Optimization of 2D Heterogeneous Lenses via BFGS and Volume Integral Equation Method

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***Abstract*— In this paper we apply a quasi-Newton optimization algorithm called BFGS to optimize heterogeneous lenses in 2D by using the volume integral equation method. Different preliminary designs are presented with frequency selective focal point.**

I. INTRODUCTION

Volume integral equations have been widely used to solve electromagnetic problems with heterogeneous dielectrics. See [1][2]. In all cases we need a fast and efficient convolution with the free-space Green's function algorithm. In this paper we apply the recently developed method (see [3][4]) to analyze homogeneous and inhomogeneous lens problems.

II. MATHEMATICAL FORMULATION

Maxwell's equations for inhomogeneous medium in time harmonic regime are the following:

$$\begin{aligned} \nabla \times \boldsymbol{E}(\boldsymbol{x}) &= i\omega\mu(\boldsymbol{x})\boldsymbol{H}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) &= -i\omega\epsilon(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}) \end{aligned} \tag{1}$$

In general we can write the solution as a sum of two terms: $\boldsymbol{E} = \boldsymbol{E}^{\text{inc}} + \boldsymbol{E}^{\text{scat}}, \boldsymbol{H} = \boldsymbol{H}^{\text{inc}} + \boldsymbol{H}^{\text{scat}}$ the incoming field and the scattered field where the incoming field $\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}}$ verifies the free space Maxwell's equations and the scattered field $\boldsymbol{E}^{\text{scat}}, \boldsymbol{H}^{\text{scat}}$ verifies the radiation condition at infinity. This problem can be reformulated by a volume integral equation (see [4]). In this paper we will consider z-invariant geometries with TE waves and $\mu(\boldsymbol{x}) = \mu_0$. In that case, the differential equation can be rewritten as:

$$\Delta u(\boldsymbol{x}) + \omega^2\epsilon(\boldsymbol{x})\mu_0 u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^2 \tag{2}$$

and the scalar $u(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) \cdot \hat{\mathbf{z}}$

In that case, the solution can be written as:

$$u^{\text{scat}}(\boldsymbol{x}) = \int_D g_k(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})d\boldsymbol{y} \tag{3}$$

where the source σ verifies the following volume integral equation:

$$\begin{aligned} \sigma + (\omega^2\epsilon(\boldsymbol{x})\mu_0 - \omega^2\epsilon_0\mu_0) \int_D g_k(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})d\boldsymbol{y} = \\ = -(\omega^2\epsilon(\boldsymbol{x})\mu_0 - \omega^2\epsilon_0\mu_0)u^{\text{inc}}(\boldsymbol{x}) \end{aligned} \tag{4}$$

And the Green's function for 2D is $g_k(\boldsymbol{r}) = \frac{1}{4i}H_0(kr)$. Here, $r = \|\boldsymbol{r}\|_2$ and H_0 denotes the zeroth order Hankel function of the first kind.

III. WINDOWED SPECTRAL DISCRETIZATION

A critical step in all integral equation problems is the convolution with the free space Green's function:

$$u(\boldsymbol{x}) = \int_D g_k(\boldsymbol{x} - \boldsymbol{y})\,\sigma(\boldsymbol{y})\,d\boldsymbol{y}. = g_k(\boldsymbol{x}) * \sigma(\boldsymbol{x}) \tag{5}$$

Using the convolution theorem and a uniformly equispaced grid one can use the FFT to evaluate fast the convolution integral with a cost $O(Nlog(N))$.

that is

$$u(\boldsymbol{x}) = g_k(\boldsymbol{x}) * f(\boldsymbol{x}) = \mathscr{F}^{-1}\left(G_k(\boldsymbol{s}) \cdot \mathscr{F}(\sigma)\right) \tag{6}$$

where

$$G_k(\boldsymbol{s}) = \frac{1}{|\boldsymbol{s}|^2 - k^2} \tag{7}$$

$\mathscr{F}$ here denotes the Fourier transform. The principal difficulty in employing Fourier methods is the singularity $\frac{1}{|\boldsymbol{s}|^2-k^2}$ in the integrand. It is possible to compute accurately this integral using the nonuniform FFT with high order quadrature rules for the singularity (see [5] and the references therein). In this paper we use the method presented in [3] to compute fast integral convolutions for any free space Green's function in 2D and 3D for any elliptic PDE.

Let us suppose that the working space is the unit box $D \subset \mathbf{R}^d$. Then, the maximum distance between any source and target point in D is $\sqrt{2}$ in 2D and $\sqrt{3}$ in 3D. We define

$$g_k^L(\boldsymbol{r}) = g_k(\boldsymbol{r})\text{rect}\left(\frac{\text{r}}{2\text{L}}\right) \tag{8}$$

with rect(x) defined to be the characteristic function for the unit interval:

$$\text{rect(x)} = \begin{cases} 1 & \text{for } |x| < 1/2 \\ 0 & \text{for } |x| > 1/2. \end{cases}$$

If we set $L > \sqrt{d}$ in d dimensions, then the solution (5) is clearly indistinguishable from

$$u(\boldsymbol{x}) = g_k(\boldsymbol{x}) * f(\boldsymbol{x}) = g_k^L(\boldsymbol{x}) * \sigma(\boldsymbol{x}) = \mathscr{F}^{-1}\left(G_k^L(\boldsymbol{s}) \cdot \mathscr{F}(\sigma)\right) \tag{9}$$

where now $G_k^L(\boldsymbol{s})$ is a smooth function (see [3])

Since g_k^L is compactly supported, the Paley-Wiener theorem implies that its Fourier transform G_k^L is entire (and C^∞). Moreover, it is straightforward to compute analytically the functions $G_k^L(\boldsymbol{s})$ in 2D and 3D. Finally, the discretization by the trapezoidal rule on the domain $[-\frac{N}{2}, \frac{N}{2}]^d$ permits rapid evaluation using nothing more than the FFT.

IV. Numerical Examples

This analysis machinery can be used to optimize different goal functions and obtain differnet lens designs. Next we apply the method to analyze different lens problems in 2D. In all cases, the incoming wave is a plane wave with electric field z polarized.

First we have a standard inhomogeneous gradient-index dielectric flat lens.

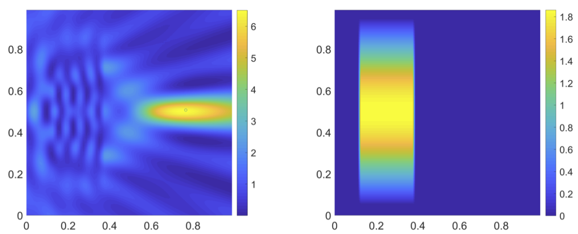

Fig. 1. Left: total field for incoming (left to right) plane wave. Right: variable index $\epsilon(\boldsymbol{x})$ of the flat lens

Next we apply the optimization method BFGS (see [6]) to obtain the optimal function $\epsilon(\boldsymbol{x})$ that maximices the amplitude of the field in the focal point:

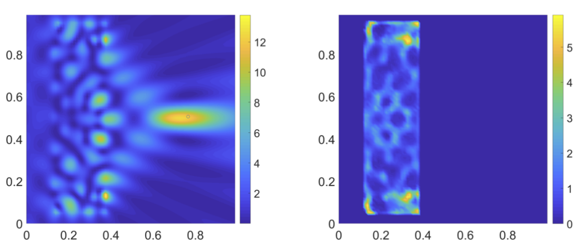

Fig. 2. Left: total field for incoming (left to right) plane wave. Right: variable index $\epsilon(\boldsymbol{x})$ of the flat lens with the resulting optimal function

Notice that the value of $|\boldsymbol{E}(\boldsymbol{x})|$ at the focal point is a factor of 2 higher in the lens optimized by the BFGS method (figure 2) than in the first case (figure 1).

In figure 3 we see a lens optimized to obtain maximum focus in the upper focal point at frequency f_1, and maximum focus in the lower focal point at frequency $f_2 = 1.1 \cdot f_1$

V. Conclusion

We apply the volume integral equation method in 2D together with the optimization method BFGS to obtain the optimal distribution $\epsilon(\boldsymbol{x})$ of a flat lens to achieve different goals, like frequency selective focal point.

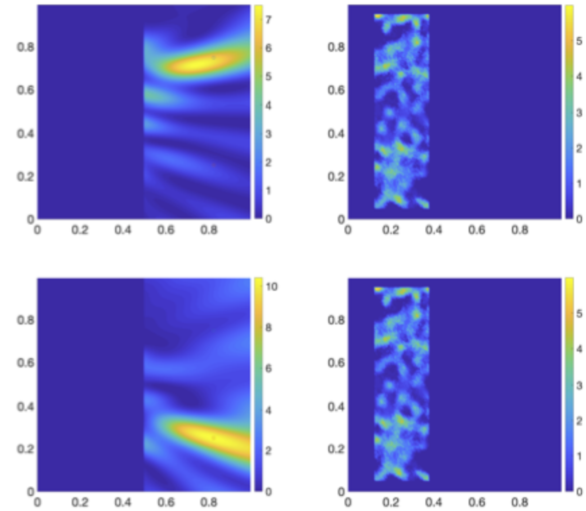

Fig. 3. Left: total field for incoming (left to right) plane wave at frequency f_1 and $f_2 = 1.1 \cdot f_1$. Right: variable index $\epsilon(\boldsymbol{x})$ of the flat lens with the resulting optimal function (same for both frequencies)

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