Two conformal projections for constant-height surface to plane mapping

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Abstract. Regions at high elevations may require specific mapping solutions other than the conventional ellipsoid-to-grid projections which produce high discrepancies between ground and projected distances. These particular solutions are known as low-distortion projections (LDPs). They can be realized by making use of an Elevated Ellipsoid (EE) or a Constant-height Surface (ChS) above the ellipsoid as the reference surface, or by means of a scaled projection. No conformal projections have been derived so far for the ChS-to-plane transformation. This article aims to solve this situation by deriving the formulation of Direct and Transverse Mercator-type projections for ChS-to-plane conformal mapping.

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Introduction

The design of map projections and the choice of corresponding parameters (e.g. true scale parallels) are usually made to produce minimum distortions in the reference ellipsoid to plane transformation for an area of interest. A preliminary computation has to be done before projection in order to *reduce* or project the measurements, which are taken close to the earth surface, onto the reference ellipsoid. This reduction process can take the form of a simple and handy ground-to-ellipsoid scale factor. It is normally assumed that the earth surface and the reference ellipsoid are relatively close to each other so that the requirement of low distortions in the ellipsoid-to-plane transformation entails low distortions for the complete ground-to-plane transformation. These distortions can be known and taken into account by the user in each particular case, but the question is whether a particular projection which is optimal in terms of ellipsoid-to-plane transformation can be considered optimal for the complete ground-to-plane transformation or, on the contrary, be significantly improved for this latter purpose. This seems to be the case of highly elevated regions, for which specific solutions, known as low-distortion projections (LDPs) have been derived in the past (Billings 2013a, 2013b, Armstrong *et al*. 2017, Dennis 2018, 2019).

The distance between two points reduced to the ellipsoid, which will be subsequently projected onto the plane, is a clearly defined magnitude that is measured along the geodesic line passing through both endpoints. Conversely, at a certain height, above or below the ellipsoid, the notion of horizontal distance becomes ambiguous: *horizontal distance* may mean the distance projected onto the local geodetic horizon of the first point, or the horizon of the second point, or be defined for a sort of average height, or with an alternative definition. This type of ambiguity can cause some confusion when using LDPs. To overcome this, an *elevated ellipsoid* (EE) of semi-axes

$$a' = a + h_0 \quad (1)$$

$$b' = b + h_0 \quad (2)$$

can be used, where *a* and *b* are the semi-axes of the original reference ellipsoid and h_0 is the desired elevation (Rollins and Meyer 2019). Alternatively, a scale factor can be applied to semiaxis *a* while retaining the same eccentricity for the ellipsoid. This approach was already used in the three Michigan zones of the State Plane Coordinate System of 1927 (Coast & Geodetic Survey 1979; Burkholder 1980; Lusch 2005, Dennis 2018). These ideas have been presented in other many different occasions (e.g. Burkholder 1993, Armstrong *et al*. 2017, Rollins and Meyer 2019). However, some may find it difficult to use them: as an example, the original implementation of the Wisconsin Coordinate Reference System (Wisconsin State Cartographer's Office 2015) used an enlarged and elevated ellipsoid but was later replaced by the approach of changing the projection scale only. This example may indicate that the strategy of scaling the projection with no change in the reference ellipsoid is preferred at present since it does not entail any increase of complexity, as with the EE and the Constant-height Surface (ChS), while having a similar performance.

By contrast, Rollins and Meyer (2019) sustain that a ChS is a suitable elevated reference surface for constructing LDPs in places at high elevations, while taking into account that the LDP design should also consider the total linear distortion and not only height. Field distances can be reduced to a ChS by using an adaptation of the widely used scale factor formula $(1+h/R)^{-1}$ (Stem 1990) as $(1+(h–h_0)/R)^{-1}$ with h the mean ellipsoidal height of the endpoints, h_0 the ellipsoidal height of the ChS and R the mean of the radii of curvature in both points for the particular direction. Being more specific, we will use R as the average of the Euler's radius of curvature in both endpoints for the corresponding geodetic azimuths, although the result does not depend strongly on the definition chosen for R.

A ChS is not an ellipsoid. No mapping from the reference ellipsoid to the grid, whether that mapping is conformal or not, can be conformal if used to map from a ChS to the grid but only *nearly conformal*, as shown in Rollins and Meyer (2019), since the curvatures on the ChS cannot equal those on the ellipsoid in general. Indeed, this is true not only for ChSs but for all models other than the reference ellipsoid, including EEs, gravitational equipotential surfaces, planes, etc., since the curvatures of these surfaces differ from those of the reference ellipsoid. As Rollins and Meyer (2019) say, there still could be a conformal direct mapping (ChS $\rightarrow$ plane), which has not been presented so far.

The present paper overcomes this situation by presenting:

- A Direct Mercator-type conformal projection (for use in very low-latitude areas)
- A Transverse Mercator-type conformal projection (for general use except near the poles)

Constant-height Surface to grid conformal projections

A ChS is not an ellipsoid (neither the original ellipsoid to a scale different from 1 nor the elevated ellipsoid referred to before) but a different closed surface that is constructed by prolonging the normals at every point of the ellipsoid a distance h_0. The radii of curvature of the normal sections in the nouth-south direction and east-west directions are, respectively (Rollins and Meyer, 2019)

$$\rho' = \rho + h_0 \qquad (3)$$

$$\nu' = \nu + h_0 \qquad (4)$$

where ρ and ν are the principal radii of curvature of the original reference ellipsoid, which are given, respectively, by

$$\rho = \frac{a(1-e^2)}{(1-e^2 sin^2\varphi)^{3/2}} \qquad (5)$$

$$\nu = \frac{a}{\sqrt{1-e^2 sin^2\varphi}} \qquad (6)$$

with major semi-axis *a*, eccentricity *e* and geodetic latitude of the point φ.

We derive in the following subsections two conformal Mercator-type projections for ChS-to-plane mapping by elaborating on the original ideas that led to the formulae of Direct and Transverse Mercator-type projections of the reference ellipsoid.

Direct Mercator-type conformal projection for ChS to plane

The Direct Mercator projection for the transformation (ellipsoid → plane) is obtained by

$$x = a\Delta\lambda \qquad (7)$$

$$y = a\psi \qquad (8)$$

where *a* is the major semi-axis of the ellipsoid, $\Delta\lambda$ is the increment of geodetic longitude with respect to the origin of longitudes (i.e. $\Delta\lambda = \lambda - \lambda_0$ where λ is the point longitude and λ_0 is the central meridian longitude) and ψ is the isometric latitude, also called Mercator parameter (Osborne, 2013, p.111 Eq. 6.1), which can be defined as

$$\psi = \int_0^{\varphi} \frac{\rho}{\nu cos\varphi} d\varphi \qquad (9)$$

For the case of the ellipsoid the integral results in

$$\psi = ln \left| tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\left(\frac{1 - esin\varphi}{1 + esin\varphi}\right)^{e/2} \right| \qquad (10)$$

with *e* the eccentricity of the ellipsoid and φ the geodetic latitude.

Now we apply this idea to construct a conformal projection for the transformation (ChS → plane). Using primed symbols for the ChS and unprimed for the reference ellipsoid, we obtain, analogously to Eq. (9)

$$\psi' = \int_0^{\varphi} \frac{\rho'}{\nu' cos\varphi} d\varphi \qquad (11)$$

or considering Eqs. (3) and (4)

$$\psi' = \int_0^{\varphi} \frac{\rho + h_0}{(\nu + h_0) cos\varphi} d\varphi \qquad (12)$$

with the expressions for ρ and ν given in Eqs. (5) and (6).

It is convenient to introduce the following change of variable

$$e_2 = \frac{e^2}{1+\frac{h_0}{a}} \tag{13}$$

which will simplify to e^2 for the case of zero elevation.

Although a solution in closed form can be found for the integral in Eq.(12), it is handier to use the following expression obtained after expanding ψ' in powers of e_2

$$\psi' = ln\left|\frac{cos\frac{\varphi}{2}+sin\frac{\varphi}{2}}{cos\frac{\varphi}{2}-sin\frac{\varphi}{2}}\right| - e_2 sin\varphi - \left(\frac{1}{3}+\frac{1}{2}\frac{h_0}{a}\right)e_2{}^2 sin^3\varphi - \left(\frac{1}{5}+\frac{21}{40}\frac{h_0}{a}\right)e_2{}^3 sin^5\varphi - \frac{1}{7}e_2{}^4 sin^7\varphi \tag{14}$$

where the terms amounting to less than 0.00001" for latitudes up to 80º and heights up to 4000m have been disregarded.

The Direct Mercator (conformal) projection for the transformation (ChS $\rightarrow$ plane) is then given by the formulation

$$x = a\Delta\lambda \tag{15}$$

$$y = a\psi' \tag{16}$$

with ψ' given in Eq. (14). If desired, a scale coefficient k_0 could be applied at the equator. To do so Eqs. (15) and Eq. (16) need be multiplied by k_0. Not including k_0 in the formulas (i.e. k_0 = 1) means the projection has unit linear scale factor at the Equator.

An algorithm for the inverse computation of geodetic coordinates from grid coordinates can be found in the Appendix.

This projection is conformal, therefore its linear distortion is the same in every direction from a point. We can do an easy check of conformality by computing the linear distortions in the meridian and the parallel for different values of the latitude φ and height h_0. The results depicted in Table 1 have been obtained by the corresponding expressions Eqs. (45) and (46) given in the Appendix (though the latter is the recommended expression due to its simplicity). We only find negligible discrepancies, i.e. close to the machine working precision for low latitudes and still negligible for mid or high latitudes. It must be noted, however, that this projection is only of practical use within a very narrow bound near the equator (advisably below 2º of latitude at the most) due to the very large distortions introduced. For general use, except very near the poles, a suitable projection is introduced in the following section.

Table 1 here

Transverse Mercator-type conformal projection for ChS to plane

For constructing the Transverse Mercator projection we initially resort to the idea that is generally used for the derivation of the formulae for the ellipsoid-to-plane transformation (see e.g. Osborne, 2013) now applied to the ChS-to-plane transformation. Let f' be a complex function mapping the complex plane (ψ', $\Delta\lambda$) onto the complex plane (y,x)

$$y + ix = f'(\psi' + i\Delta\lambda) =$$

$$= f'(\psi') + i\Delta\lambda f'^{(1)}(\psi') - \frac{\Delta\lambda^2}{2} f'^{(2)}(\psi') - i\frac{\Delta\lambda^3}{6} f'^{(3)}(\psi') + \frac{\Delta\lambda^4}{24} f'^{(4)}(\psi') + i\frac{\Delta\lambda^5}{120} f'^{(5)}(\psi') - \frac{\Delta\lambda^6}{720} f'^{(6)}(\psi') + \cdots \quad (17)$$

where we have used the expansion around the central meridian, i.e. $(\psi' + i0)$ or simply written ψ', so that $f'^{(n)}(\psi')$ denotes the nth derivative of f' evaluated at ψ'. If function f' and its derivatives $f'^{(n)}$ exist and are nonzero then the resulting transformation will be conformal.

Equating real and imaginary parts on both sides of Eq. (17) we obtain

$$x = \Delta\lambda f'^{(1)}(\psi') - \frac{\Delta\lambda^3}{6} f'^{(3)}(\psi') + \frac{\Delta\lambda^5}{120} f'^{(5)}(\psi') + \cdots \quad (18)$$

$$y = f'(\psi') - \frac{\Delta\lambda^2}{2} f'^{(2)}(\psi') + \frac{\Delta\lambda^4}{24} f'^{(4)}(\psi') - \frac{\Delta\lambda^6}{720} f'^{(6)}(\psi') + \cdots \quad (19)$$

Now we apply the condition that the central meridian ($\Delta\lambda$ = 0, therefore $x = 0$ $y = f'(\psi')$) be a line of scale coefficient k_0

$$y = f'(\psi') = k_0 \int_0^{\varphi} \rho' d\varphi \quad (20)$$

This has the effect of univocally defining function *f′*. Substituting Eq. (3) into Eq. (20) yields

$$f'(\psi') = k_0 \int_0^{\varphi} (\rho + h_0) d\varphi = k_0 \int_0^{\varphi} \rho d\varphi + k_0 h_0 \varphi \quad (21)$$

The first term in the right-hand side is k_0 times the length of meridian arc from the equator to latitude φ on the reference ellipsoid, which is called $m(\varphi)$ in Osborne (2013), so that

$$f'(\psi') = k_0 m(\varphi) + k_0 h_0 \varphi \quad (22)$$

in contrast with

$$f(\psi) = k_0 m(\varphi) \quad (23)$$

which is the function *f* used in the Transverse Mercator for ellipsoid-to-plane transformation. Hence

$$f'(\psi') = f(\psi) + k_0 h_0 \varphi \quad (24)$$

Now we compute the corresponding derivatives of function *f′* given by Eq. (24).

$$f'^{(1)}(\psi') = \frac{df'(\psi')}{d\psi'} = \frac{df'(\psi')}{d\psi} \frac{d\psi}{d\psi'} = \left(\frac{df(\psi)}{d\psi} + k_0 h_0 \frac{d\varphi}{d\psi} \right) \frac{d\psi}{d\psi'} \quad (25)$$

The first term inside the right-hand side parenthesis is the first derivative of the function *f* with respect to variable ψ which is used in the standard Transverse Mercator for ellipsoid-to-plane transformation, i.e. $f^{(1)}(\psi)$, and

$$\frac{d\varphi}{d\psi} = \frac{\nu cos\varphi}{\rho} \quad (26)$$

as it can be derived from Eq. (9). It can be demonstrated that no significant error is committed if we take $\frac{d\psi}{d\psi'}$ as unity given that we will multiply these derivatives by increments of longitude much smaller than one ($\Delta\lambda \ll 1 \approx 57.2957795°$). Therefore

$$f'^{(1)}(\psi') = f^{(1)}(\psi) + k_0 h_0 \frac{\nu cos\varphi}{\rho} \quad (27)$$

For computing subsequent derivatives we can take ν equal to ρ in the last term. Then

$$f'^{(2)}(\psi') = \frac{df'^{(1)}(\psi')}{d\psi'} = \frac{df'^{(1)}(\psi')}{d\psi} \frac{d\psi}{d\psi'} = \left(\frac{df^{(1)}(\psi)}{d\psi} + k_0 h_0 \frac{d(cos\varphi)}{d\psi} \right) = f^{(2)}(\psi) + k_0 h_0 \frac{d(cos\varphi)}{d\varphi} \frac{d\varphi}{d\psi} =$$

$$= f^{(2)}(\psi) - k_0 h_0 sin\varphi \frac{\nu cos\varphi}{\rho} \quad (28)$$

where again we have taken $\frac{d\psi}{d\psi'}$ as unity, and we can take ν equal to ρ in the last term, so that

$$f'^{(2)}(\psi') = f^{(2)}(\psi) - k_0 h_0 sin\varphi cos\varphi \qquad (29)$$

Similarly

$$f'^{(3)}(\psi') = \frac{df'^{(2)}(\psi')}{d\psi'} = \frac{df'^{(2)}(\psi')}{d\psi}\frac{d\psi}{d\psi'} = \left(\frac{df^{(2)}(\psi)}{d\psi} - k_0 h_0 \frac{d(sin\varphi cos\varphi)}{d\psi}\right) = f^{(3)}(\psi) - k_0 h_0 \frac{d(sin\varphi cos\varphi)}{d\varphi}\frac{d\varphi}{d\psi} =$$

$$= f^{(3)}(\psi) - k_0 h_0 (cos^2\varphi - sin^2\varphi)\frac{\nu cos\varphi}{\rho} \qquad (30)$$

where again $\frac{d\psi}{d\psi'}$ has been taken as unity and we can take ν equal to ρ in the last term. Therefore

$$f'^{(3)}(\psi') = f^{(3)}(\psi) - k_0 h_0 (cos^2\varphi - sin^2\varphi) cos\varphi \qquad (31)$$

Now, for computation of subsequent derivatives the influence of the last term can be neglected as well as $\frac{d\psi}{d\psi'}$ be taken as unity, so that

$$f'^{(4)}(\psi') = f^{(4)}(\psi) \qquad (32)$$

$$f'^{(5)}(\psi') = f^{(5)}(\psi) \qquad (33)$$

$$f'^{(6)}(\psi') = f^{(6)}(\psi) \qquad (34)$$

Inserting the function Eq. (24) and its derivatives – Eqs. (27), (29), and (30) to (34) – into Eqs. (18) and (19) and recalling that

$$x_e = \Delta\lambda f^{(1)}(\psi) - \frac{\Delta\lambda^3}{6} f^{(3)}(\psi) + \frac{\Delta\lambda^5}{120} f^{(5)}(\psi) + \cdots \qquad (35)$$

$$y_e = f(\psi) - \frac{\Delta\lambda^2}{2} f^{(2)}(\psi) + \frac{\Delta\lambda^4}{24} f^{(4)}(\psi) - \frac{\Delta\lambda^6}{720} f^{(6)}(\psi) + \cdots \qquad (36)$$

are the equations for the ellipsoid-to-plane Transverse Mercator projection, we can obtain the corresponding equations for the ChS-to-plane Transverse Mercator projection as

$$x = x_e + \Delta\lambda k_0 h_0 \frac{\nu cos\varphi}{\rho} + \frac{\Delta\lambda^3}{6} k_0 h_0 (cos^2\varphi - sin^2\varphi) cos\varphi \qquad (37)$$

$$y = y_e + k_0 h_0 \varphi + \frac{\Delta\lambda^2}{2} k_0 h_0 sin\varphi cos\varphi \qquad (38)$$

Needless to say, the coordinates of the standard ellipsoid-to-plane Transverse Mercator projection (x_e, y_e) have to had been computed with the same design parameters (k_0, false easting, false northing...) than the corresponding coordinates of the ChS-to-plane Transverse Mercator projection (x, y). Obviously, (x, y) are simplified to (x_e, y_e) for zero height h_0.

As it is demonstrated in the Appendix the scale factor of the ChS-to-plane Transverse Mercator projection is independent of height h_0 and therefore equal to the scale factor of the standard ellipsoid-to-plane Transverse Mercator projection.

$$k = k_e \quad (39)$$

An algorithm for the inverse computation of geodetic coordinates from grid coordinates can also be found in the Appendix.

Conclusions

Direct and Transverse Mercator-type projections for ChS-to-plane conformal mapping have been derived. The formulation has been kept relatively simple and approximate enough to guarantee a degree of conformality of the order of 10^{-9} or better for the sensible range of application (latitudes from 0 to 80º, heights from 0 to 3000m, increments of longitudes up to 3º). While the Direct Mercator projection is of very limited use since its distortions become too large except for a very thin band (advisably below 2º) centered in the equator, the Transverse Mercator projection presented is of general use except very near the poles (less than 5º or 10º) and can be applied to highly elevated areas, such as the mountainous western regions of the United States, where conventional ellipsoid-to-plane projections may produce intolerable discrepancies between ground and grid distances.

Data Availability Statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix. Additional derivations

This section contains additional derivations for the two ChS-to-grid projections presented above. In particular, the derivations provide the corresponding scale factors, demonstrate the fulfillment of Cauchy-Riemann equations for conformality and present algorithms for the inverse computation, namely the determination of latitude and longitude in the ChS of ellipsoid height h_0 from grid coordinates *x, y*.

Direct Mercator for ChS to plane

Cauchy-Riemann equations are necessary and sufficient conditions for any projection to be conformal (Snyder 1987, p.27). They can be written as

$$\begin{aligned} x_\lambda &= y_\psi \\ x_\psi &= -y_\lambda \end{aligned} \tag{40}$$

where x_λ, y_λ, x_ψ, y_ψ are the partial derivatives of the functions defining the map projection with respect to λ and ψ, being (x,y) and (λ,ψ) two isometric coordinate systems.

It is easy to demonstrate that the ChS-to-grid Direct Mercator projection, defined by Eqs. (15) and (16), where the isometric latitude is denoted by ψ', fulfills these conditions, since they reduce to

$$\begin{aligned} a &= a \\ 0 &= 0 \end{aligned} \tag{41}$$

Similarly to Baselga (2018, 2019) we can define a scale factor k in the infinitesimal neighborhood of a point as the ratio of the projected distance on the grid defined by Eqs. (15) and (16), ds_g, to the original distance on the constant-height surface ds_{ChS}, which can be computed with the general expression

$$k = \frac{ds_g}{ds_{ChS}} = \frac{\sqrt{(x_\varphi{}^2 + y_\varphi{}^2)d\varphi^2 + (x_\lambda{}^2 + y_\lambda{}^2)d\lambda^2 + 2(x_\varphi x_\lambda + y_\varphi y_\lambda)d\varphi d\lambda}}{\sqrt{(\rho + h_0)^2 d\varphi^2 + (\nu + h_0)^2 cos^2\varphi d\lambda^2}} \tag{42}$$

where x_φ, y_φ, x_λ, y_λ are the partial derivatives of the functions defining the map projection with respect to φ and λ, and $d\varphi$ and $d\lambda$ are the geographic coordinate differences between two infinitesimally close points, φ_i, λ_i and $\varphi_j = \varphi_i + d\varphi$, $\lambda_j = \lambda_i + d\lambda$.

Scale factors in the meridian and the parallel, k_m and k_p, can be respectively obtained with $d\lambda$ = 0 and $d\varphi$ = 0 as

$$k_m = \frac{\sqrt{x_\varphi{}^2 + y_\varphi{}^2}}{\rho + h_0} \tag{43}$$

$$k_p = \frac{\sqrt{x_\lambda{}^2 + y_\lambda{}^2}}{(\nu + h_0)cos\varphi} \tag{44}$$

For the case of the ChS-to-grid Direct Mercator projection, defined by Eqs. (15) and (16), these scale factors result in

$$k_m = \frac{a\psi'_\varphi}{\rho + h_0} \tag{45}$$

$$k_p = \frac{a}{(\nu + h_0)cos\varphi} \tag{46}$$

where ψ'_φ denotes the partial derivative of the isometric latitude ψ', given by Eq. (14), with respect to the geodetic latitude φ. The second expression is much simpler to compute than the first one and we know they must provide the same result since in any conformal projection the scale factor is direction independent, $k = k_m = k_p$ (Snyder 1987, p.24); therefore, we will prefer the latter expression for computing the scale factor due to its simplicity.

$$k = \frac{a}{(\nu + h_0)cos\varphi} \tag{47}$$

Just in case one might want to compute the first expression, for example for the purpose of verification, the partial derivative ψ'_{φ} is equal to A in the Eq. (54) below.

Finally, while Eqs. (15) and (16) provide the direct formulation for the ChS-to-grid projection, the inverse computation, that is, the determination of geodetic coordinates (φ, λ) from grid coordinates (*x, y*) can be obtained as follows.

First, it is immediate to determine the increment of longitude with respect to the origin meridian (of longitude λ_0) from Eq. (15) as

$$\Delta\lambda = \frac{x}{a} \tag{48}$$

and then the geodetic longitude as

$$\lambda = \lambda_0 + \Delta\lambda \tag{49}$$

It is also immediate to obtain the isometric latitude ψ' from Eq. (16) as

$$\psi' = \frac{y}{a} \tag{50}$$

Now, to obtain the geodetic latitude φ from the isometric latitude ψ' we will use the following fastly-convergent iterative procedure. First, the crude approximation

$$\psi' \approx \varphi(1 - e_2) \tag{51}$$

obtained by first-order series expansion of Eq. (14) around $\varphi = 0$ permits to obtain a first approximate value for the latitude, denoted here by φ_0

$$\varphi_0 = \frac{\psi'}{1 - e_2} \tag{52}$$

where one should recall the definition of e_2 in Eq. (13).

A first-order series expansion of Eq. (14) this time around $\varphi = \varphi_0$ provides

$$\psi' \approx \psi'_0 + A(\varphi - \varphi_0) \tag{53}$$

where ψ'_0 is obtained by substituting $\varphi = \varphi_0$ into Eq. (14) and

$$A = \frac{cos\left(\frac{\varphi_0}{2}\right)^2 + sin\left(\frac{\varphi_0}{2}\right)^2}{\left(cos\frac{\varphi_0}{2} - sin\frac{\varphi_0}{2}\right)\left(cos\frac{\varphi_0}{2} + sin\frac{\varphi_0}{2}\right)} - e_2 cos\varphi_0 - {e_2}^2\left(1 + \frac{3h_0}{2a}\right)cos\varphi_0 sin^2\varphi_0 - {e_2}^3\left(1 + \frac{21h_0}{8a}\right)cos\varphi_0 sin^4\varphi_0 - {e_2}^4 cos\varphi_0 sin^6\varphi_0 \qquad (54)$$

A new latitude value can therefore be obtained by

$$\varphi = \varphi_0 + \frac{\psi' - \psi'_0}{A} \qquad (55)$$

This new value is now taken as initial latitude φ_0 for the subsequent iteration so that it is introduced into Eq. (14) to obtain ψ'_0 and used in Eq. (54) to obtain a new value for *A*. Both of them are introduced into Eq. (55) to obtain a refined latitude value. As said, this procedure converges fast, usually in just two iterations for low latitudes, as with the numerical example below, or a few more iterations for medium or high latitudes (where this projection is not recommended, recall the corresponding high distortions shown for these latitudes in Table 1).

Some numerical values for an example of use of the ChS-to-grid Direct Mercator projection follow: given the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101, $e = \sqrt{2f - f^2}$), geodetic coordinates φ = 20º and λ = 6º, origin of longitudes λ_0 = 3º, and ellipsoid height h_0 = 2000m of the ChS, the direct computation is solved by Eq. (15) and Eq. (14) into Eq. (16) resulting in *x* = 333958.472m and *y* = 2258428.227m. The scale factor, computed by means of Eq. (47), is 1.06342769. The inverse computation, starting with these values for *x* and *y*, produce the initial geodetic coordinates φ = 20º and λ = 6º.

Transverse Mercator for ChS to plane

It is easy to demonstrate that the ChS-to-grid Transverse Mercator projection, defined by Eqs. (18) and (19), where the isometric latitude is denoted by ψ', fulfills Cauchy-Riemann conditions, Eq. (40), since the partial derivative of *x*, given in Eq. (18), with respect to λ

$$x_\lambda = f'^{(1)}(\psi') - \frac{3}{6}\Delta\lambda^2 f'^{(3)}(\psi') + \frac{5}{120}\Delta\lambda^4 f'^{(5)}(\psi') - \cdots \qquad (56)$$

and the partial derivative of *y*, given in Eq. (19), with respect to ψ'

$$y_{\psi'} = f'^{(1)}(\psi') - \frac{\Delta\lambda^2}{2} f'^{(3)}(\psi') + \frac{\Delta\lambda^4}{24} f'^{(5)}(\psi') - \cdots \qquad (57)$$

produce the same series. Analogously, the partial derivative of *x* with respect to ψ'

$$x_{\psi'} = \Delta\lambda f'^{(2)}(\psi') - \frac{\Delta\lambda^3}{6} f'^{(4)}(\psi') + \frac{\Delta\lambda^5}{120} f'^{(6)}(\psi') - \cdots \qquad (58)$$

and the partial derivative of *y* with respect to λ

$$y_{\lambda} = -\frac{2}{2}\Delta\lambda f'^{(2)}(\psi') + \frac{4}{24}\Delta\lambda^3 f'^{(4)}(\psi') - \frac{6}{720}\Delta\lambda^5 f'^{(6)}(\psi') + \cdots \qquad (59)$$

are series equal except for a sign change, as required by the second condition of Cauchy-Riemann, Eq. (40).

Now, we obtain the scale factor of the ChS-to-grid Transverse Mercator projection. As in Eq. (42) the desired scale factor is defined as the ratio of the grid distance ds_g to the distance on the constant-height surface ds_{ChS}. We can also compute other two differential distances: the distance on the surface of the ellipsoid ds_e and the grid distance obtained using the formulation of the standard Transverse Mercator projection ds_{g0}. Dividing and multiplying by these two distances we can transform a little bit the equation for the scale factor

$$k = \frac{ds_g}{ds_{ChS}} = \frac{ds_g}{ds_{g0}} \frac{ds_{g0}}{ds_e} \frac{ds_e}{ds_{ChS}} \qquad (60)$$

The transformation from the ellipsoid to the grid using the formulation of the standard Transverse Mercator projection has a scale factor which we have previously denoted as k_e, this is precisely the second factor required in the equation above

$$\frac{ds_{g0}}{ds_e} = k_e \qquad (61)$$

The last factor in Eq. (60) is the ratio of the distance on the ellipsoid to the distance on the constant-height surface. Using the first fundamental forms of the ellipsoid and the constant-height surface we can write

$$\frac{ds_e}{ds_{ChS}} = \frac{\sqrt{\rho^2 d\varphi^2 + \nu^2 cos^2\varphi d\lambda^2}}{\sqrt{(\rho + h_0)^2 d\varphi^2 + (\nu + h_0)^2 cos^2\varphi d\lambda^2}} \qquad (62)$$

Since the projection is conformal the distortion is independent of the direction (Snyder 1987, p.24). We can consider $d\lambda$ = 0, for simplicity, to analyze the distortion along the meridian, therefore

$$\frac{ds_e}{ds_{ChS}} = \frac{\rho}{\rho + h_0} \tag{63}$$

Finally, the first factor required in the right-hand side of Eq. (60) is the ratio of the distance using coordinates in the ChS Transverse Mercator grid, (*x*, *y*) in Eqs. (37) and (38), to the distance using coordinates in the standard Transverse Mercator grid, (x_e, y_e). Using the corresponding first fundamental forms we can write

$$\frac{ds_g}{ds_{go}} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{{dx_e}^2 + {dy_e}^2}} \tag{64}$$

where *dx*, *dy* denote differences in the ChS Transverse Mercator grid coordinates of two neighboring points in the same meridian (recall that we are considering $d\lambda$ = 0) and analogously for dx_e, dy_e for their corresponding coordinates in the standard Transverse Mercator grid. These coordinate differences are due to a difference in latitude $d\varphi$ only, therefore $dy^2 \gg dx^2$ and ${dy_e}^2 \gg {dx_e}^2$, and no significant error is committed if for this ratio we make the consideration that $dx^2 + dy^2 \approx dy^2$ and ${dx_e}^2 + {dy_e}^2 \approx {dy_e}^2$.

Therefore, we have

$$\frac{ds_g}{ds_{go}} = \frac{dy}{dy_e} \tag{65}$$

Considering Eq. (38) we can write

$$dy = \frac{\partial y_e}{\partial \varphi} d\varphi + \frac{\partial}{\partial \varphi}(k_0 h_0 \varphi) d\varphi + \frac{\partial}{\partial \varphi}\left(\frac{\Delta\lambda^2}{2} k_0 h_0 sin\varphi cos\varphi\right) d\varphi \tag{66}$$

The last term can be safely neglected, being the increment of longitude with respect to the central meridian a value well below one ($\Delta\lambda \ll 1 \approx 57.2957795°$). Therefore

$$dy = \frac{\partial y_e}{\partial \varphi} d\varphi + k_0 h_0 d\varphi \tag{67}$$

Considering that

$$dy_e = \frac{\partial y_e}{\partial \varphi} d\varphi \quad (68)$$

Eq. (65) reads

$$\frac{ds_g}{ds_{go}} = \frac{\frac{\partial y_e}{\partial \varphi} + k_0 h_0}{\frac{\partial y_e}{\partial \varphi}} \quad (69)$$

To evaluate the derivative of coordinate *y* in the standard Transverse Mercator projection we recall that its leading term in *y* is the length along the meridian m(φ) (Osborne 2013, p.122, Eq. 7.29), that is $y_e \approx m(\varphi)$ or, if a central scale factor k_0 is used

$$y_e \approx k_0 m(\varphi) \quad (70)$$

For an infinitesimal change in latitude $d\varphi$ the corresponding meridian length can be computed as

$$dm(\varphi) = \rho d\varphi \quad (71)$$

With $\frac{dm(\varphi)}{d\varphi} = \rho$ from Eq. (71) we can compute from Eq. (70)

$$\frac{\partial y_e}{\partial \varphi} = k_0 \frac{dm(\varphi)}{d\varphi} = k_0 \rho \quad (72)$$

which, upon substitution in Eq. (69), yields

$$\frac{ds_g}{ds_{go}} = \frac{k_0 \rho + k_0 h_0}{k_0 \rho} = \frac{\rho + h_0}{\rho} \quad (73)$$

Now, introducing Eqs. (73), (61) and (63) into (60) we obtain

$$k = \frac{ds_g}{ds_{ChS}} = \frac{\rho + h_0}{\rho} k_e \frac{\rho}{\rho + h_0} = k_e \quad (74)$$

That is, the scale factor of the ChS-to-plane Transverse Mercator projection is equal to the scale factor of the standard ellipsoid-to-plane Transverse Mercator projection. This result can be confirmed by an alternative computation procedure completely independent from the demonstration above which relies only on the defining equations of the projection: using numerical determination of partial derivatives of Eqs. (37) and (38) (i.e. by computing coordinates for small increments of latitude and longitude) and subsequent calculation of the scale factors by Eqs. (43) and (44) we obtain negligible discrepancies of the order of 10^{-9} or below, partly attributable to the truncation in the series in Eqs. (37) and (38), which confirm this result.

Finally, while Eqs. (37) and (38) provide the direct formulation for the ChS-to-grid projection, the inverse computation, that is, the determination of geodetic coordinates (φ, λ) from grid coordinates (x, y) can be obtained as follows.

First, as a starting approximation we obtain initial latitude and longitude values (φ, λ) by means of the inverse formulas for the standard Transverse Mercator projection of the reference ellipsoid using (x, y) as if they were indeed (x_e, y_e).

From Eqs. (37) and (38) we can write

$$x_e = x - \Delta\lambda k_0 h_0 \frac{\nu cos\varphi}{\rho} - \frac{\Delta\lambda^3}{6} k_0 h_0 (cos^2\varphi - sin^2\varphi) cos\varphi \quad (75)$$

$$y_e = y - k_0 h_0 \varphi - \frac{\Delta\lambda^2}{2} k_0 h_0 sin\varphi cos\varphi \quad (76)$$

Now we start the following simple iterative procedure: introducing the approximate latitude and longitude values obtained before in these two equations along with the known coordinates (x, y), we obtain new values (x_e, y_e) from which we can obtain refined values for the latitude and the longitude by means of the inverse formulas for the standard Transverse Mercator projection. This procedure has a fast convergence, for instance, for the following numerical example in the second iteration the coordinate differences are already of the order of 10^{-9}.

Some numerical values for an example of use of the ChS-to-grid Transverse Mercator projection follow: given the GRS80 ellipsoid (a = 6378137, f = 1/298.257222101, $e = \sqrt{2f - f^2}$), geodetic coordinates φ = 40º and λ = 6º, origin of longitudes λ_0 = 3º, and ellipsoid height h_0 = 2000m of the ChS, central scale factor k_0 = 0.9996 and false easting of 500000m, the direct computation is solved by Eq. (37) and Eq. (38) resulting in x = 756180.159m and y = 4433466.111m. The scale factor, computed by means of Eq. (74), is 1.00040750. The inverse computation, starting with these values for *x* and *y*, produce the initial geodetic coordinates φ = 40º and λ = 6º.

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Tables

Table 1. Conformality test for Direct Mercator projection for ChS (of height h_0) to plane: k_m and k_p denote the linear distortion in the meridian and parallel directions, respectively. For comparison, the scale value of the Direct Mercator projection for the reference ellipsoid along the meridian k_{em} and its difference with respect to its scale along the parallel k_{ep} are also given.

Latitude (deg)	h_0 (m)	k_m	k_m - k_p	k_e	k_{em} - k_{ep}
0	0	1	0	1	0
0	1000	0.99984324	-1.11E-16	1	0
0	2000	0.99968653	0	1	0
0	3000	0.99952986	0	1	0
20	0	1.06376102	2.22E-15	1.06376102	2.22E-15
20	1000	1.06359432	2.66E-15	1.06376102	2.22E-15
20	2000	1.06342769	2.44E-15	1.06376102	2.22E-15
20	3000	1.06326110	2.22E-15	1.06376102	2.22E-15
40	0	1.30360069	3.02E-13	1.30360069	3.02E-13
40	1000	1.30339662	3.01E-13	1.30360069	3.02E-13
40	2000	1.30319261	3.01E-13	1.30360069	3.02E-13
40	3000	1.30298867	3.01E-13	1.30360069	3.02E-13
60	0	1.99497290	2.14E-12	1.99497290	2.14E-12
60	1000	1.99466095	2.14E-12	1.99497290	2.14E-12
60	2000	1.99434910	2.13E-12	1.99497290	2.14E-12
60	3000	1.99403735	2.13E-12	1.99497290	2.14E-12
80	0	5.74004558	2.07E-12	5.74004558	2.07E-12
80	1000	5.73914869	2.07E-12	5.74004558	2.07E-12
80	2000	5.73825208	2.07E-12	5.74004558	2.07E-12
80	3000	5.73735575	2.07E-12	5.74004558	2.07E-12