leq Aut(N)$.

Moreover, the following properties hold:

(1) $G = AN = BN = AB$.
Moreover, $|N||A \cap B| = |G/N||N \cap A||N \cap B|$.
(2) One of the subgroups, say B, is a π'-group, i.e. $B_\pi = 1$, and $\pi(G/N) \subseteq \pi'$.
(3) A is neither a π-group nor a π'-group, i.e. $A_\pi \neq 1$ and $A_{\pi'} \neq 1$.
(4) A_π is a Hall π-subgroup of N (and of G), and $|\pi(A)| \geq 3$.
Moreover, N satifies E_π, but not D_π neither $E_{\pi'}$. Further, $|\pi \cap \pi(N)| \geq 2$ and $|\pi' \cap \pi(N)| \geq 2$.
(5) The subgroup A_π is not nilpotent.
(6) Let $S \leq A$ be an s-group, for a prime number s. If $s \in \sigma$, with $\sigma \in \{\pi, \pi'\}$, then $\pi(|A : C_A(S)|) \subseteq \sigma$ and $C_A(S)$ is not a σ-group.
(7) Either A or B is non-soluble.
Moreover, for each prime $r \in \pi$ and any other prime $s \in \pi(A \cap N)$, there exists a soluble subgroup of N of order divisible by r and s.
(8) $|\pi' \cap \pi(N)| \geq 3$ and $|\pi(N)| \geq 5$. Moreover, $\pi(G) = \pi(N)$.
(9) Assume that N is a simple group of Lie type over the field $GF(q)$ of characteristic p. Then the following assertions hold:
  a) $p \in \pi'$.
  b) If N is of Lie rank $l > 1$, then $p \in \pi(B \cap N)$.
  c) Assume that there exists an element $a \in A \cap N$ of prime order r and a subgroup X in G containing A such that $C_{X \cap N}(a)$ is an abelian p'-subgroup. Then $r \in \pi$, A is soluble, $A \cap N$ is a p'-group and $|A|_p \leq q$.
  If, in addition, $C_X(a)$ is a p'-group, then A is a p'-group.

Proof We prove first statements (1)-(4).

Recall that $A_\pi B_\pi$ is a Hall π-subgroup of G by Theorem 1.

If N is any non-trivial normal subgroup of G, then $G/N = (AN/N)(BN/N)$ satisfies the hypotheses of the Main Theorem. Since G is a minimal counterexample to this theorem, we get that G/N is a D_π-group. If in addition N were a D_π-group, then G would be a D_π-group by Lemma 4, which is not the case. Then N is not a D_π-group and, in particular, we deduce also that $O_\pi(G) = O_{\pi'}(G) = 1$.

By Lemma 1, it follows that $[A_\pi^G, B_\pi^G] = 1$.

We consider now the case that $A_\pi \neq 1$ and $B_\pi \neq 1$. We notice that $A_\pi^G \cap B_\pi^G = 1$. Otherwise, if N is a minimal normal subgroup of G contained in $A_\pi^G \cap B_\pi^G$, then $[N, N] = 1$, i.e. N is abelian, and then either $N \leq O_\pi(G) = 1$ or $N \leq O_{\pi'}(G) = 1$, a contradiction.

Let H be a π-subgroup of G. We aim to prove that $H \leq (A_\pi B_\pi)^g$ for some $g \in G$, to get a contradiction and derive that either $A_\pi = 1$ or $B_\pi = 1$. By minimality of G, both factor groups G/A_π^G and G/B_π^G satisfy the property D_π. So we may assume, up to conjugacy, that :

$$H \leq A_\pi B_\pi A_\pi^G = B_\pi A_\pi^G$$

and also that:

$$H \leq (A_\pi B_\pi B_\pi^G)^g = A_\pi^{ab} B_\pi^{ab} B_\pi^G = A_\pi^b B_\pi^G$$

for some $g = ab$ with $a \in A$, $b \in B$, since B_π^G is normal in G and A_π is normal in A. Now applying again that A_π^G, B_π^G are normal in G and B_π is normal in B, and the fact that $A_\pi^G \cap B_\pi^G = 1$, we deduce:

$$\begin{aligned} H \leq B_\pi A_\pi^G \cap A_\pi^b B_\pi^G = (B_\pi A_\pi^G \cap A_\pi B_\pi^G)^b = \\ = (A_\pi(B_\pi A_\pi^G \cap B_\pi^G))^b = (A_\pi B_\pi (A_\pi^G \cap B_\pi^G))^b = (A_\pi B_\pi)^b \end{aligned}$$

as aimed. So, without loss of generality, we can consider $B_\pi = 1$. Also, if G would satisfy E_π and $E_{\pi'}$, then G would be a D_π-group (see [4, Note added in proof]), a contradiction. Hence $A_\pi \neq 1$ and $A_{\pi'} \neq 1$.

Let N be a minimal normal subgroup of G, and let $X(N)$ be the factorizer of N in $G = AB$. Assume that $X(N)$ is a proper subgroup of G. Then the minimality of G implies that $X(N)$ satisfies D_π. Since $N \trianglelefteq X(N)$ we have that N is a D_π-group. But also G/N satisfies D_π. Hence we get that G is a D_π-group by Lemma 4, a contradiction. Therefore, $G = X(N)$ and, applying [1, Lemma 1.1.4], we get $G = AB = AN = BN$. Now a straightforward computation shows that $|N||A \cap B| = |G/N||N \cap A||N \cap B|$. Moreover, since $A_\pi N/N \leq AN/N = BN/N$ and B is a π'-group, it follows that $A_\pi N/N$ is the trivial subgroup, that is, $A_\pi \leq N$. If M were another minimal normal subgroup of G, then it would be $A_\pi \leq N \cap M = 1$, a contradiction. Consequently, G has a unique minimal normal subgroup, say N, which is non-abelian.

Hence, $N \cong S \times \cdots \times S$ is a direct product of copies of a non-abelian simple group S. Since $G = A_{\pi'} N$, it follows that $A_{\pi'}$ permutes transitively the non-abelian simple direct factors of N and centralizes A_π, which implies that N is a simple group.

Thus we have proved that $G = AB = AN = BN$ is an almost simple group, i.e. $N \leq G \leq \mathrm{Aut}(N)$ with N a non-abelian simple group, $A_\pi \neq 1$, $A_{\pi'} \neq 1$, $B_\pi = 1$ and $B_{\pi'} \neq 1$. In particular, $A_\pi \leq N$, and A_π is a Hall π-subgroup of N, and also of G, since $G/N = BN/N$ is a π'-group. Moreover, N is not a D_π-group by Lemma 4. In particular, $|\pi| \geq 2$ and thus $|\pi(A)| \geq 3$. Also, N is not an $E_{\pi'}$-group, since it is an E_π-group (see [4, Note added in proof]). Hence statements (1)-(4) hold.

(5) Assume that A_π is nilpotent. Then, since this is a Hall π-subgroup of G, it follows that G is a D_π-group, by Lemma 3, a contradiction.

(6) Assume that S is an s-subgroup of A, with $s \in \sigma$, where $\sigma \in \{\pi, \pi'\}$. The first statement is clear since $A_{\sigma'} \leq C_A(S)$. Consequently, if $C_A(S)$ were a σ-group, A would be a σ-group, a contradiction by (3).

(7) Assume now that both A and B are soluble. From [14, Theorem] it is known that N is isomorphic to one of the groups in the set:

$$\mathfrak{M} = \{L_2(q), q > 3; L_3(q), q < 9; L_4(2), M_{11}, PSp_4(3), U_3(8)\}.$$

All factorizations for such a group $G = AB$, with A and B soluble, can be found in [24, Proposition 4.1]. In fact, as stated in this reference, the groups $L_3(5)$, $L_3(7)$ and $L_4(2) \cong A_8$ can be omitted.

For any of the groups N in $\mathfrak{M} \setminus \{L_2(q), q > 3;\ M_{11}, L_3(5), L_3(7), L_4(2)\}$, we get a contradiction with the assertion in (4) that $|\pi \cap \pi(N)| \geq 2$ and $|\pi' \cap \pi(N)| \geq 2$. When N is $L_3(2) \cong L_2(7)$, $L_3(3)$ or $PSp_4(3)$, the contradiction arises because $|\pi(N)| = 3$. For the case $N = L_3(8)$, by [24, Proposition 4.1] factorizations exist for G only when $N.3 \leq G$, so $3 \in \pi(G/N) \subseteq \pi'$ by (2). Moreover, by order arguments, $3 \in \pi(N \cap A) \cap \pi(N \cap B)$. Since N has a Sylow s-subgroup of order $s = 73$, which is self-centralizing in N, we get by (6) that $s \notin \pi(A)$ and so $s \in \pi'$. This means that $\{2, 3, s\} \subseteq \pi' \cap \pi(N)$, which leads to a contradiction since $|\pi(G)| = |\pi(N)| = 4$. For the cases $N = L_3(4)$ and $N = U_3(8)$ we can argue in an analogous way by considering the primes $s = 7$ and $s = 19$, respectively.

The case $N = M_{11}$ is discarded because, if it satisfies E_π for a set of odd primes, then it satisfies D_π (compare [29, Theorem 8.2 (Table 3)] with [29, Theorem 6.9, Condition II]).

Consider now the case $N = L_2(q)$, with $q > 3$. If p is the characteristic, then $p \notin \pi$ by (6), because a Sylow p-subgroup is self-centralizing in G (see for instance [15, 1.17]). From [25, Tables 1,3] it follows that, apart from some exceptional cases when $q \in \{11, 23\}$ which can be discarded by similar arguments as used above, the maximal soluble subgroups X and Y of G such that $G = XY$ satisfy the condition $\{X \cap N, Y \cap N\} = \{N_N(P), D_{\nu(q+1)}\}$, with P a Sylow p-subgroup of N, $|N_N(P)| = \epsilon q(q-1)$ with $\epsilon = (q-1, 2)^{-1}$, and $D_{\nu(q+1)}$ a dihedral group of order $\nu(q+1)$ with $\nu = (2, p)$. The case that $A \cap N \leq D_{\nu(q+1)}$ can be discarded because this means that A_π is abelian, which contradicts (5). In the other case, when $A \cap N \leq N_N(P)$, since the centralizer of any p-element in N is a p-group, and p should divide $|N \cap A|$ by order arguments as $B \cap N \leq D_{\nu(q+1)}$, it follows that $A_\pi = 1$, a contradiction.

Hence, either A or B is non-soluble.

Now, assume that $r \in \pi$ and $s \in \pi(A \cap N)$. If $s \in \pi$, then A_π is a soluble subgroup of N of order divisible by r and s. If $s \in \pi'$, we can consider the soluble subgroup $A_\pi \times (A_s \cap N)$ for A_s a Sylow s-subgroup of A. Hence (7) holds.

(8) From (7), we deduce that either $N \cap A$ or $N \cap B$ is non-soluble, because $G/N \cong A/(N \cap A) \cong B/(B \cap N)$ and $G/N \cong Out(N)$ is soluble. Hence $|\pi' \cap \pi(N)| \geq 3$ and $|\pi(N)| \geq 5$.

Our next aim is to prove that $\pi(G) = \pi(N)$. Assume that $\sigma := \pi(G) \setminus \pi(N) \neq \emptyset$, and take a prime number $s \in \sigma$. Note that $\sigma \subseteq \pi'$, since N contains A_π, which is a Hall π-subgroup of G. We have $|N||A \cap B| = |G/N||N \cap A||N \cap B|$. Hence $|G : A \cap B| = |N|^2/|N \cap A||N \cap B|$, and so $(s, |G : A \cap B|) = 1$. Then $A \cap B$ contains a Sylow s-subgroup of G, say S.

Let $\pi_0 := \pi' \setminus \{s\}$. Again since $G/N \cong A/(N \cap A)$ is a soluble π'-group, we may choose a Hall π_0-subgroup of A, say $\tilde{A}$, such that $A_{\pi'} = \tilde{A}S$, and so $A = A_\pi \times \tilde{A}S$. Let $\tilde{G} := \tilde{A}N = (A_\pi \times \tilde{A})N$. Consider now $\tilde{B} := B \cap \tilde{G} = B \cap \tilde{A}N$. Since $A_{\pi'}N = BN$ and B also contains S we can deduce that $B = B \cap A_{\pi'}N = S(B \cap \tilde{A}N) = \tilde{B}S$

and $\tilde{B}$ is a Hall π_0-subgroup of B. Moreover $(A_\pi \times \tilde{A}) \cap B = \tilde{A} \cap B = \tilde{A} \cap \tilde{B}$. Since $(|S|, |N \cap A|) = 1 = (|S|, |N \cap B|)$ it is easy to see that $|\tilde{G}| = |G|/|S| = |(A_\pi \times \tilde{A})S\tilde{B}|/|S| = |A_\pi \times \tilde{A}||\tilde{B}|/|(A_\pi \times \tilde{A}) \cap \tilde{B}| = |(A_\pi \times \tilde{A})\tilde{B}|$. Hence $\tilde{G} = (A_\pi \times \tilde{A})\tilde{B}$ is a subgroup of G, which is a product of two π-decomposable groups, and contains N. If $\tilde{G} < G$, then by the choice of G we deduce that N is a D_π-group, which is a contradiction. This implies that $\sigma = \emptyset$ and the assertion (8) follows.

(9) Assume finally that N is a simple group of Lie type over the field $GF(q)$ of characteristic p. If P is any Sylow p-subgroup of N, then $C_{\mathrm{Aut}(N)}(P)$ is a p-group (see [15, 1.17]), so it follows from (6) that $p \in \pi'$.

Now, assume that N has Lie rank $l > 1$, and $|N \cap B|$ is not divisible by p. Applying that $|N||A \cap B| = |G/N||N \cap A||N \cap B|$ and Lemma 6 we get, as in the proof of [21, Lemma 12], that $|N|_p/|N \cap A|_p \leq |G/N|_p \leq |Out(N)|_p \leq q$. Now, if P is a Sylow p-subgroup of $N \cap A$, we deduce by [21, Lemma 6] that $C_A(P)$ is a p-group, and so we get a contradiction by (6).

Finally, assume that there exists an element $a \in A \cap N$ of prime order r and a subgroup X of G such that $A \leq X$ and $C_{X \cap N}(a)$ is an abelian p'-subgroup. If $r \in \pi'$, then $A_\pi \leq C_{X \cap N}(a)$ and hence A_π is abelian, a contradiction with (5). Then $r \in \pi$, and $A_{\pi'} \cap N \leq C_{X \cap N}(a)$ is an abelian p'-group. It follows that $A \cap N$ is soluble, and so is A. Moreover, since $p \in \pi'$, $A \cap N$ is a p'-group. In this case $|A|_p = |AN/N|_p = |G/N|_p \leq q$ by Lemma 6. The last assertion is straightforward. Hence (9) holds. □

For a group G as in Proposition 1, and its unique minimal normal subgroup N, we have the following results:

Lemma 7 *N is not a sporadic simple group.*

Proof By [25, Theorem C] if N is a sporadic simple group, $N \trianglelefteq G \leq \mathrm{Aut}(N)$, and G is factorized, we have that

$$N \in \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS, He, Ru, Suz, Fi_{22}, Co_1\}.$$

Now, by comparing [29, Theorem 8.2 (Table 3)] with [29, Theorem 6.9, Condition II], we can deduce that any of the above groups being an E_π-group for a set of odd primes π, is also a D_π-group. So this case is not possible, by Proposition 1(4). □

Lemma 8 *N is not an alternating group of degree $n \geq 5$.*

Proof By [29, Theorem 8.1], if π is a set of odd primes with $|\pi| > 1$, then any alternating group is not an E_π-group. So this case can be ruled out, by Proposition 1(4). □

Lemma 9 *N is not an exceptional group of Lie type.*

Proof By [25, Theorem B], if N is an exceptional group of Lie type, $N \trianglelefteq G \leq \mathrm{Aut}(N)$ and G is factorized, then

$$N \in \{G_2(q), q = 3^c;\ F_4(q), q = 2^c; G_2(4)\}.$$

If $N = G_2(q)$, then N is a D_π-group for any set of primes π such that $2, p \notin \pi$ by [29, Theorem 6.9, Condition II], a contradiction.

Let $N = F_4(q), q = 2^c$. In this case all possible factorizations $G = AB$ (not only the maximal ones) with subgroups A, B not containing N are as follows: For $\{X, Y\} = \{A, B\}$, $X \cap N = Sp_8(q)$ and $Y \cap N \in \{{}^3D_4(q), {}^3D_4(q).3\}$ and $N = (A \cap N)(B \cap N)$. Since 2 divides $(|A \cap N|, |B \cap N|)$ and each of these subgroups has a Sylow 2-subgroup containing its centralizer in the corresponding subgroup, it follows that N is a π'-group, which is a contradiction. □

3.1 The almost simple case for classical groups of Lie type

In the sequel, $G = AB$ will be a minimal counterexample to the Main Theorem, which is an almost simple group with socle N (i.e. $N \leq G \leq \mathrm{Aut}(N)$), and N is a classical simple group of Lie type over a field $GF(q)$ of prime characteristic p, with $q = p^e$, e a positive integer. Information about the structure of such G is collected in Proposition 1. In particular, recall that $G = AN = BN = AB$ and B is a π'-group.

For such a group G, we will use the knowledge about the non-trivial factorizations $G = XY$, where X and Y are maximal subgroups of G not containing N, which appears in [25, Theorem A, Tables 1-4]. In the referred tables and the corresponding proofs an explicit description of the subgroups $X_0 \trianglelefteq X \cap N$, $Y_0 \trianglelefteq Y \cap N$ is given. We notice that the notation used in [25] (see page 5) is X_A and X_B, instead of X_0 and Y_0, respectively, the latter being more convenient for our purposes. We point out that in general $X_0 = X \cap N$ and likewise $Y_0 = Y \cap N$, as can be seen in [25, 2.2.5]; in the exceptional cases, we give information about $|X \cap N : X_0|$ or $|Y \cap N : Y_0|$, respectively.

For any appearence of a pair (X, Y), we will distinguish two cases: either $A \leq X, B \leq Y$, or $A \leq Y, B \leq X$. Note that if we assume, for instance, that $A \leq X, B \leq Y$, then the following facts hold:
- $G = AY = BX$.
- $G = XN = YN$.
- $\pi \subseteq \pi(X)$, and A_π is a Hall π-subgroup of X.
- $|G : Y| = |N : N \cap Y|$ divides $|A|$.
- $|G : X| = |N : N \cap X|$ divides $|B|$.
- $X = A(B \cap X)$, $Y = B(A \cap Y)$, and both X, Y satisfy D_π.

We will keep as close as possible the notation in [25] for the subgroups of the almost simple groups. In particular, if $\hat{N}$ is a classical linear group on the vector space V with centre Z (so that $N = \hat{N}/Z$ is a classical simple group) and $\hat{N} \trianglelefteq \hat{G} \leq GL(V)$, for any subgroup X of $\hat{G}$ we will denote by $\hat{}X$ the subgroup $(XZ \cap \hat{N})/Z$ of N. Also we will use the notation P_i, $1 \leq i \leq m$, N_i, N_i^ϵ $(\epsilon = \pm)$, for stabilizers of subspaces as described in [25, 2.2.4].

We will use frequently, without further reference, the fact that if N is a simple group of Lie type over $GF(q)$, $q = p^e$, $n \geq 3$ is an integer, and $(q, n) \neq (2, 6)$, then q_n does not divide $|Out(N)|$ (see [25, 2.4 Proposition B]). Also the information about elements whose order is a primitive prime divisor appearing in Lemma 5 will be used eventually without reference.

In the proof of the next lemmas we will refer to Proposition 1(x) *just as* 1(x). *Moreover, unless otherwise stated,* 1(9) *will mean* 1(9)c).

Lemma 10 *N is not isomorphic to $L_m(q)$.*

Proof We do not consider here the cases when $L_m(q)$ is isomorphic to an alternating group, which have been discarded in Lemma 8.

Recall that $|\pi(N)| \geq 5$ by 1(8).

We analyze first the cases $m \leq 3$. If $N = L_2(q)$, all possible factorizations appearing in [25, Tables 1,3] can be ruled out, except possibly for $q = 29$ or $q = 59$, because either $|\pi(N)| \leq 4$ which contradicts 1(8), or both X and Y are soluble, contradicting 1(7). When $q \in \{29, 59\}$, there exists also a factorization $G = XY$ with $X \cap N = P_1$, $Y \cap N = A_5$, the alternating group of degree 5. In both cases it holds that $q \in \pi'$ by 1(9), $\{3, 5\} \subseteq \pi$ and $A \leq A_5$. But the fact that A_5 does not have Hall $\{3, 5\}$-subgroups leads to a contradicton.

Consider now $N = L_3(q)$, so $|N| = (1/(q-1, 3))q^3(q^3-1)(q^2-1)$ and $|Out(N)| = 2(q-1, 3)e$. The cases when $q \leq 8$ are excluded by 1(8). From [25, Tables 1,3] we know that all factorizations $G = XY$ satisfy that for one of the factors, say X, $|N \cap X|$ divides $\frac{q^3-1}{q-1} \cdot 3$, which is not divisible by p if $p \neq 3$. Since $p \in \pi(N \cap B)$ by 1(9) this forces that $A \leq X$ when $p \neq 3$. For the case $p = 3$, we get that $|N : N \cap Y|_3 \leq q/3 < q^2$, so $C_G(N \cap Y)$ is a p-group because of [21, Lemma 6], and this also implies that $A \leq X$. Since $q^3 - 1$ divides $|G : Y|$ we get that $r = q_3 \in \pi(A)$. But from the known structure of $X \cap N$ (this group is the normalizer of a Singer cycle of order $(q^3-1)/(q-1)$), it is clear that an element of order 3 cannot centralize an element of order dividing $(q^3 - 1)/(q - 1)$. This means that $A_\pi \leq A \cap N$ is contained in a Singer cycle, and so it is abelian, which contradicts 1(5).

Assume from now on that $m \geq 4$. In this case both q_m and q_{m-1} exist, except when $(m, q) = (6, 2)$ or $(7, 2)$.

Take first $N = L_6(2)$, so $|N| = 2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$ and $|Out(N)| = 2$. This group has a Sylow 31-subgroup which is self-centralizing in G. This forces that $31 \in \pi(B) \subseteq \pi'$. Hence, according to [25, Table 1], it holds $B \leq Y$ with $Y \cap N = P_1$ or P_5, and $A \leq X$, with $X \cap N = \hat{}GL_a(2^b)$, $ab = 6, b$ prime, or $X \cap N = PSp_6(2)$. In all cases $2^6 - 1 = 7 \cdot 3^2$ divides $|A|$. The cases $X \cap N = \hat{}GL_3(2^2).2$ and $X \cap N = PSp_6(2)$ can be ruled out by applying that a Sylow 7-subgroup of such a group is self-centralizing and $7 \in \pi(A) \cap \pi(B) \subseteq \pi'$, which forces that A is a π'-group. If $X \cap N = \hat{}GL_2(2^3).3$, by order arguments it holds $\{3, 5\} \subseteq \pi(B)$ and so $\pi = \{7\}$, a contradiction.

Take now $N = L_7(2)$, so $|N| = 2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$ and $|Out(N)| = 2$. In this case $r = q_m = 127$ and N has a self-centraling Sylow r-subgroup, which forces that $r \in \pi(B)$. Then, by [25, Table 1], it holds that $A \leq Y$ with $Y \cap N = P_1$ or P_6, and $B \leq X$ with $X \cap N = \hat{}GL_1(2^7).7 = 127.7$, a contradiction with 1(9)b) because $p = 2$ should divide $|N \cap B|$.

So we may assume from now on that both q_m and q_{m-1} exist. We analyze next the different factorizations $G = XY$ appearing in [25, Table 1].

Case $X \cap N = \hat{}GL_a(q^b).b$ (with $m = ab$, b prime), $Y \cap N = P_1$ or P_{m-1}.

Assume first that $A \leq X$ and $B \leq Y$. Then $r = q_m$ divides $|G : Y|$, and so $q_m \in \pi(A)$. If we take an element $a \in A \cap N$ of order r it holds that $C_N(a)$ is contained in a torus of order $(q^m-1)/((m, q-1)(q-1))$. Hence, by 1(9), we deduce that $r \in \pi$, and A is soluble. From [2, Lemma 2.5] the order of a soluble subgroup of N whose order is divisible by r should divide either $m(q^m-1)$, or $2m^2(m+1)(q-1)l$

with $q = p$, $r = m+1$, $m = 2^l$. But in this last case, $r = m+1$ is the only primitive prime divisor of such group with respect to the pair (q,m) and so, applying [25, 2.4 Proposition D] it is $(q,m) \in \{(2,4),(2,10),(2,12),(2,18),(3,4),(3,6),(5,6)\}$, but this contradicts the fact that m is a power of 2 and $|\pi(G)| \geq 5$. Hence $A \cap N$ is a soluble subgroup of order dividing $m(q^m - 1)$. Indeed, by [24, Theorem 1.1], such a soluble group is contained in $\hat{}GL_1(q^m).m$, the normalizer of a Singer cycle in N and, moreover, since B is non-soluble it contains $SL_{m-1}(q)$. Now, if there would exist a prime divisor t of $|A_\pi|$ such that t is not a primitive prime divisor of $q^m - 1$, then t would divide $|B|$, which is a π'-group, a contradiction. Hence any prime divisor t of $|A_\pi|$ is indeed a primitive prime divisor of $q^m - 1$, and so $t > m$ and $\pi \subseteq \pi((q^m - 1)/((m, q-1)(q-1)))$. Therefore, since A_π is the unique Hall π-subgroup of the soluble group $A \cap N$, from the known structure of the normalizer of a Singer cycle (cf. [12, II. 7.3 Satz]) we get that A_π is contained in a Singer cycle and so it is abelian, which contradicts 1(5).

Let suppose now that $A \leq Y$, $B \leq X$. Then $q^{m-1} - 1$ divides $|G : X|$ and so $s = q_{m-1} \in \pi(A)$. If we consider an element b of order s in $A \cap N$ it holds by Lemma 5 that $C_N(b)$ is contained in an abelian subgroup of order $(q^{m-1} - 1)/(q-1, m)$. Therefore, by 1(9), it holds that $s \in \pi$, $A \cap N$ is a p'-group and $|A|_p \leq q$. But this is a contradiction since $|G : X|_p \geq q^2$, and $|G : X|$ divides $|A|$.

Case $X \cap N = PSp_m(q).a$, $m = 2k$ even, $m \geq 4$, $a \leq 2$. Here $Y \cap N = P_1, P_{m-1}$ or $Y \cap N = Stab(V_1 \oplus V_{m-1})$ (and in this case G contains a graph automorphism).

Assume first that $A \leq X$. Then $q^m - 1$ divides $|G : Y|$, and so we deduce that $r = q_m \in \pi(A)$. Since $m = 2k$ is even, the centralizer of an element of order r in $X \cap N$ is an abelian group of order $\frac{q^k+1}{(2,q-1)}$, by Lemma 5. By 1(9), we can deduce that $r \in \pi$ and A is soluble. But by [2, Lemma 2.8] the order of a soluble subgroup of $PSp_m(q)$ whose order is divisible by r should divide either $2k(q^k + 1)$, or $16k^2(q-1)r\log_2(2k)$ with $q = p$, $r = 2k+1 = m+1$, and k a power of 2. In the last case, it holds that r is the only primitive prime divisor of $q^m - 1$, and so by [25, 2.4 Proposition D] we get a contradiction as above. Hence $|A \cap N|$ should divide $2k(q^k + 1)$, which leads to a contradiction by order arguments since $q^m - 1$ divides $|A \cap N|$.

Assume now that $B \leq X$ and $A \leq Y$. Then $q^{m-1} - 1$ divides $|G : X|$ and so $|A|$. This means that $s = q_{m-1} \in \pi(A)$ and applying 1(9) and arguing as above we get a contradiction, because here also $|G : X|_p \geq q^2$.

Case $X \cap N = \hat{}GL_{m/2}(q^2).2$, $Y \cap N = Stab(V_1 \oplus V_{m-1})$, with $q \in \{2,4\}$, and $m \geq 4$ even (here G contains a graph automorphism).

Assume first that $A \leq X$, $B \leq Y$. Clearly, $|G : Y|$ is divisible by $r = q_m$, so that $r \in \pi(A)$. The centralizer in N of an element a of order r in $N \cap A$ is contained in a torus of order $(q^m - 1)/((m, q-1)(q-1))$, and moreover, there is no any outer automorphism centralizing an element of order r (here we are currently assuming that $m \leq 4$, so $(m, q-1) = 1$). Hence, by 1(9) we deduce that A is a $2'$-group, which is a contradiction by order arguments (here $2 \in \pi(G/N)$).

If $A \leq Y$, $B \leq X$, then $s = q_{m-1} \in \pi(A)$ and arguing as above we get a contradiction with 1(9).

Finally, from [25, Table 3] for the case $m = 5$, $q = 2$ there exists also a factorization $G = XY$ with $X \cap N = 31.5$ and $Y \cap N = P_2$ or P_3. Since a Sylow

31-subgroup of N is self centralizing in G, it holds $31 \notin \pi(A)$ and so $A \leq Y$. But then $N \cap B$ is a p'-group, which contradicts 1(9) b). $\square$

Lemma 11 *N is not isomorphic to $U_m(q)$, $m \geq 3$.*

Proof Assume that $N = U_m(q)$, $m \geq 3$. Recall that $p \in \pi'$, by 1(9)a).

Case m odd. From [25, Theorem A, Corollary 2], the only groups G such that $N \leq G \leq \mathrm{Aut}(N)$ which are factorizable appear for $N = U_3(3)$, $U_3(5)$, $U_3(8)$ or $U_9(2)$. By 1(8), $|\pi(N)| \geq 5$. Thus the only possible case would be $N = U_9(2)$. Note that $|N| = 2^{36} \cdot 3^{11} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 43$ and $|Out(N)| = 6$. From [25, Table 3], there exists a unique factorization of this group such that $X \cap N = J_3$ and $Y \cap N = P_1$, where P_1 is a parabolic maximal subgroup of N.

Assume first that $A \cap N \leq X \cap N = J_3$. Note that $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$. In this case, $\{3, 5\} \subseteq \pi(B) \subseteq \pi'$, by order arguments. Hence $\pi \subseteq \{17, 19\}$. But J_3 has no subgroups of order $17 \cdot 19$ (cf. [7, p. 82]), so we get a contradiction. Therefore, we may assume that $A \cap N \leq Y \cap N = P_1$, a parabolic subgroup. In such case, $\{2, 3, 5, 17, 19\} \subseteq \pi'$, and a Hall π-subgroup of N has order dividing $7 \cdot 11 \cdot 43$. But there are no subgroups of order $7 \cdot 11$ in $U_9(2)$ (see for example [28, Theorem 8.8]), and any subgroup of order 43 is self-centralizing, so this case cannot occur.

Case m even, $m = 2k$, $k \geq 2$. It follows from [25, Tables 1, 3] (and with the same notation) that one of the maximal subgroups in the factorization of G with $N \leq G \leq \mathrm{Aut}(N)$, say X, has the property $X \cap N = N_1 = U_{2k-1}(q)$, unless $N = U_4(2)$ or $U_4(3)$. Since $|U_4(2)| = 2^6 \cdot 3^4 \cdot 5$ and $|U_4(3)| = 3^6 \cdot 2^7 \cdot 5 \cdot 7$, these possibilities are excluded by 1(8).

Apart from some exceptional cases, that we check below, since N_1 is a unitary group of odd dimension, it is known from [25, Tables 1, 3] that any subgroup H such that $N_1 \leq H \leq \mathrm{Aut}(N_1)$ has no proper factorization with factors not containing N_1.

Assume that $A \leq X$, and so $X = A(X \cap B)$. Now note that $X = N_G(N_1)$, so $X/C_G(N_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(N_1)$. Therefore $X/C_G(N_1)$ has no proper factorizations. If $N_1 \cong N_1 C_G(N_1)/C_G(N_1)$ were contained either in the π-decomposable group $AC_G(N_1)/C_G(N_1)$ or in the π'-group $(X \cap B)C_G(N_1)/C_G(N_1)$, it would follow that $N_1 = X \cap N$ would be a π'-group, a contradiction as $A \cap N \leq N_1 = X \cap N$. Hence it holds that either $X = AC_G(N_1)$ or $X = (X \cap B)C_G(N_1)$, and we can argue like in the proof of [21, Lemma 17] to get also a contradiction.

The exceptional cases, when $N = U_{2k}(q)$ and $X \cap N = N_1 = U_{2k-1}(q)$ is factorized, appear for $N_1 = U_3(3)$, $U_3(5)$, $U_3(8)$ or $U_9(2)$, by [25, Table 3]. The case $N = U_4(3)$ is excluded by 1(8) as mentioned above. Hence we should study the cases $N = U_4(5)$, $U_4(8)$ and $U_{10}(2)$.

For $N = U_4(5)$ we have $|X \cap N| = |U_3(5)| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$. If $A \leq X$, then by order arguments, it holds $\pi = \{7\}$, a contradiction. By similar arguments, we get a contradiction for $N = U_3(8)$ when $X \cap N = N_1 = U_2(8)$ and $A \leq X$.

Suppose $N = U_{10}(2)$ and $A \leq X$, so that $A \cap N \leq N_1 = U_9(2)$. In this case, the centralizer in G of any element $a \in A \cap N$ of order $r = 19$ is an abelian $2'$-group (recall that a field automorphism does not centralize any element of order r). By 1(9), A is a $2'$-group. This leads to a contradiction by order arguments in all cases, except possibly when $Y \cap N = P_5$ is a parabolic subgroup. In this last case, $|G : Y| = \prod_{i=2}^{5}(q^{2i-1} + 1)(q + 1)$ for $q = 2$ (see [25, 3.3.3]), and this number divides $|A|$. Then we can consider the prime $s = q_{10} = 11 \in \pi(A)$ and an element of order s in $A \cap N$ whose centralizer in N is an abelian group. Since $s \in \pi(B) \subseteq \pi'$, this means that A_π is abelian, which contradicts 1(5).

Therefore we may assume in all cases that $A \leq Y$, $B \leq X$. It follows that $|G : X|$ divides $|A|$. By [25, 3.3.3], it holds that

$$|N : N \cap X| = |G : X| = q^{m-1}(q^m - 1)/(q + 1).$$

Then there exists a maximal torus T_1 of order $(q^m - 1)/((m, q+1)(q+1))$ in N (cf. [28, Lemma 1.2]) and an element $a \in A \cap N$ of order $r = q_m$, such that $C_N(a) \leq T_1$. Since T_1 is an abelian p'-group, it follows by 1(9) that $A \cap N$ is a p'-group and $|A|_p \leq q$. But this is a contradiction, since q^{m-1} divides $|A|$. ◻

Lemma 12 *N is not isomorphic to $PSp_{2m}(q)$, neither to $P\Omega_{2m+1}(q)$, $m \geq 2$.*

Proof By comparing [29, Theorem 8.8 (Table 7)] (see also [9, Theorem 4.3]) with [29, Theorem 6.9, Condition V], we can deduce that if $N \in \{PSp_{2m}(q), P\Omega_{2m+1}(q)\}$ is an E_π-group for a set of primes π with $2, p \notin \pi$, then it is also a D_π-group, so we get a contradiction. ◻

Lemma 13 *N is not isomorphic to $P\Omega^-_{2m}(q)$, $m \geq 4$.*

Proof We consider first the factorizations appearing in Table 1 in [25] (with the same notation).

For the case $N = P\Omega^-_8(2)$ there exists only a factorization with $X \cap N = \Omega^-_m(4).2 \cong L_2(16).2$, $Y \cap N = N_1 = PSp_6(2)$, which arises when $G = \mathrm{Aut}(N) = N.2$ (see [25, 5.1.16]). Here $|G : X| = 2^7 \cdot 3^3 \cdot 7$ and $|G : Y| = 2^3 \cdot 17$. The case $A \leq X$ cannot occur, since in this case $17 \in \pi(A)$ and N has a Sylow 17-subgroup which is self-centralizing in G, a contradiction by 1(6). Hence, we may assume that $A \leq Y$ and so $7 \in \pi(A)$. But a Sylow 7-subgroup (of order 7) is self-centralizing in $Y \cap N$, so by 1(9) we deduce that $|A|_2 \leq 2$, which is a contradiction since 2^7 divides $|A|$.

We assume from now on $m \geq 4$, and $m \geq 5$ for $q = 2$. Here q_{2m} always exists.

Case $X \cap N = \hat{\ }GU_m(q)$, m odd. In such case, $Y \cap N = P_1$, or $Y \cap N = N_1$.

Suppose first that $A \leq X$, so that $\pi(|G : Y|) \subseteq \pi(A)$ and $r = q_{2m} \in \pi(A)$. If a is an element of order r in $N \cap A$, then its centralizer in $X \cap N$ is contained in a torus of order dividing $(q^m + 1)$, and so $r \in \pi$, A is soluble and $A \cap N$ is a p'-group, by 1(9). If $Y \cap N = N_1$, then $|G : Y| = (1/(2, q-1))q^{m-1}(q^m + 1)$ by [25, 3.5] and this gives a contradiction by order arguments. If $Y \cap N = P_1$, then $(q^{m-1} - 1)(q^m + 1)$ divides $|G : Y|$, and so $|A|$. In particular, $t = q_{m-1} \in \pi(A \cap N)$ (this prime exists if $(m, q) \neq (7, 2)$, since $m \geq 4$). Then by 1(7) N would have a soluble maximal subgroup of order divisible by t and r, so they should divide $2m(q^m + 1)$ by [2, Lemma 2.8]. This can only happen if $m = t$ prime and $q \in \{2, 3, 5\}$ (by [25, 2.4, Proposition D]), but this contradicts the fact that $A \cap N$ is soluble and its order also divides $2m(q^m + 1)$, which is not the case. In the case when $(m, q) = (7, 2)$ we get also a contradiction because $G \leq N.2$ and the order of a maximal soluble subgroup of N divisible by $r = 51$ is not divisible by $q^{m-1} - 1 = 7 \cdot 9$.

Hence, we may assume that $A \leq Y$. This means that $s = q_{2m-2} \in \pi(A)$ (recall that m is odd). In this case there is a torus $T \leq N$ of order $(q^{m-1}+1)(q-1)/(4, q^m + 1)$ containing the centralizer in N of an element $a \in A \cap N$ of order s. This is again a contradiction, by 1(9).

Case $X \cap N = \Omega_m^-(q^2).2$, $Y \cap N = N_1$. Here $G = \mathrm{Aut}(N) = XY$ and $q \in \{2, 4\}$ (see [25, 3.5.1]). Note that $|G : Y| = q^{m-1}(q^m + 1)$.

If $q = 2$, then $G = O_{2m}^-(2) = N.2$ and $X \cap N = \Omega_m^-(4).2$. Suppose first that $A \leq X$, $B \leq Y$. Since $|G : Y|$ divides $|A|$, it follows that $r = q_{2m} \in \pi(A)$. If $a \in A \cap N$ is an element of order r, then $C_N(a)$ is contained in a torus of order $(q^m + 1)$. This provides a contradiction by 1(9).

Now we may assume that $A \leq Y$, so that $s = q_{2m-2} \in \pi(A)$ (recall that $m \geq 5$ in this case). Let a be an element of order s in $A \cap N$. It follows that $C_N(a)$ is contained in a torus of order $(q^{m-1}+1)(q+1)/(4, q^m+1)$. This leads again to a contradiction by 1(9), because $|N \cap B|_2 |G/N|_2 < |N|_2$ since $N \cap B \leq X \cap N = \Omega_m^-(4).2$.

If $q = 4$, then $G = N.4$ and $X \cap N = \Omega_m^-(16).2$. Suppose first that $A \leq X$, and so $|G : Y|$ divides $|A|$. Then $r = q_{2m} \in \pi(A)$. But if a is an element of order r in $N \cap A$, then $C_N(a)$ is contained in a torus of order $(q^m + 1)$, a contradiction by 1(9).

Now we may assume that $A \leq Y$. So that $s = q_{2m-2} \in \pi(A)$. Let a be an element of order s in $A \cap N$. It follows that $C_N(a)$ is contained in a torus of order $(q^{m-1} + 1)(q + 1)$. Then $s \in \pi$ and so $A \cap N$ is of odd order, by 1(9), which is not the case.

Case $X \cap N = GU_m(4), Y \cap N = N_2^+$. Here $G = \mathrm{Aut}(N) = N.4$, and m is odd.

In this case, by [25, 3.5.2(c)], for $q = 4$, it holds:

$$|G : Y| = (1/2)\frac{q^{2m-2}(q^m+1)(q^{m-1}-1)}{(q-1)}.$$

If $A \leq X$, then $r = q_{2m} \in \pi(A)$. Then there exists an element $a \in A \cap N$ of order r such that $C_N(A)$ is contained in a torus of order $q^m + 1$. Then, by 1(9), $A \cap N$ is of odd order and $|A|_2 \leq 4$, which is a contradiction since $|G : Y|$ divides $|A|$.

Therefore $A \leq Y$ and $s = q_{2m-2} \in \pi(A)$. Since N contains a maximal torus of order $(q^{m-1} + 1)(q - 1)$, which is the centralizer of an element $a \in A \cap N$ of order s, this means that $s \in \pi$ and $A \cap N$ is of odd order. In fact, there is no field automorphism centralizing an element of order s, hence $C_G(a)$, and so A, is of odd order. But this is again a contradiction, because $|G/N| = 4$ in this case.

It remains to consider the factorization for the case $N = P\Omega_{10}^-(2)$ which appears in [25, Table 3]. In this case $X \cap N = A_{12}$ and $Y \cap N = P_1$. Since the alternating group A_{12} does not contain Hall π-subgroups with $2 \notin \pi$ and $|\pi| \geq 2$, it is $A \leq Y$, and $B \leq X$. In this case $17 \in \pi(A)$, and G has a self-centralizing Sylow 17-subgroup, which contradicts 1(6).

The lemma is proved. □

Lemma 14 *N is not isomorphic to $P\Omega_{2m}^+(q), m \geq 4$.*

Proof Assume first that $N = P\Omega_{2m}^+(q)$, for $m > 4$, and consider the factorizations appearing in [25, Table 1]. Let $d = (4, q^m - 1)$.

Case $X \cap N = N_1$. Here $|G : X| = \frac{1}{(2,q-1)} q^{m-1}(q^m - 1)$.

Suppose first that $A \leq X$ and $B \leq Y$. We distinguish the different possibilities for $Y_0 \trianglelefteq Y \cap N$. Unless otherwise specified, we recall that $Y_0 = Y \cap N$.

$Y \cap N = P_m$ or $Y \cap N = P_{m-1}$ (stabilizers of totally singular m-subspaces). Then $|G : Y| = \prod_{i=1}^{m-1}(q^i + 1)$ (see [25, 3.6.1]). Moreover $|G : Y|$ divides $|A|$, and $r = q_{2m-2} \in \pi(A)$ (this prime exists since we are currently assuming that $m > 4$).

There exists an element $a \in A \cap N$ of order r such that $C_N(A)$ is contained in a torus of order $(1/d)(q^{m-1}+1)(q+1)$ (see Lemma 5). By 1(9), $r \in \pi$ and $A \cap N$ is a p'-group. If $(q,m) \neq (2,5)$, then there exists a prime $s = q_{2m-4} \in \pi(A)$. Then N must have a soluble subgroup of order divisible by r and s, which is not the case by [2, Lemma 2.8]. If $(q,m) = (2,5)$, then $|G:Y| = 3^3 \cdot 5 \cdot 17$. But N has no subgroup of order $5 \cdot 17$, a contradiction.

$Y \cap N = \hat{\ }GL_m(q).2$. Note that $G \geq N.2$, when m is odd. Since $|G:Y|$ divides $|A|$, we have $r = q_{2m-2} \in \pi(A)$ again. If $a \in A \cap N$ is an element of order r, then $C_N(a)$ is contained in a torus of order $(1/d)(q^{m-1}+1)(q+1))$. Hence $|A|_p \leq q$, by 1(9). But this is not the case, since $|G:Y|_p > q$.

$Y \cap N = \hat{\ }GU_m(q).2$, m even. In this case $s = q_{m-1} \in \pi(A)$ and the centralizer in N of an element $a \in A \cap N$ of order s is contained in a torus of order $(1/d)(q^{m-1}-1)(q-1)$. Hence, by 1(9), it holds $|A|_p \leq q$, which is not possible since $|G:Y|$ divides $|A|$.

$Y \cap N = \Omega^+_m(4).2^2, q = 2$, $m = 2k$ even. Then $r = q_{2m-2} \in \pi(A), s = q_{2m-4} \in \pi(A)$. Arguing as in the previous cases, we deduce that $r \in \pi$. But N does not contain any soluble subgroup whose order is divisible by r and s by [2, Lemma 2.8], so we obtain a contradiction with 1(7).

$Y \cap N = \Omega^+_m(16).2^2, q = 4$, m even. Then again $r = q_{2m-2} \in \pi(A)$, $s = q_{2m-4} \in \pi(A)$ and we obtain a contradiction as above.

$Y_0 = PSp_2(q) \otimes PSp_m(q)$, where $m = 2k$ even, $q > 2$. Here Y_0 has index 1 or 2 in $Y \cap N$ (cf. [23, Table 3.5.E, Prop. 4.4.12]). In this case also $r = q_{2m-2} \in \pi(A)$. If $a \in A \cap N$ is an element of order r, then $C_N(A)$ is contained in a torus T_1 of order $(1/d)(q^{m-1}+1)(q+1))$. By 1(9), $r \in \pi$ and $|A|_p \leq q$, which is not case, since $|G:Y|$ divides $|A|$.

Now assume that $A \leq Y$, $B \leq X$ (for the case $X \cap N = N_1$ under consideration). This implies that $|G:X| = (1/(2,q-1))q^{m-1}(q^m-1)$ divides $|A|$.

If m is odd, we consider $s = q_m \in \pi(A)$. In this case $C_N(a)$ for $a \in A \cap N$ of order s is contained in a torus of order $(1/d)(q^m-1)$. By 1(9), it can be deduced $|A|_p \leq q$, which is not the case since q^{m-1} divides $|A|$.

We distinguish now the different possibilities for $Y_0 \trianglelefteq Y \cap N$ when m is even. Unless otherwise specified, it holds $Y_0 = Y \cap N$.

$Y \cap N = P_m$ or P_{m-1}. In this case we have that $s = q_m \in \pi(A)$ and the centralizer in $Y \cap N$ of an element $a \in A \cap N$ of order s is contained in an abelian subgroup of $Y \cap N$ of order $q^m - 1$. Hence, by 1(9), $|A|_p \leq q$, which is a contradiction since q^{m-1} divides $|A|$.

$Y \cap N = \hat{\ }GL_m(q).2$, m even. We can argue as in the previous case, since $s = q_m \in \pi(A)$ and the centralizer in $Y \cap N$ of an element $a \in A \cap N$ of order s is again an abelian p'-group, which implies that $|A_p| \leq q$ and leads to a contradiction.

$Y \cap N = \hat{\ }GU_m(q).2$, $m = 2k$ even. We can consider a prime $u = q_k \in \pi(A)$. Then the centralizer in $Y \cap N$ of an element $a \in A \cap N$ of order u is contained in a torus of order $(q^k+1)/(q+1,m)(q+1)$, which is a p'-group. Therefore, by 1(9) we get that $|A_p| \leq q$, and this is a contradicion because q^{m-1} divides $|A|$.

$Y_0 = PSp_2(q) \otimes PSp_m(q)$, m even, $q > 2$. If $a \in A \cap N$ is an element of order $r = q_m$, then its centralizer in $Y \cap N$ is contained in $T \times L_2(q)$, where T is an abelian p'-group (recall $PSp_2(q) \cong L_2(q)$). If $r \in \pi$, then $|A \cap N|_p \leq q$, which gives a contradiction since q^{m-1} divides $|A|$ and $m > 4$. Now, if $r \in \pi'$, then A_π is a Hall π-subgroup of $T \times L_2(q)$, so it is abelian, a contradiction.

$Y \cap N = \Omega_m^+(q^2).2^2, q \in \{2,4\}$, $m = 2k$. In these cases we consider the prime $t = q_k \in \pi(A)$ and a torus in $Y \cap N$ of order dividing $(q^k - 1)$, which is an abelian $2'$-group that contains the centralizer in $Y \cap N$ of an element of order t in $A \cap N$. Applying 1(9) we get that $|A_p| \leq q$, which is a contradiction.

Case $X \cap N = P_1$. Here $Y \cap N = \hat{}GU_m(q).2$, m even.

If $A \leq X$ and $B \leq Y$, we argue as in the case when $X \cap N = N_1$ and $Y \cap N = \hat{}GU_m(q).2$, m even, by considering an element in A of order $s = q_{m-1}$ and a torus containing its centralizer to get a contradiction with 1(9).

If $A \leq Y, B \leq X$, then $|G : X|$ divides $|A|$. Therefore $r = q_{2m-2}$, $t = q_m \in \pi(A)$. The centralizer of an element $a \in A \cap N$ of order t in $Y \cap N$ is contained in an abelian subgroup of order $q^m - 1$. Hence, by 1(9), $t \in \pi$. But there is no soluble subgroup in N of order divisible by r and t, by [2, Lemma 2.8], so we obtain a contradiction with 1(7).

Case $X \cap N = N_2^-$. Here $|G : X| = (1/2)q^{2m-2}(q^m - 1)(q^{m-1} - 1)/(q + 1)$, and $(X/Z(X))' \cong P\Omega_{2m-2}^-(q)$.

First assume $A \leq X$, $B \leq Y$.

Let $Y \cap N = \hat{}GL_m(q).2$, $q \in \{2,4\}$. For $q = 2$, $G \geq N.2$ if m is odd, and for $q = 4$, $G \geq N.2$ and $G \neq O_{2m}^+(4)$. Since $|G : Y|$ divides $|A|$, it holds $r = q_{2m-2} \in \pi(A)$. The centralizer in N of an element of order r in $N \cap A$ is an abelian $2'$-group. Hence, by 1(9), it holds that $|A|_2 \leq q$, which is not the case by order arguments.

Assume now that $Y \cap N = P_m$ or P_{m-1}. Since $|G : Y| = \prod_{i=1}^{m-1}(q^i + 1)$, it holds that $r = q_{2m-2}, s = q_{2m-4} \in \pi(A)$ (s exists when $(q,m) \neq (2,5)$). Again the centralizer in N of an element of order r in $N \cap A$ is an abelian p'-group, so $r \in \pi$ by 1(9). But then by 1(7) N should have a soluble subgroup of order divisible by r and s, which is not the case (see [2, Lemma 2.8]). The case $(q,m) = (2,5)$ is excluded as above in the Case $X \cap N = N_1$.

Now we may suppose $A \leq Y$, $B \leq X$.

Consider $s = q_{m-1}$ when m is even, and $s = q_m$ when m is odd. In any case, $s \in \pi(A)$, since $|G : X|$ divides $|A|$.

For any choice of $Y \cap N$, we can find a torus T in N, which is an abelian p'-group, and $C_N(a) \leq T$ for an element a of order s in $A \cap N$ (see Lemma 5). By 1(9), $s \in \pi$ and $|A|_p \leq q$, a contradiction since $|G : X|$ divides $|A|$.

Case $X \cap N = N_2^+$. In this case $Y \cap N = \hat{}GU_m(4), q = 4, m$ even, and $G = N.2$. Here we have $|G : X| = \frac{1}{2}q^{2m-2}(q^m - 1)(q^{m-1} + 1)/(q - 1)$, by [25, (3.6.3 c)].

If $A \leq X$, we can take $s = q_{m-1} \in \pi(A)$. Then, by Lemma 5, there exists a torus of order $(1/d)(q^{m-1} - 1)(q - 1)$ containing the centralizer in N of an element of $A \cap N$ of order s. Hence, by 1(9), it holds $|A|_p \leq q$, which is not possible since $|G : Y|$ divides $|A|$.

Assume now $A \leq Y$, so $|G : X|$ divides $|A|$. In this case, $r = q_{2m-2} \in \pi(A)$, and again we have that there exists a torus of order $(1/d)(q^{m-1} + 1)(q + 1)$ which contains $C_N(A)$ for an element $a \in A \cap N$ of order r. Hence we get a contradiction with 1(9), because q^{2m-2} divides $|G : X|$, and so it divides $|A|$.

Next we consider some extra factorizations appearing in [25, Tables 2, 3].

Let $N = P\Omega_{16}^+(q)$, $X \cap N = \Omega_9(q).a$, $a \leq 2$, $Y \cap N = N_1 = \Omega_{15}(q)$. If $A \leq X$, we consider $r = q_{14} \in \pi(A)$. If $A \leq Y$ we consider $s = q_8 \in \pi(A)$ and a torus in $Y \cap N$ containing the centralizer of an element of order s in $A \cap N$. The contradiction arises in both cases applying 1(9), because in any case $|A|_p > q$.

Let $N = \Omega^+_{24}(2), X \cap N = N_1 = \Omega_{23}(2), Y \cap N = Co_1$. If $A \leq X$, then we consider $r = q_{22} \in \pi(A)$, to get a contradiction as in previous cases by 1(9). If $A \leq Y$, since the only Hall π-subgroups of Co_1 with $|\pi| \geq 2$ appear for $\pi = \{11, 23\}$ and a Sylow 23-subgroup in this group is self-centralizing, we get a contradiction by 1(6).

Consider now the case $m = 4$, i.e. $N = P\Omega^+_8(q)$.

By 1(8), we can assume that $q \neq 2$, since $|\pi(P\Omega^+_8(2))| = 4$.

According to [25, Table 4], apart from some exceptional cases considered below, the possible maximal subgroups X, Y are such that one of the following holds:

(a) $X \cap N = \Omega_7(q)$, $Y \cap N = \Omega_7(q)$;
(b) $X \cap N = \Omega_7(q)$, $Y \cap N =$ P_1, P_3 or P_4;
(c) $X \cap N = \Omega_7(q)$, $Y \cap N = \hat{}((q+1)/d \times \Omega^-_6(q)).2^d$;
(d) $X \cap N = \Omega_7(q)$, $Y \cap N = \hat{}((q-1)/d \times \Omega^+_6(q)).2^d$, $q > 2$, $G = N.2$ if $q = 3$;
(e) $X \cap N = \Omega_7(q)$, $Y \cap N = \Omega^-_8(q^{1/2})$, q a square;
(f) $X \cap N = \Omega_7(q)$, $Y \cap N = (PSp_2(q) \times PSp_4(q)).2$, q odd;
(g) $X \cap N = \hat{}((q+1)/d \times \Omega^-_6(q)).2^d$, $Y \cap N = P_1, P_3$ or P_4.

where $d = (2, q-1)$.

First note that, by checking all possibilities for $X \cap N$ and $Y \cap N$ in (a)-(g), it holds that $q^4 - 1$ divides $(|N \cap A|, |N \cap B|)$ by order arguments. This means in particular that $s = q_4 \in \pi(q^4 - 1) \subseteq \pi'$, and therefore $A_\pi \leq C_N(a)$ for any element $a \in A$ of order s.

Note that the case $A \cap N \leq Y \cap N = (PSp_2(q) \times PSp_4(q)).2$ cannot occur since $\pi(PSp_2(q) \times PSp_4(q)).2) \subseteq \pi(q^4 - 1) \cup \{p\} \subseteq \pi'$. For any other choice of $Z = X, Y$ in (a)-(g), if we assume that $A \leq Z$, we can find a suitable torus in $Z \cap N$ whose order is divisible by s, and which contains the centralizer in $Z \cap N$ of an element $a \in A \cap N$ of order s. Since a torus is an abelian p'-group, we get by 1(9) that $s \in \pi$, which is a contradiction. For the case $\Omega_7(q)$ we can consider, for instance, a torus of order dividing $(q^2 + 1)(q + 1)$. When $Z \cap N$ is one of the groups $\hat{}((q+1)/d \times \Omega^-_6(q)).2^d$, $\hat{}((q-1)/d \times \Omega^+_6(q)).2^d$, or one of the parabolic subgroups P_1, P_3, P_4, we can take a torus of order dividing $(q^2 + 1)(q - 1)(q + 1)$. Finally, we can consider a torus of $\Omega^-_8(q^{1/2})$, q a square, of order $(q^2 + 1)/(4, q^2 + 1)$.

For the cases $q = 3$ and $q = 4$ there are some extra factorizations in [25, Table 4], and in all cases it holds that $q^4 - 1$ divides $(|N \cap A|, |N \cap B|)$, and $s = q_4 \in \pi(A)$. (note that $q_4 = 5$ for $q = 3$, and $q_4 = 17$ for $q = 4$). We can argue as above in all cases except possibly when $N = P\Omega^+_8(3)$, $X \cap N = 2^6.A_8$, $Y \cap N \in \{P_1, P_3, P_4\}$, $G \geq N.2$, and $A \leq X$. But the alternating groups do not have Hall π-subgroups for $|\pi| \geq 2$, so this case cannot occur. □

The Main Theorem is now proved.

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