RACSAM (2021) 115:63
https://doi.org/10.1007/s13398-020-00996-z

ORIGINAL PAPER

Sets of periods for chaotic linear operators

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Received: 12 August 2020 / Accepted: 22 December 2020 / Published online: 9 February 2021

Abstract
We provide a complete characterization of the possible sets of periods for Devaney chaotic linear operators on Hilbert spaces. As a consequence, we also derive this characterization for linearizable maps on Banach spaces.

Keywords Chaotic operators · Hypercyclic operators · Periodic points

Mathematics Subject Classification Primary 47A16; Secondary 37D45 · 37B99 · 37C25

1 Introduction

Let X be a infinite dimensional separable complex Banach space. Let $L(X)$ denote the set of linear and continuous operators from X to X. Given an operator $T \in L(X)$, a vector $x \in X\backslash\{0\}$ is called *n-periodic for T*, for some $n \in \mathbb{N}$, if $T^n x = x$ and $T^m x \neq x$ for each $1 \leq m < n$. An integer $n \in \mathbb{N}$ is a *period for T*, if T admits an n-periodic vector. The set of periods for T is denoted by

$$\mathcal{P}(T) := \{n \in \mathbb{N} : n \text{ is a period for } T\}.$$

The set of periods of a map is strongly related with its chaotic behavior, as it was firstly observed by Li and Yorke in [24]. There is a wide study of the periodic structures for chaotic (and non-chaotic) discrete dynamical systems. The most celebrated result, which was the starting point for a huge line of research, was the theorem of Šarkovskiĭ [27] that established a total order in $\mathbb{N} \cup \{2^\infty\}$ so that, for every continuous map $f : I \to I$ on an interval I, there is $n \in \mathbb{N} \cup \{2^\infty\}$ with $\mathcal{P}(f) = \{m \in \mathbb{N} ; \ n \trianglerighteq m\}$ (where $\trianglerighteq$ is Sharkovsky's order)

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and, for every $n \in \mathbb{N} \cup \{2^\infty\}$, there is a continuous map $f_n : [0,1] \to [0,1]$ such that $\mathcal{P}(f_n) = \{m \in \mathbb{N} \,;\ n \trianglerighteq m\}$. That is, he completely characterized the possible sets of periods for interval maps.

Šarkovskiĭ's theorem motivated researchers to characterize the sets of periods for certain classes of self maps. As a sample of this effort, we mention here the characterizations of the sets of periods for some polynomials and rational functions on $\mathbb{C}$ in the early work [7], for interval and circle maps [4–6], for continuous self maps of the real line $\mathbb{R}$ [16], for triangular maps on a cartesian product of arbitrary spaces [2] or of compact intervals [14], for tree maps [3] and also for different classes of automorphisms [17,18].

Concerning linear operators, the sets of periods for linear operators on $\mathbb{C}^n$, $\mathbb{R}^n$ and ℓ^2 were characterized in [1], see also [23], and on a Hilbert space in [15].

We point out that an earlier version of our results was fundamental for the study of the structure of the set of periods of chaotic strongly continuous semigroups by Muñoz, Seoane and Weber in [25,26].

According to Devaney a dynamical system, consisting of a continuous map on a metric space, is said to be chaotic if it is topologically transitive, its periodic points form a dense set and it has sensitive dependence on the initial conditions. Subsequently, the third condition was showed to be redundant (see [8]). The concept of chaos was associated to the behavior of certain nonlinear dynamical systems. However, chaos may occur also in linear systems, provided they are infinite-dimensional. A theory of the dynamics of linear operators on infinite-dimensional spaces has been extensively studied for over twenty years. An overview of the state of the art in the area of linear chaos can be found in the monographs [10,22].

An operator $T : X \to X$ on a Banach space X is called *topologically transitive* if, for any $U, V \subset X$ non-empty open sets there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Within this context, transitivity is equivalent to *hypercyclicity*, that is, the existence of vectors $x \in X$ whose orbit under T is dense in X. In such a case, x is called a hypercyclic vector for T. The operator T is said to be *Devaney chaotic* if it is hypercyclic and admits a dense set of periodic points.

In the context of operators with a chaotic dynamics, a natural question arises: *Is there a chaotic operator on any separable Hilbert space with a prescribed set of periods?* In this note, we give a positive answer to this question and, as a consequence, we also show how to extend this result to linearizable maps on Banach spaces.

2 Periodic structure of chaotic operators

It is well-known that whenever an operator T on a complex space admits nontrivial n-periodic points, then there are eigenvectors whose eigenvalue is an n-root of the unity (see, for instance, see [13]). In fact, the set of periodic points of T is the vector space $\operatorname{span}\{x \in X \,/\, \exists n \in \mathbb{N},\ \exists \lambda \in \mathbb{C} : \lambda^n = 1\,,\ Tx = \lambda x\}$.

Given $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we say that λ is a *primitive n-th root* of 1 if $\lambda^n = 1$ and $\lambda^m \neq 1$ for $1 \leq m < n$. We denote by

$$\Lambda_n := \{\lambda \in \mathbb{C} \ : \ \lambda \text{ is a primitive} n-\text{th root of } 1\} \subset \Gamma_n := \{\lambda \in \mathbb{C} \ : \ \lambda^n = 1\}.$$

For λ in the unit circle $\mathbb{T}$ and $\varepsilon > 0$, we denote by $I_{\lambda,\varepsilon}$ the open arc of the unit circle of length ε centered at λ.

The following characterizes the set of periods for a chaotic operator on a separable Hilbert space. We were inspired by a result of Bayart and Grivaux [9], where they showed that an operator on a separable complex Banach space with sufficiently many eigenvectors associated to eigenvalues of modulus 1 is hypercyclic. As a consequence, they constructed hypercyclic operators with prescribed unimodular point spectrum, see Theorem 2.11 and Lemma 2.12 in [9]. We notice that (i) below was also observed in [1]. Also notice that in [1] the sets of periods for an operator included always 1 since the vector 0 was also considered to determine the sets of periods. In our case, for convenience, we only consider periodic vectors $x \in X\backslash\{0\}$. This is a slight difference in (i) below.

Theorem 1 (i) *If $A \subset \mathbb{N}$ is a set of periods for an operator on a Banach space X, then A contains the least common multiple (lcm) of each pair of elements in A.*
(ii) *If $A \subset \mathbb{N}$ is infinite and contains the lcm of each pair of elements in A, then there exists a chaotic operator $T \in L(\ell^2)$ such that $\mathcal{P}(T) = A$.*

Proof (i) Suppose that $A = \mathcal{P}(T)$ for some operator $T \in L(X)$. Let $n, m \in A$, $p = \text{lcm}(m, n)$, and $x_1, x_2 \in X$ be n and m periodic vectors, respectively. These vectors can be expressed as a linear combination of eigenvectors corresponding to n-roots and m-roots of unity, see e.g. [[13], Sec.3]:

$$x_1 = \sum_{i=1}^{k_n} \alpha_i y_i, \qquad \alpha_i \neq 0, \quad i = 1, \ldots, k_n,$$

$$x_2 = \sum_{j=1}^{k_m} \beta_j z_j, \qquad \beta_j \neq 0, \quad j = 1, \ldots, k_m.$$

Let $\lambda_i \in \Gamma_n$ be the eigenvalue of y_i, $i = 1, \ldots k_n$, and let $\lambda'_j \in \Gamma_m$ be the eigenvalue of z_j, $j = 1, \ldots, k_m$. We have that there are $n_i, m_j \in \mathbb{N}$, such that $\lambda_i \in \Lambda_{n_i}$ and $\lambda'_j \in \Lambda_{m_j}$, for $i = 1, \ldots, k_n$, $j = 1, \ldots, k_m$. Since x_1 is n-periodic and x_2 is m-periodic, we get $n = \text{lcm}(n_1, \ldots, n_{k_n})$ and $m = \text{lcm}(m_1, \ldots, m_{k_m})$. We define $x := \sum_{i=1}^{k_n} y_i + \sum_{j=1}^{k_m} z_j$, where we identify $y_i = z_j$ if $\lambda_i = \lambda'_j$ for some i and j. Finally, since $p = \text{lcm}(n_1, \ldots, n_{k_n}, m_1, \ldots, m_{k_m})$, we deduce that x is a p-periodic vector for T and $p \in A$.
(ii) Fix $\theta \in]1, 4/3[\backslash\mathbb{Q}$. We select a non decreasing sequence $(n_m)_m$ of positive integers such that $\lim_m \frac{n_m}{m} = \theta/2$ and $m/2 < n_m < 4m/6$ for all $m > 4$. A well-known easy consequence of the Prime Number Theorem and the fact that $(n_m)_m$ is non decreasing and unbounded yields that, for every $\varepsilon > 0$, there is $m_\varepsilon \in \mathbb{N}$ such that, if $m \geq m_\varepsilon$, then there is a prime number p satisfying $n_m < p < (1 + \varepsilon)n_m$. Applying this result to $(1/k)_k$, $k \in \mathbb{N}$, we get an increasing sequence of positive integers $(m_k)_k$, with $m_1 > 4$, such that for each $k \in \mathbb{N}$ and for $m \in \mathbb{N}$ with $m_k \leq m < m_{k+1}$, there exists a prime number $p_{k,m}$ satisfying

$$n_m < p_{k,m} < (1 + \frac{1}{k})n_m.$$

Now define

$$p_m := \begin{cases} 1 & \text{if } 1 \leq m < m_2 \\ p_{k,m} & \text{if } m_k \leq m < m_{k+1} \text{ for } k \geq 2 \end{cases}$$

to get a sequence of integers $(p_m)_m$, with p_m prime for $m \geq m_2$, such that $\lim_m \frac{p_m}{m} = \theta/2$. We observe that p_m and m are coprime for $m \geq m_2$. Otherwise either $p_m = m$ or $p_m \leq m/2$, which yields a contradiction with the selection of $(n_m)_m$. This shows that $\eta_m := e^{2\pi i \frac{p_m}{m}} \in \Lambda_m$, $m \in \mathbb{N}$. Moreover, $\lim_m \eta_m = e^{\pi\theta i} \in \mathbb{T}\backslash e^{i\pi\mathbb{Q}}$.

For convenience, sort positive integers not in A in increasing order and denote them as

$$A^{\mathrm{c}} := \mathbb{N}\backslash A = \{j_1 < j_2 < \dots\}.$$

Claim: There exists a sequence $\{U_k\}_k$ with the following properties:

1. $\cup_{k=1}^{\infty} U_k$ consists of a countable union of pairwise disjoint open arcs in $\mathbb{T}$, not sharing endpoints, where all endpoints have the form $e^{i\phi\pi}$ with $\phi \in]0, 2[\backslash\mathbb{Q}$,
2. $\cup_{k=1}^{\infty} \Lambda_{j_k} \subset \cup_{k=1}^{\infty} U_k$,
3. $\eta_m \notin \cup_{k=1}^{\infty} U_k$ for $m \neq j_k$ and $k \in \mathbb{N}$.

We prove the claim by induction. Since $e^{i\theta\pi} \notin \Lambda_{j_1}$ and Λ_{j_1} is finite, set $\delta = d(e^{i\theta\pi}, \Lambda_{j_1})$ and find $M_1 \in \mathbb{N}$ such that

$$\eta_m \in B(e^{i\theta\pi}, \delta/2), \ m > M_1.$$

Compute

$$\begin{aligned} \varepsilon_1 &= \min\{d(\lambda, \mu), \ \lambda, \mu \in \Lambda_{j_1}, \ \lambda \neq \mu\}, \\ \varepsilon_2 &= \min\{d(\lambda, \eta_m), \ \lambda \in \Lambda_{j_1}, \ 1 \le m \le M_1, \ m \neq j_1\}, \end{aligned}$$

and take $0 < \varepsilon < \min\{\delta, \varepsilon_1, \varepsilon_2\}/2$. For each $\lambda \in \Lambda_{j_1}$ we associate an open arc $I_{\lambda,\varepsilon}$ containing λ, with diameter ε, and with the property that both endpoints of the arc have the form $e^{i\phi\pi}$ with $\phi \in]0, 2[\backslash\mathbb{Q}$. By construction, the family of open arcs $\{I_{\lambda,\varepsilon}, \ \lambda \in \Lambda_{j_1}\}$ is pairwise disjoint and no endpoint is shared by two arcs.

Suppose we have a sequence $\{U_i\}_{i=1}^{k-1}$ with the following properties:

1. $\cup_{i=1}^{k-1} U_i$ consists of a finite union of pairwise disjoint open arcs not sharing endpoints where all endpoints have the form $e^{i\phi\pi}$ with $\phi \in]0, 2[\backslash\mathbb{Q}$,
2. $\cup_{i=1}^{k-1} \Lambda_{j_i} \subset \cup_{i=1}^{k-1} U_i$,
3. $\eta_m \notin \cup_{i=1}^{k-1} U_i$ for $m \neq j_i$ and $1 \le i \le k-1$. In order to complete the induction process we have to construct U_k. Since $e^{i\theta\pi} \notin \cup_{i=1}^{k} \Lambda_{j_i}$, set $\delta_k = d(e^{i\theta\pi}, \cup_{i=1}^{k} \Lambda_{j_i})$ and find $M_k \in \mathbb{N}$ such that

$$\eta_m \in B(e^{i\theta\pi}, \delta/2), \ m > M_k.$$

Compute

$$\begin{aligned} \varepsilon_{3k-2} &= \min\{d(\lambda, \mu), \ \lambda, \mu \in \Lambda_{j_k}, \ \lambda \neq \mu\}, \\ \varepsilon_{3k-1} &= \min\{d(\lambda, e^{i\phi\pi}), \ \lambda \in \Lambda_{j_k}, \ e^{i\phi\pi} \text{ is an endpoint of } \cup_{i=1}^{k-1} U_i\}, \\ \varepsilon_{3k} &= \min\{d(\lambda, \eta_m), \ \lambda \in \Lambda_{j_k}, \ 1 \le m \le M_k, \ m \notin \{j_1, j_2, \dots, j_k\}\}, \end{aligned}$$

and take $0 < \varepsilon < \min\{\delta_k, \varepsilon_{3k-2}, \varepsilon_{3k-1}, \varepsilon_{3k}\}/2$. For each $\lambda \in \Lambda_{j_k} \backslash \cup_{i=1}^{k-1} U_i$ we associate an open arc $I_{\lambda,\varepsilon}$ containing λ, with diameter ε, and with the property that both endpoints of the arc have the form $e^{i\phi\pi}$ with $\phi \in]0, 2[\backslash\mathbb{Q}$. Define

$$U_k := \bigcup_{\lambda \in \Lambda_{j_k} \backslash \cup_{i=1}^{k-1} U_i} I_{\lambda,\varepsilon},$$

and the claim is proved. Observe that in the case where $\mathbb{N}\backslash A$ is finite, we have that $U_k = \emptyset$ after a finite number of induction steps. Define now $U := \cup_{k=1}^{\infty} U_k$, which is an open subset of $\mathbb{T}$ and, by construction, $\mathbb{T}\backslash U$ does not have isolated points (i.e., it is a perfect set) and the primitive roots of the unity $\mu_m \in \mathbb{T}\backslash U$ if and only if $m \in A$. To finish our proof we will need the Kalisch operator, that was used for the first time in [9] in the context of chaotic properties in linear dynamics. More precisely, let $K : L^2[0, 2\pi] \to L^2[0, 2\pi]$ be defined as

$$Kf(\theta) = e^{i\theta} f(\theta) - \int_0^\theta ie^{it} f(t)dt, \quad \theta \in [0, 2\pi].$$

It is well known (and easy to see) that, for any $\lambda \in \mathbb{T}\backslash\{1\}$, $\lambda = e^{i\beta}$ with $\beta \in]0, 2\pi[$, we have

$$\ker(K - \lambda I) = \text{span}(f_\lambda), \quad \text{where} \quad f_\lambda := \mathbf{1}_{[\beta,2\pi]}.$$

W.l.o.g. we suppose that $1 \in U$. Otherwise a suitable rotation of K does the job. We consider the Hilbert space $H := \overline{\text{span}}\{f_\lambda \,;\, \lambda \in \mathbb{T}\backslash U\}$, which is K-invariant. Since $\mathbb{T}\backslash U$ is a compact perfect set and, by continuity of the map $\lambda \mapsto f_\lambda$, the set of eigenvectors of $T := K|_H$ associated to roots of unity is dense in H, T is chaotic (see, e.g., [10, Section 5.5.3] or [20, Fact 3.5]). Moreover, $\sigma_p(T) = \sigma(T) = \mathbb{T}\backslash U$, where $\sigma(T)$ and $\sigma_p(T)$ denote the spectrum and the point spectrum of the operator T respectively. Given $m \in A$, we have that $\eta_m \in \Lambda_m \cap \sigma_p(T)$. Since all separable infinite dimensional Hilbert spaces are equivalent, we replace H by ℓ^2 from now on. Let $x \in \ell^2$ be an eigenvector of T associated to η_m, then x is m-periodic, and $A \subseteq \mathcal{P}(T)$.

On the other hand, if $m \in \mathcal{P}(T)$, proceeding as in (i), we find a finite family $x_1, \ldots, x_k \in \ell^2$ of eigenvectors of T such that if $\lambda_1, \ldots, \lambda_k$ are the respective associated eigenvalues and $m_1, \ldots, m_k \in \mathbb{N}$ are so that $x_i \in \Lambda_{m_i}$, $i = 1, \ldots, k$, then $m = \text{lcm}(m_1, \ldots, m_k)$. If $m_i \in A, i = 1, \ldots, k$, then the hypothesis on A imply that $m \in A$. If there was some $m_j \notin A$, then $\lambda_j \in U \cap \sigma_p(T)$ which is a contradiction.

Remark 2 The Kalisch type operator K of the proof was used several times in connection with ergodic theory in linear dynamics (see, e.g., [9,10,20]). In particular, when the spectrum of $T := K|_H$ is perfect, T admits an invariant ergodic measure with full support (see, e.g., [10, Section 5.5.3]). Also, Grivaux and Matheron observed in [19, Theorem 1.10] that, in this case, T is densely distributionally chaotic. As a consequence, we have that, if $A \subset \mathbb{N}$ is infinite and contains the lcm of each pair of elements in A, then there exists an operator $T \in L(\ell^2)$ such that $\mathcal{P}(T) = A$, and T is ergodic and (densely) distributionally chaotic (therefore, Li-Yorke chaotic). We refer the reader to [11] and [12] for the basic notions and results on distributional and Li-Yorke chaos, respectively, in linear dynamics.

The most important properties of dynamical systems are preserved under conjugacy, and so does the set of periods. If we take this into account, and call a continuous map $f : X \to X$ to be *linearizable* if it is topologically conjugate to an operator $T \in L(Y)$ on a, possibly different, Banach space Y, one can consider the problem of studying the set of periods for linearizable maps. A very useful result to face this problem is stated in [21, page 357]. As a consequence of the Anderson-Kadec Theorem, any separable infinite-dimensional Banach space is homeomorphic to the Hilbert space ℓ^2. Together with our Theorem 1, this shows a very powerful result for linearizable maps.

Theorem 3 *Given any separable infinite-dimensional Banach space X and an infinite set $A \subset \mathbb{N}$, there exists a chaotic linearizable map $f : X \to X$ such that $\mathcal{P}(f) = A$ if and only if A contains the least common multiples of all pairs of elements in A.*

Acknowledgements This work was supported by MICINN and FEDER, Projects MTM2016-75963-P and PID2019-105011GB-I00. The second and third authors were also supported by Generalitat Valenciana, Project PROMETEO/2017/102. We would to that the referees whose careful reading and observations produced an improvement in the presentation of the article.

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