COMPUTING OPTIMAL DISTANCES TO PARETO SETS OF MULTI-OBJECTIVE OPTIMIZATION PROBLEMS IN ASYMMETRIC NORMED LATTICES

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Abstract. Given a finite dimensional asymmetric normed lattice, we provide explicit formulae for the optimization of the associated (non-Hausdorff) asymmetric "distance" among a subset and a point. Our analysis has its roots and finds its applications in the current development of effective algorithms for multi-objective optimization programs. We are interested in providing the fundamental theoretical results for the associated convex analysis, fixing in this way the framework for this new optimization tool. The fact that the associated topology is not Hausdorff forces us to define a new setting and to use a new point of view for this analysis. Existence and uniqueness theorems for this optimization are shown. Our main result is the translation of the original abstract optimal distance problem to a clear optimization scheme. Actually, this justifies the algorithms and shows new aspects of the numerical and computational methods that have been already used in visualization of multi-objective optimization problems.

1. Introduction

Asymmetric norms have been recently introduced as a way of incorporating new metric tools for helping the decision maker in the graphical analysis of Pareto fronts associated to multi-objective optimization techniques (see [4]). Roughly speaking, there are two different approximations to the general matter of solving multi-objective optimization problems. In the first one, all the (real valued) objectives are simultaneously optimized, giving a Pareto front in which the decision maker choose the best solution according to his preferences. The second option is to provide an aggregation function combining all the objectives in such a way that it contains the criterion of the decision maker; once this function is given, the problem becomes a real valued optimization problem to which all the usual mathematical machinery can be applied. The present paper must be understood as a contribution to the second theoretical setting. Interested readers in the first approximation are referred to [11] and [23] for more details; see also [21, 22] and the references therein for recent developments using asymmetric distance notions.

To be more precise, the analysis of the set of solutions of a multi-objective optimization problem is usually based on the expertise of the decision maker, who compare the elements of the Pareto set trying to get the best option for his particular purpose. A considerable effort has been made for giving the required tools in recent years, mainly concerning the development of new visualization procedures. For example, in [24] the reader can find an explanation of a visualization

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technique based on the so called level diagrams; in [25] a general presentation of the existing methods is provided.

Although this way of understanding the analysis of Pareto fronts is fruitful for solving applied problems, it is also necessary to give precise theoretical arguments to show that these techniques provide safe optimization procedures. In order to do this, it is useful to understand the decision-making rule as an optimization of the (real) function defined as the composition of the original vector-valued objective map and an aggregation function. The second one might represent all the requirements that are taken into account by the decision maker. In the case that we consider in this paper, the aggregation function is given by an asymmetric norm, what concerns the metric properties of the Euclidean space. This fact forces to study optimization in the non-standard context of the asymmetric normed linear lattices and best approximation to convex sets in these spaces.

The asymmetric norm that we consider allows to unify in a single mathematical object the notions of dominance —a partial order relation in the n-dimensional space—, and (Euclidean) distance among points. Essentially, the (quasi) distance among two points provided by an asymmetric norm measures "how far is one point to dominate the other point". This "distance" is not symmetric, since the distance from one point x to another point y does not coincide with the distance from y to x. That is, it is a genuine quasi distance. This non-standard metric idea may be used for constructing a new aggregation function that would take into account the notions of dominance and the Euclidean distance as well as the preferences of the designer. The construction of such a specific aggregation function is not the aim of the present paper. We assume that the particular asymmetric norm has been already chosen by the decision maker, and the solution of the multi-objective optimization problem is found by solving the mono-objective problem defined by composing with this aggregation function.

An additional problem appears when we try to find the best approximation from a point to a set with respect to an asymmetric norm. In contrast to what happens when a standard Euclidean norm is considered and due to the fact that the induced topology is not Hausdorff, there are a lot of points in the set that attain the minimum distance to a given point even if the set is compact and convex. It is needed to fix an adequate definition of optimal distance from a given point to a subset with respect to an Euclidean asymmetric norm for assuring that the problem admits a unique solution. This will be done by adapting some previous theoretical developments, centering our attention in finite dimensional vector lattices with compatible Euclidean norms that are used for defining asymmetric norms in the canonical way: as the norm of the positive parts of the vectors.

The ideas of the present paper are structured as follows. After the preliminary Section 2, the main results of the article are presented in Sections 3 and 4. Section 3 is devoted to explain the rules for computing q-nearest points and optimal distance points in asymmetric normed lattices. It must be noticed that the definition of these sets of points are not a direct generalization of the notions that are used in the case of normed spaces. Essentially, as we said before, the reason is that the topology defined by a lattice asymmetric norm is not Hausdorff. This leads to the need of translating the question and the to a more abstract setting, using topological techniques that are not usual in a classical optimization framework. It must be said that most of the results that we use here have been obtained previously in theoretical articles from an abstract topological point of view. We will use the results —obtained mainly by Cobzaş— that can be found in [5, 6, 8, 10, 17] (for a systematic presentation see [7]). Together with some classical results of convex analysis, these papers allow to translate some general aspects of the convex optimization to this non-Hausdorff setting for providing some clear computational rules, that are explained in Section 4. Therefore, the main contributions of the present article are twofold:

(i) To establish the main existence and uniqueness theorems regarding optimal distance points in the present setting. They are given in Section 3.2 (Theorem 3.11 and Corollary 3.12).

(ii) To adapt and translate the known abstract results into practical tools for applying them in multi-objective optimization problems. This can be found in Section 3.1 and Section 4.

2. Preliminaries

Consider a Pareto set A of solutions of a multi-objective problem that belongs to a finite dimensional linear lattice —typically, $\mathbb{R}^n$ endowed with an order $\leq$ and a compatible lattice norm—. We must point our that we use the term "lattice" in the present paper as an ordered vector space whose norm is compatible with the order (see for example [3, 19]). The ordering in the lattice represents the criterion for establishing a hierarchy among the elements of the solution set, in the sense that, if $x, y \in A$ and $x \leq y$, then x dominates y. In terms of the optimization process, this means that the point x is preferred to y as an optimal final solution of the problem by the decision maker. *A priori,* all the elements of A provide a valid solution to the original problem. However, the decision maker needs to choose one of them based on geometric arguments associated to the structure of the problem, and using information that is not explicitly included in the requirements of the optimization problem. These arguments are often of the following type: (a) If two elements x and y of A are considered, x is a better solution than y if x dominates y in the natural ordering of the space. (b) If x and y are not ordered —that is, neither $x \leq y$ nor $y \leq x$—, then x is better than y if it is closer to a fixed selected point that is considered the optimum limit for the process (utopia) —normally, the origin of the coordinate system—.

As we said in the Introduction, a new way of finding an equilibrium among these two criteria for choosing the best solution has been recently introduced. It is based on a topological structure that considers the ordering in the linear lattice as a preference criterion for defining the —in general not Hausdorff— topology by what is called an asymmetric distance (see [4]). Let us explain briefly the mathematical structure that is used. The asymmetric lattice norm induced by a finite dimensional normed lattice $(\mathbb{R}^n, \leq, \|\cdot\|)$ is a function given by

$$q(x) := \|x \vee 0\|, \quad x \in \mathbb{R}^n;$$

here "$\vee$" denotes the maximum of "x" and 0 in the vector lattice order (see for example the first pages in [19]). It defines a quasi-distance d by means of the formula

$$d(x, y) = q(y - x) = \|(y - x) \vee 0\|, \quad x, y \in \mathbb{R}^n;$$

it is clear that $d(x, y)$ does not coincide with $d(y, x)$ in general, so the quasi-distance is certainly not a distance. It is also easy to see that, for $n \geq 2$, the induced topology cannot be Hausdorff; in fact, it only satisfies the First Axiom of separability (T_1, see [9, 14]).

The main idea that these tools allow to model is the following: $d(x,y)$ measures the "minimal normed distance that y must be translated in order to dominate x". For example, in the natural coordinatewise ordering of $\mathbb{R}^n$ endowed with the standard Euclidean norm, the point $y = (1,0)$ must be translated to $(0,0)$ to dominate $x = (0,1)$, and the minimal distance is then equal to 1.

In this sense, we have that $d(x,y) = 0$ if $y \leq x$, that is, if y is already dominating x and it does not need to be moved. This rule is the key for providing effective geometric tools of visualization for helping the decision maker to choose a concrete result of all the ones appearing in the Pareto set A.

However, answers to the two main theoretical problems appearing in a optimization procedure —existence and uniqueness of optimal solution— has not explicitly provided yet. As we said in the Introduction, to give concrete answers to these problems and to fix the requirements for the lattice asymmetric norms for satisfying the results are the aims of this paper.

In order to do this, we will use some fundamental results of the recently developed theory of asymmetric normed linear spaces. We will center our attention in the finite dimensional case with Euclidean asymmetric norms, in order to assure the uniform convexity and the smoothness that are classically required for proving the uniqueness of the optimal points of shortest distance among convex subsets. We will see that these requirements are also adequate in the asymmetric case, allowing to prove the expected results of existence and uniqueness. Some recent developments on the geometry of finite dimensional asymmetric normed spaces can be found in [18]; the setting established there completes the information required for constructing optimization tools, as in the classical case.

Let us introduce now the necessary definitions and fundamental results that will be needed in the paper.

2.1. ***Asymmetric normed spaces.*** Let X be a linear space. An asymmetric norm q on X is function $q : X \to [0,\infty)$ satisfying that for $x, y \in X$,

(1) $q(tx) = tq(x)$ for $t \geq 0$,

(2) $q(x + y) \leq q(x) + q(y)$ and

(3) $q(x) = 0 = q(-x)$ if and only if $x = 0$.

Such an asymmetric norm defines a translation invariant topology when the quasi-distance $d(x,y) = q(y - x)$, $x, y \in X$, is considered, that is given by the countable basis of neighborhoods defined by the balls $B_n(x) = \{y \in X : d(x,y) = q(y - x) \leq 1/n\}$. The resulting topological quasi-metric structure is called an asymmetric normed linear space. The induced topology has in general weak separation properties (it is not in general Hausdorff), and the expression $\hat{q}^s(x) := \max\{q(x), q(-x)\}$, $x \in X$, gives always a norm on X. As the reader will notice later on, we will define the associated norm q^s in a different way in order to obtain an equivalent norm preserving the smoothness properties of the original Euclidean norm. This structure is nowadays well-known; the reader can find all the information that is needed for this paper in the monograph [7] and the references therein. A linear operator $T : X \to Y$ among asymmetric normed linear spaces (X,q) and (Y,p) is continuous if and only if there is a constant $k > 0$ such that $p(T(x)) \leq k\,q(x)$ for all $x \in X$ (see for example Proposition

2.1.2 in [7]). The dual space X_q^b of an asymmetric normed space is the space of all functionals $\varphi : X \to \mathbb{R}$ that are q-bounded as $\varphi(x) \leq k\, q(x)$ for a certain constant k. The "norm" $\|\varphi|_q$ of φ is the infimum of all constants k satisfying this inequality. It must be pointed out that X_q^b is not in general a linear space but a cone: this is why we use the symbol $\|\varphi|_q$, since it is not strictly speaking a norm. Such a function is sometimes called a cone-norm.

2.2. ***Euclidean asymmetric normed lattices.*** We will consider normed linear lattice structures as starting point. Recall that a Banach lattice is an ordered linear space $(X, \leq)$ —sometimes called a Riesz space—, with a norm $\|\cdot\|$ that is compatible with the order: that is, if $x, y \in X$ and $|x| \leq |y|$, then $\|x\| \leq \|y\|$. If an order $\leq$ is given, then the lattice operations $x \vee y$ (maximum), $x \wedge y$ (minimum) and $|x|$ (absolute value) are always defined (see for example [19, pp.1,2], [3, Ch.1] or [20, Ch.2]). On the other hand, for defining an order in the finite dimensional space $\mathbb{R}^n$ in such a way that it gives a Banach lattice structure, it is enough to consider a convex cone C non containing a subspace and with non empty interior. Then, for $x, y \in \mathbb{R}^n$, we have that $x \leq y$ if and only if $y - x \in C$. In this paper, we will consider orders given by cones generated by a basis $\mathcal{B} = \{b_1, ..., b_n\}$ of $\mathbb{R}^n$, that is, cones like

$$C = \Big\{ \sum_{i=1}^{n} \lambda_i b_i : \lambda_i \geq 0,\, i = 1, ..., n \Big\}. \tag{1}$$

We will consider asymmetric norms defined as $q(x) = \|x \vee 0\|$, where $\|\cdot\|$ is a lattice norm as defined above. The notation $x \vee 0$ will be often used in the paper: if $x = (x_1, ..., x_n) \in \mathbb{R}^n$, this means

$$x \vee 0 = (x_1, ..., x_n) \vee 0 = (\max\{x_1, 0\}, ..., \max\{x_n, 0\}).$$

Note that we have required that if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, —where $|x|$ and $|y|$ are the modulus of the elements x and y defined by the ordering—. We will say in this case that the associated structure $(\mathbb{R}^n, q, \leq)$ —where q is defined using $\|\cdot\|$—, is an asymmetric normed linear lattice. If moreover the norm $\|\cdot\|$ is an Euclidean norm in $\mathbb{R}^n$ —that is, a norm defined by a scalar product "$\cdot$" by $\|x\| = \sqrt{x \cdot x}$—, we will say that the space $(\mathbb{R}^n, q, \leq)$ is an Euclidean asymmetric normed linear lattice. This will be the class of spaces that we will consider in this paper. Note that in this case an Euclidean norm can be given by considering the formula

$$q^s(x) := \sqrt{q(x)^2 + q(-x)^2}, \quad x \in \mathbb{R}^n.$$

The study of asymmetric normed linear lattices started in the nineties as the first attempt for understanding linear structures endowed with non-symmetric topologies. In the seminal papers [2, 12], this is one of the natural asymmetric norms that were considered. Several papers can be found on this specific current topic (see [1, 10, 13] and the references therein); as far as we know, the last one is [9], where a systematic study of this family of asymmetric norms is developed.

We will use the following class of compatible asymmetric norms over finite dimensional lattices $(\mathbb{R}^n, \leq)$. Let $\mathcal{B} := \{b_i : i = 1, ..., n\}$ be a basis of $\mathbb{R}^n$; so, if C is the convex cone generated by $\mathcal{B}$, then $x \leq y$ if and only if $y - x \in C$. Let us write $(x_1^i, ..., x_n^i)$ for the coordinates of b_i with respect to the canonical basis of $\mathbb{R}^n$, $0 \leq i \leq n$. Consider the matrix M defined by the coordinates of the vectors

$b_1, ..., b_n$ as columns, that is,

$$M = \begin{bmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{bmatrix}.$$

Note that this matrix defines a scalar product $\cdot_{\mathcal{B}}$ in $\mathbb{R}^n$ with the property that the elements of $\mathcal{B}$ constitute an orthonormal basis of the space. Therefore, easy linear algebra arguments allow to obtain the following fact.

Lemma 2.1. *The Euclidean asymmetric lattice norm associated to the basis $\mathcal{B}$ is given by*

$$q(x) = \sqrt{\max\{\alpha_1, 0\}^2 + ... + \max\{\alpha_n, 0\}^2}$$

$$= \sqrt{\Big((x_1, ..., x_n)(M^{-1})^T \vee 0\Big) \cdot (M^{-1}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$$

for $(x_1, ..., x_n)$ being the coordinates of $x \in \mathbb{R}^n$ with respect to the canonical basis and $(\alpha_1, ..., \alpha_n)$ the coordinates of x with respect to the basis $\mathcal{B}$. Moreover, q is compatible with the order defined by the cone (1) generated by the basis $\mathcal{B}$ of $\mathbb{R}^n$.

Proof. Take $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Then we have that

$$q(x) = \sqrt{\max\{\alpha_1, 0\}^2 + ... + \max\{\alpha_n, 0\}^2}$$

$$= \sqrt{\max\{\alpha_1, 0\}\alpha_1 + ... + \max\{\alpha_n, 0\}\alpha_n} = \sqrt{(\alpha_1, ..., \alpha_n) \vee 0) \cdot (\alpha_1, ..., \alpha_n)^T}.$$

Replacing $(\alpha_1, ..., \alpha_n)$ by $(x_1, ..., x_n)(M^{-1})^T$ as well as $(\alpha_1, ..., \alpha_n)^T$ by

$$M^{-1}(x_1, ..., x_n)^T,$$

we get the result. □ □

As we said before, we will center our attention in this class of Euclidean asymmetric norms. Let us remark again that the symbol $\leq$ does not denote always the canonical order in $\mathbb{R}^n$ but the order compatible with the asymmetric norm q, for which the positive cone is defined by the basis $\mathcal{B}$ as in (1). Note that the equality

$$\{z \in \mathbb{R}^n : q(z) = 0\} = \{z : z \leq 0\}, \tag{2}$$

holds, where $\leq$ is the order generated by this cone. This set is relevant for the characterization of the topological properties of an asymmetric normed space (see Eq.(2.4.3) in [7, §.2.4.2], or [17]).

Example 2.2. *Suppose that the decision maker considers that the direction of the OX axis is more relevant for the optimization than the OY axis, at a rate of 2 to 1. This can be modeled by defining the new basis $\mathcal{B}_0$ as $\mathcal{B}_0 = \{(2, 0), (0, 1)\}$, and so*

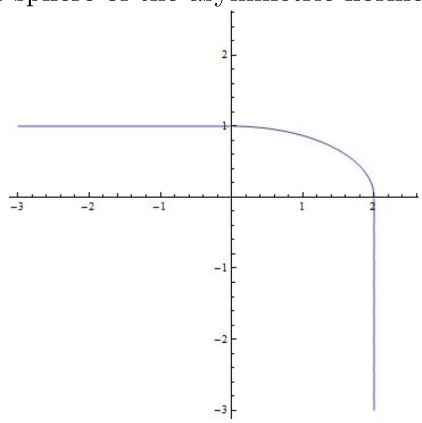
$$M_0 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Euclidean asymmetric norm associated to $\mathcal{B}_0$ is then

$$q_0(v) = \sqrt{\Big((x_1, x_2)(M^{-1})^T \vee 0\Big)(M^{-1}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \sqrt{\frac{1}{4}\max\{x_1, 0\}^2 + \max\{x_2, 0\}^2},$$

for $v = (x_1, x_2) \in \mathbb{R}^2$. The unit ball of this asymmetric norm is defined by the points below the curve in Figure 2.2.

FIGURE 1. Unit sphere of the asymmetric normed space $(\mathbb{R}^2, q_0)$.



3. q-NEAREST POINTS AND OPTIMAL DISTANCE POINTS IN ASYMMETRIC NORMED LATTICES: THE OPTIMIZATION SETTING

We will use the explained theoretical geometric structure for defining the following optimization framework.

- Consider a set of solutions $A \subseteq \mathbb{R}^n$ of a multi-objective optimization problem, that is, the associated Pareto set.
- Consider a point x_0 not belonging to A and fixed by the analyst, representing an optimal solution to the problem from the point of view of the geometrical properties of the lattice and taking into account the original problem, *in case it were a solution of the problem. Typically, x_0 is not so.* We can consider without loss of generality that $x_0 = 0$; otherwise, it is always possible to translate x_0 to the origin of the space. The translation invariant nature of the topology associated to an asymmetric norm allows to do it (see [7]).
- A point $x \in A$ is considered a better solution than $y \in A$ whenever x dominates y. In case x and y are not comparable —that is, $x \not\leq y$ and $y \not\leq x$—, then we have to understand the values of the distances $d(x, y) = q(y-x)$ and $d(y, x) = q(y-x)$ as follows: $d(x, y) = q(y-x)$ is the "distance that y must be translated in order to dominate x". Therefore x is a better solution than y if $d(x, y) \leq d(y, x)$.

In the model, the order in the underlying Banach lattice is fundamental, since it contains the information about the dominance relation. The decision maker must choose the relevant directions in which dominance makes sense due to the properties of the optimization problem. After the definition of the set of relevant directions —n linearly independent vectors $b_1, ..., b_n$— they can be used for defining a basis as $\mathcal{B}$ in Section 2 —. The order is given by the cone C defined as in (1).

3.1. ***q-nearest points in asymmetric normed lattices.*** Consider a subset A of an asymmetric normed lattice $(\mathbb{R}^n, \leq, q)$. The weakest notion of best approximation from A to a certain point x in the space is given by the so called q-nearest points of A to x. Let us introduce some notation. We mainly follow [7] and the references therein.

Recall that, broadly speaking, the point x from which we want to compute the distance to A typically dominates a big part of A in the optimization sense, what means that $x \leq y$ for a lot of elements y of A. This implies that $d(y, x) = 0$ for these points, what makes unnecessary to compute the optimal distance in this "direction of the space" and justifies to focus our analysis on the symmetric distance $d(x, y)$ as we explain in what follows.

Let $A \subset \mathbb{R}^n$ and $x \notin A$. Define the non-negative real number

$$d_q(x, A) := \inf\{q(y - x) : y \in A\},$$

and the set

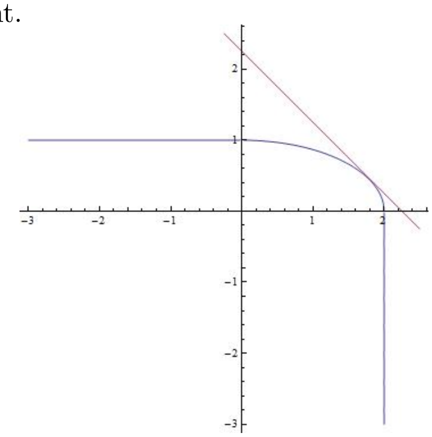
$$P_A(x) := \{y \in \mathbb{R}^n : q(y - x) = d_q(x, A)\}.$$

Definition 3.1. *If $P_A(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$, then it is said that A is q-proximinal. In this case, each $y \in P_A(x)$ is said to be a q-nearest point to x in A.*

The notion of q-nearest point in a given set to a point is central in this paper. Although the set $P_A(x)$ is not in general defined just by a single point, sometimes this situation appears in a natural way. Let us show this in the next example.

Example 3.2. *Consider again the asymmetric norm q_0 defined in Example . Suppose that the convex set A is the Pareto front of a 2-objective optimization problem given by the segment defined by the line $x_1 + x_2 = 9/4$ and $x_1 \in [-1/5, 5/2]$. Then a direct calculation shows that $P_A(0)$ is nonempty and contains only the point $(9/5, 9/20)$ (see Figure 3.2). This is the q-nearest point in the Pareto front A to the point 0 this case. Note that the nearest point with respect to the canonical Euclidean norm is $(9/8, 9/8)$.*

FIGURE 2. Unit sphere of the asymmetric norm q_0 together with the Pareto front.



The characterization of this kind of points can be done for asymmetric norms in terms of the actions of the functionals of the dual space as in the case of normed spaces; the rest of this section is devoted to do that. This is rather surprising, since as we said in Section 2.1 the dual space defined as the set of bounded/continuous linear functionals $\varphi : (X, q) \to \mathbb{R}$ —in the sense that satisfy an inequality as $\varphi(x) \leq k\, q(x), \quad x \in X$, for a certain constant $k > 0$—, is not in general a linear space. However, its structure and the main separation properties that it provides by acting on sets of X are nowadays well-known (see [2, 5, 6, 8, 15, 16]). At least for the case of a convex set A the following results can be stated. They give descriptions of the q-nearest points in terms of the elements of the dual space in a Hahn-Banach fashion.

Theorem 2.5.2 in [7]. *For a non-empty convex subset A of an asymmetric linear space (X, q) and an element $x \in X$, we have that*

$$d_q(x, A) = \sup_{\|\varphi|_q \leq 1} \inf_{y \in A} \varphi(y - x).$$

Moreover, if $d_q(x, A) > 0$, then there exist $\varphi_0 \in X_q^b$, $\|\varphi_0|_q = 1$, such that $d_q(x, A) = \inf\{\varphi_0(y - x) : y \in A\}$, that means that the supremum in the formula above is attained.

From this result, it can be deduced the following characterization of q-nearest points in terms of functionals of the dual space.

Theorem 2.5.3 in [7]. *Let A be a non-empty set of the asymmetric normed space (X, q), $y \in A$ and $x \in X$. If there is a functional $\varphi \in X_q^b$ such that $\|\varphi|_q = 1$, $\varphi(y - x) = q(y - x)$ and $\varphi(y) = \inf \varphi(A)$, then y is a q-nearest point to x in A. Conversely, if A is convex, $d_q(x, A) > 0$ and y is a q-nearest point to x in A, then there is a functional $\varphi \in X_q^b$ satisfying the conditions in the previous paragraph.*

For using these arguments for constructing an optimization tool, is better to give a version in terms of separating functionals of the unit sphere of dual space X_q^b. The following result gives the adequate version, in which the geometric separation in clearly established.

Theorem 2.5.5 in [7]. *Let A be a non-empty set of the asymmetric normed space (X, q), $y_0 \in A$ and $x \in X$. If for every $y \in A$ there is a functional $\varphi = \varphi_y$ in the unit ball of X_q^b such that $\varphi(y_0 - x) = q(y_0 - x)$ and $\varphi(y_0 - y) \leq 0$, then y_0 is a q-nearest point to x in A.*

Conversely, if A is convex and $y_0 \in A$ is such that $q(y_0 - x) = d_q(x, A) > 0$, then for every $y \in A$ there is a functional φ_y (in fact an extreme point of the unit ball of X_q^b) satisfying the conditions above.

Let us come back now to our setting. Recall that we are considering finite dimensional real lattices $(\mathbb{R}^n, \leq)$ for which the asymmetric norm is given by the formula $q(x) = \|x \vee 0\|$, $x \in \mathbb{R}^n$, where $\| \cdot \|$ is a lattice norm compatible with the order $\leq$ (see the definition of q at the beginning of Section 2). The abstract elements in the previous results can be now fixed in a much more concrete way. We will see as a starting point that, essentially, *only positive functionals are continuous.* Note that in this context "continuity" means continuity from the space $(\mathbb{R}^n, q)$ to the asymmetric normed space $(\mathbb{R}, \max\{r, 0\})$; indeed, $p(r) := \max\{r, 0\}$, $r \in \mathbb{R}$ is clearly an asymmetric norm in $\mathbb{R}$.

First of all, let us establish the optimal way of representing *duality* on asymmetric normed lattices as the ones we are concerned in this paper. As we said in Section 2, the order in the space $(\mathbb{R}^n, q, \leq)$ is defined by the positive cone given by the preference directions fixed at the beginning, and that define a basis $\mathcal{B}$ of $\mathbb{R}^n$. The natural way of representing the elements of the dual space is as the vectors $\varphi := (a_1, \cdots, a_n) \in \mathbb{R}^n$ whose actions on the elements $x = (x_1, \cdots, x_n)$ of $(\mathbb{R}^n, q, \leq)$ are bounded, that is

$$\varphi(x) = \langle \varphi, x \rangle \leq k q(x)$$

for a certain $k > 0$ independent of x. Next lemma provides specific information for our case.

Lemma 3.3. *Consider an asymmetric normed lattice $(\mathbb{R}^n, q, \leq)$ for an asymmetric norm q belonging to our class. Then a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ that is represented in the canonical basis as $\varphi = (a_1, ..., a_n)$ belongs to $(\mathbb{R}^n)^b_q$ if and only if*

$$(a_1, ..., a_n)(M^{-1})^T \geq 0.$$

Taking into account that $(\beta_1, ..., \beta_n) = (a_1, ..., a_n)(M^{-1})^T$, where $(\beta_1, ..., \beta_n)$ are the coordinates of φ in the dual basis of $\mathcal{B}$, this condition can be written also as $(\beta_1, ..., \beta_n) \geq 0$.

Proof. Let us show first the "only if" part of the proof. Due to the structure of the space and taking into account that the scalar product in the underlying Euclidean space is given by

$$x \cdot y = (x_1, \cdots, x_n)(M^{-1})^T \cdot (M^{-1}) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad x, y \in \mathbb{R}^n,$$

the duality among a vector x with coordinates $(x_1, \cdots, x_n)$ —in the canonical basis— and a functional $\varphi = (a_1, \cdots, a_n)$ must also be written as

$$\varphi(x) = (x_1, \cdots, x_n)(M^{-1})^T \cdot (M^{-1}) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad x \in \mathbb{R}^n, \ \ \varphi \in (\mathbb{R}^n)^b_q.$$

Therefore, using Lemma 2.1, the condition $\varphi(x) \leq k\, q(x)$ can be written as

$$\varphi(x) = (x_1, \cdots, x_n)(M^{-1})^T \cdot (M^{-1}) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\leq k \sqrt{\Big((x_1, \cdots, x_n)(M^{-1})^T \vee 0\Big) \cdot (M^{-1}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}.$$

Using the coordinates $(\alpha_1, \cdots, \alpha_n)$ of the vector x in the basis $\mathcal{B}$ as explained in Section 2, we obtain

$$(\alpha_1, \cdots, \alpha_n) \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \leq k \sqrt{\sum_{i=1}^{n} \max\{\alpha_i, 0\}^2} \tag{3}$$

where $\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix} = (M^{-1})\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$ are the coordinates of φ in the dual basis $\mathcal{B}^*$, which coincides with $\mathcal{B}$. Since (3) must hold for every $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$, we have that the boundedness/continuity requirement is given by the inequality

$$\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix} = (M^{-1})\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} \geq 0,$$

and so the result is proved.

For the converse, just consider the continuity inequality given by equation (3): since $(\beta_1, ..., \beta_n) \geq 0$, using Cauchy-Schwarz inequality we obtain for all $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$

$$(\alpha_1, \cdots, \alpha_n) \cdot \begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix} \leq \sum_{i=1}^n \max\{\alpha_i, 0\}\beta_i \leq \sqrt{\sum_{i=1}^n \max\{\alpha_i, 0\}^2} \cdot \sqrt{\sum_{i=1}^n \beta_i^2}$$

what gives the result. □ □

This allows to write the following concrete representation of the dual space $(\mathbb{R}^n)^b_q$. Note that, using the representation of the vector x in the new basis $\mathcal{B}$

$$x \sim (\alpha_1, \cdots, \alpha_n) = (x_1, \cdots, x_n)(M^{-1})^T$$

and the usual duality in the sequence space ℓ^2 with this representation, we obtain the following result.

Lemma 3.4. *The dual space is the normed cone $((\mathbb{R}^n)^b_q, q^*)$ represented (in the canonical basis) by the vectors*

$$(\mathbb{R}^n)^b_q = \Big\{\varphi = (a_1, ..., a_n) \sim (\beta_1, \cdots, \beta_n) \in \mathbb{R}^n : (\beta_1, \cdots, \beta_n)$$

$$= (a_1, ..., a_n)(M^{-1})^T \geq 0\Big\},$$

and with a cone-norm given by

$$q^*(\varphi) := \|\varphi|_q = \Big(\sum_{i=1}^n \beta_i^2\Big)^{\frac{1}{2}}.$$

Proof. This is just a consequence of the following equalities. Take

$$q^*(\varphi) := \|\varphi|_q = \sup\left\{\varphi(x) = (\beta_1, \cdots, \beta_n) \cdot M^{-1}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\right\}, \tag{4}$$

where the supremum is computed over all vectors x satisfying

$$q(x) = \sqrt{\Big((x_1, ...x_n)(M^{-1})^T \vee 0\Big) \cdot M^{-1}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}} \leq 1.$$

Then, in view of (4) and taking into account that all β_i are non-negative the duality in the Euclidean space gives

$$\sup\left\{\varphi(x) = (\beta_1, \cdots, \beta_n) \cdot \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} : q(x) = \sqrt{\sum_{i=1}^n \max\{\alpha_i, 0\}^2} \leq 1\right\}$$

$$= \Big(\sum_{i=1}^n |\beta_i|^2\Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^n \beta_i^2\Big)^{\frac{1}{2}}. \qquad \square$$

$\square$

These results allow to write the characterization of q-nearest points in terms of functionals as follows. Note that we can assume that the selected point x to which we want to find the q-nearest distance point can be identified with the origin 0.

Corollary 3.5. *For a non-empty convex subset A of an asymmetric linear space $(\mathbb{R}^n, q)$ of our class, we have that*

$$d_q(0, A) = \sup\left\{\inf\Big\{(\beta_1, \cdots, \beta_n) \cdot M^{-1} \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} : y \in A\Big\} : \sum_{i=1}^n \beta_i^2 \leq 1, \, \beta_i \geq 0\right\},$$

where $(y_1, ..., y_n)$ are the coordinates in the canonical basis of any $y \in A$.

Moreover, if $d_q(0, A) > 0$, then there exists $\varphi_0 = (\beta_1, \cdots, \beta_n)$ with

$$\beta_1 \geq 0, \cdots, \beta_n \geq 0, \quad \sum_{i=1}^n \beta_i^2 = 1,$$

such that

$$d_q(0, A) = \inf\Big\{(\beta_1, \cdots, \beta_n) \cdot M^{-1} \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} : y \in A\Big\}.$$

Proof. Just use Theorem 2.5.2 in [7], Lemma 3.3 and Lemma 3.4. $\square$ $\square$

Let us give now the specific Hahn-Banach separation theorem for our family of finite dimensional asymmetric normed lattices.

Corollary 3.6. *Let A be a non-empty set of an asymmetric normed space $(\mathbb{R}^n, q)$ of our class and $y \in A$. If there is $(\beta_1, ..., \beta_n)$ such that $\beta_1 \geq 0, \cdots, \beta_n \geq 0$, $\sum_{i=1}^n \beta_i^2 = 1$ and*

$$q(y) = (\beta_1, \cdots, \beta_n)\, M^{-1} \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} \leq (\beta_1, \cdots, \beta_n)\, M^{-1} \begin{bmatrix}z_1\\ \vdots \\ z_n\end{bmatrix}$$

for all $(z_1, ..., z_n) \in A$, then y is a q-nearest point to 0 in A.

Conversely, if A is convex, $d_q(0, A) > 0$ and y is a q-nearest point to 0 in A, then there is a vector $(\beta_1, \cdots, \beta_n)$ satisfying the conditions above.

Proof. Use Theorem 2.5.3 in [7], Lemma 3.3 and Lemma 3.4. $\square$ $\square$

Due to the fact that the norm from which the asymmetric norm q is constructed comes from a scalar product, we can give a more concrete version of the result above, that in fact provides the best tool for computing the q-nearest points. Note that the convexity of A is not necessary for (ii) $\Rightarrow$ (i).

Corollary 3.7. *Let A be a non-empty convex set of an asymmetric normed space $(\mathbb{R}^n, q)$ of our class such that $d_q(0, A) > 0$. The following statements are equivalent for a point $y = (y_1, \cdots, y_n) \in A$.*

(i) *y is a q-nearest point to 0 in A.*

(ii) *For every $z = (z_1, \cdots, z_n) \in A$ we have that*

$$\big((y_1, \cdots, y_n)(M^{-1})^T \vee 0\big) \cdot M^{-1} \begin{bmatrix} y_1 - z_1 \\ \vdots \\ y_n - z_n \end{bmatrix} \leq 0.$$

Proof. For (ii) $\Rightarrow$ (i) we can use Corollary 3.6 (or Theorem 2.5.5 in [7]), Lemma 3.3 and Lemma 3.4, taking into account that an element $(\beta_1, ..., \beta_n)$ that gives by duality the asymmetric norm $q(y)$ is $(\beta_1, ..., \beta_n) = \frac{\big((y_1,\cdots,y_n)(M^{-1})^T \vee 0\big)}{q(y)}$, since

$$\big((y_1, \cdots, y_n)(M^{-1})^T \vee 0\big) \cdot M^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = q(y)^2.$$

Note that this equation gives also that the asymmetric norm of $(\beta_1, ..., \beta_n)$ is equal to one, as can be seen by dividing the equation by $q(y)^2$ and applying Lemma 3.4.

For (i)$\Rightarrow$ (ii) just consider the converse in Corollary 3.6. Notice that due to the strict convexity of the Euclidean norm, the only (norm one) functional that provides the norm of the vector $\hat{y}^+ := M^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \vee 0$ is $\frac{\hat{y}^+}{\|\hat{y}^+\|}$. Indeed, $q(y)^2 = \|\hat{y}^+\|^2 = \hat{y}^+ \cdot \hat{y}^+$. □ □

3.2. ***Optimal distance points in asymmetric normed lattices.*** Even for the Euclidean generated asymmetric norms of our family the notion of q-nearest point is too weak for having uniqueness results for optimization problems, since for very natural (convex) examples it can be easily proved that the set of q-nearest points from 0 to a set A is far to be a singleton. This motivated the introduction in the abstract theory of the notion of optimal distance points. Let $A \subset \mathbb{R}^n$ and $x \notin A$. Let us write P_A for the set of q-nearest points to A, that is

$$P_A := P_A(0) = \Big\{ y \in \mathbb{R}^n : q(y) = d_q(0, A) = \inf \big\{ q(z) : z \in A \big\} \Big\}.$$

Recall that sometimes this set contains only one point. For instance, we shown in Example 3.2 a particular situation in which this happens. However, this may not be the case. Let us show with an easy example that in general this set is not a singleton at all. This fact motivates the introduction of the main notion of this section: optimal distance points for asymmetric Euclidean lattice norms.

Example 3.8. *Take the Euclidean canonical asymmetric lattice norm in* $\mathbb{R}^2$*, that is given by*

$$q_2((y_1, y_2)) = \sqrt{(\max\{y_1, 0\})^2 + (\max\{y_2, 0\})^2}.$$

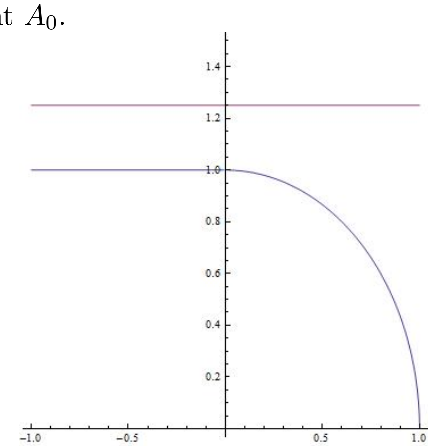
Consider that the Pareto front of an optimization problem is given by the (convex) set $A_0 = \{(y_1, 5/4) : -1 \leq y_1 \leq 1\}$*. Then for every* $(y_1, 5/4)$ *in this set,*

$$d_q((0,0), (y_1, 5/4)) = q((y_1, 5/4)) = \sqrt{\max\{y_1, 0\}^2 + (5/4)^2}.$$

The infimum of such function in A_0 *is* $5/4$*, and it is attained for all the points that satisfies that* $y_1 \leq 0$*. That is, the set of* q*-nearest points from* 0 *to* A_0 *is*

$$P_{A_0} = \{(y_1, 5/4) : -1 \leq y_1 \leq 0\}.$$

FIGURE 3. Unit sphere of the asymmetric norm q_2 together with the Pareto front A_0.



Before introducing the concept of optimal distance point, let us write some geometric/topological properties of the sets of q-nearest points. The following results can be found in [7] and in [17]; they are direct consequences of Lemma 2.3 and Proposition 2.6 in [17] for our setting. Recall that $\leq$ is the order compatible with q given by the cone generated by the elements of $\mathcal{B}$, which does not coincide necessarily with the canonical order of $\mathbb{R}^n$.

Remark 3.9. *Let* A *be a non-empty set of a finite dimensional asymmetric normed lattice* $(\mathbb{R}^n, q, \leq)$ *of our class. Then*

(i) *If* A *is a convex subset, then* P_A *is convex too.*
(ii) *If* A *is a closed subset of* $(\mathbb{R}^n, q^s)$*, then* P_A *is a closed subset of* $(\mathbb{R}^n, q^s)$*.*
(iii) *If* $y \in P_A$*, then* $\{z \in A : z \leq y\} \subset P_A$*.*
(iv) *If* $y \in \mathbb{R}^n$ *and* $\{z \in A : z \leq y\} \neq \emptyset$*, then* $q(y) \geq d(0, A)$*.*

We will introduce the concept of optimal distance point formally in what follows. This stronger notion allows to provide uniqueness results, and its analysis completes the information given by the computation of the set of q-nearest points. The following definition was originally given in [17] and can be found in Section 2.5.4 of [7] (make $x = 0$ in the definition appearing there).

Definition 3.10. *Let $(\mathbb{R}^n, q, \leq)$ an Euclidean asymmetric normed lattice of our class and write $\|\cdot\|_q$ for the associated Euclidean norm. Let $A \subseteq \mathbb{R}^n$. A q-nearest point $y \in A$ satisfying*

$$\|y\|_q \leq \|z\|_q$$

for every $z \in P_A$, is called an optimal distance point.

This definition, the previous properties of the set $P(A)$ and known results on optimal points on normed spaces allow to write the following relevant existence and uniqueness result as a direct consequence.

Theorem 3.11. *Let $A \subseteq \mathbb{R}^n$ a convex set such that $P_A \neq \emptyset$. Then*

(i) *There is at most one optimal distance point.*

(ii) *If A is q^s-closed, then such a point exists.*

Proof. Strict convexity of the norm q^s, local compactness of finite dimensional normed spaces or reflexivity of these spaces imply the following result from Theorem 2.5 in [17] (alternatively, see Theorem 2.5.9 in [7]). $\square$

$\square$

If $A \subseteq \mathbb{R}^n$, we write $c(A)$ for $\overline{co(A)}^{q^s}$, the closure of the convex hull with respect to the Euclidean norm. By changing A by this set and using again Theorem 2.5 in [17], we can formulate the following

Corollary 3.12. *Let A be a set of solutions of a multi-objective optimization problem —that is, its Pareto set—. Then there exists a unique optimal distance point from 0 to $c(A)$.*

Let us finish the section with an example. We intend to show that combining the computation of optimal distance points among the q-nearest points provides a good solution for the optimization problem explained in the Introduction. Note that all the points of P_A —the q-nearest points— are optimal if the only criterion used is the optimization of q. However, this set is too big and is not giving in general a reasonable answer to the problem, as can be easily shown.

Example 3.13. *Consider the subset A_0 of the canonical Banach lattice $\mathbb{R}^2$ given in Example 3.8.*

(i) *The optimization of the non-symmetric distance provided by q —the set of q-nearest points to 0— gives that*

$$P_{A_0} = \{(y_1, 5/4) : -1 \leq y_1 \leq 0\}.$$

We can try with other reasonable criterion; for example, take the point (y_1^0, y_2^0) of P_{A_0} that dominates all the set P_{A_0} (in the sense that $(y_1^0, y_2^0) \leq (y_1, y_2)$ for $(y_1, y_2) \in P_{A_0}$). This is the point $(-1, 5/4)$. However, the optimal distance point from 0 to P_{A_0} is $(0, 5/4)$: among all the q-nearest points, the one from which the Euclidean distance to 0 attains the minimum.

(ii) *In the example above, the optimal distance point coincides with the point of A_0 for which the minimal Euclidean distance from 0 to A_0 is attained. However this is not always the case, as can be seen in the next example. Let us define now the (convex) subset*

$$A_1 := \{(y_1, y_2) : y_2 = y_1 + 1, \ -2 \leq y_1 \leq 0\}$$

of the space considered above. We have that the Euclidean distance is

$$d_{q^s}(0, A_1) = 1/\sqrt{2},$$

and the mimimum is attained only at the point $(-1/2, 1/2) \in A_1$. *However, we have that* $P_{A_1} = \{(y_1, y_1 + 1) : -2 \leq y_1 \leq -1\}$ *to which the distance* $d_q(0, A_1) = 0$ *and that does not contain the point* $(-1/2, 1/2)$. *Clearly, the optimal distance point is in this case* $(-1, 0)$.

4. Conclusions: optimal distance points for the analysis of the Pareto sets of multi-objective optimization problems

As we explained in the introductory section, the aim of this paper is to provide suitable theoretical results concerning the existence and uniqueness of solution of optimization problems using asymmetric norms for the foundation of this recently developed technique (see [4]). The applied context in which our results work can be explained as follows.

Given a Pareto set $A \subset \mathbb{R}^n$ *of a multi-objective optimization problem, a decision maker has to choose the best point in* A *satisfying his requirements. Using the information he has about the problem, he can clearly define the adequate domination relation among points in the space* $\mathbb{R}^n$. *That is, he is able to establish the directions that should be favored in the space for fixing the right domination among points —preference directions—, and the relative intensity among them, for finally choosing the best point in* A. *This point is identified in our development with the optimal distance point of* A *in the corresponding asymmetric Euclidean linear lattice.* In the classical analysis of Pareto sets the canonical basis provides these directions, but this can be adjusted in each case by the decision maker to improve the result.

In what follows we explain an explicit algorithm for solving the problem.

(1) The decision maker chooses a basis of preference directions, that is used for defining a new Euclidean quasi norm q as explained in previous sections. This q contains the information on the domination directions that are preferred by the decision maker.

(2) *The problem is then to compute the best approximation from the subset* A *to an element* $x_0 \in \mathbb{R}^n \setminus A$ *in the asymmetric Euclidean normed lattice* $(\mathbb{R}^n, \leq, q)$, where the asymmetric norm q is defined by the vectors that provide the preference directions fixed by the decision maker.

(3) As a consequence of the asymmetric nature of the topology in $(\mathbb{R}^n, \leq, q)$ the best approximation must be understood in the following sense: *to find among the points* $y \in A$ *that satisfy that* $q(y - x_0)$ *is minimum, the ones that satisfy that the Euclidean distance* $q^s(y - x_0) = \inf\{q^s(z - x_0) : z \in A\}$ *is also minimum.* Following the notation introduced in [17], these points are *optimal points of shortest distance*, or simply *optimal distance points.* We may assume also that $x_0 = 0$ by making a translation if necessary.

(4) We want also to analyze under which assumptions on A we can assert that there exists an optimal point of shortest distance, and in this case if it is unique.

About this problem, we know by Theorem 3.11 and Corollary 3.12 that:

1. If A is convex and $P_A \neq \emptyset$, then there is at most one optimal distance point, that always **exists** if A is also closed for the Euclidean topology.
2. There exists a **unique** optimal distance point from 0 to $c(A)$, the closure of the Euclidean topology of the convex hull of A.

Let us write explicitly the mathematical optimization problem using the results of the previous section.

Let $A \subset \mathbb{R}^n$ be the Pareto set of a multi-objective optimization problem and consider the asymmetric normed lattice $(\mathbb{R}^n, \leq, q)$ of our class given by a basis $\mathcal{B}$ of preference directions defined by the decision maker. We assume that A is a non-empty convex set and $d_q(0, A) > 0$; otherwise, the convex hull may be considered in case the second requirement are satisfied for the new set. Note that, in view of formula (2) explained at the beginning of the paper, $d_q(0, A) = 0$ means that the optimum point 0 —the utopia in our setting— is attained.

We have to consider two processes:

(A) Compute the set $\mathcal{N}_A$ all the points $(y_1, ..., y_n) \in A$ that satisfy that for $z = (z_1, \cdots, z_n) \in A$ we have that

$$\big((y_1, \cdots, y_n)(M^{-1})^T \vee 0\big) \cdot M^{-1} \begin{bmatrix} y_1 - z_1 \\ \vdots \\ y_n - z_n \end{bmatrix} \leq 0.$$

As a consequence of Corollary 3.7, this set is composed by the candidates for being q-nearest points to 0.

(B) Compute the elements $(y_1, ..., y_n) \in \mathcal{N}_A$ that satisfy that

$$d_{q^s}(0, N_A) = \inf\{q^s((z_1, ..., z_n)) : (z_1, ..., z_n) \in \mathcal{N}_A\} = q^s((y_1, ..., y_n)).$$

Notice that we are assuming that such elements $(y_1, ..., y_n)$ exist.

Having this in mind, we can give a clear statement for the problem to be solved with the aim of obtaining the optimal distance points of a Pareto set.

Given a Pareto set of a multi-objective optimization problem $A \neq \emptyset$, the question to be solved by using the adequate computational methods for obtaining the optimal distance points to 0 with respect to the asymmetric norm is:

Objective: To find the **infimum** of $(y_1, ..., y_n)(M^{-1})^T \cdot (M^{-1}) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

among all points $(y_1, \cdots, y_n)$ that satisfy that

$$\big((y_1, \cdots, y_n)(M^{-1})^T \vee 0\big) \cdot M^{-1} \begin{bmatrix} y_1 - z_1 \\ \vdots \\ y_n - z_n \end{bmatrix} \leq 0$$

for all $z = (z_1, \cdots, z_n) \in A$, where all the vectors are expressed in their canonical coordinates, and M is the matrix of the basis $\mathcal{B}$ defined by the preference directions chosen by the decision maker.

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