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Some classes of finite groups and mutually permutable products

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Abstract

This paper is devoted to the study of mutually permutable products of finite groups. A factorised group $G = AB$ is said to be a mutually permutable product of its factors A and B when each factor permutes with every subgroup of the other factor. We prove that mutually permutable products of \mathcal{Y} -groups (groups satisfying a converse of Lagrange's theorem) and SC-groups (groups whose chief factors are simple) are SC-groups, by means of a local version. Next we show that the product of pairwise mutually permutable \mathcal{Y} -groups is supersoluble. Finally, we give a local version of the result stating that when a mutually permutable product of two groups is a PST-group (that is, a group in which every subnormal subgroup permutes with all Sylow subgroups), then both factors are PST-groups.

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1 Introduction and statement of results

In this paper we will deal only with finite groups.

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Many group theorists have been worried about what can be said of a group $G = G_1G_2 \cdots G_m$ which is a product of some pairwise permutable subgroups if some properties of the factors are known. For instance, a well-known theorem of Kegel and Wielandt [25, 32] says that a product of two nilpotent groups is soluble. The fact that a product of two supersoluble groups is not necessarily supersoluble, even if both factors are normal in the group, motivates the restriction of this question to factorised groups in which both factors are connected by certain stronger permutability properties. The first author and Shaalan introduced in [5] the notion of *mutually permutable product* $G = AB$ of two subgroups A and B : in a mutually permutable product, each factor permutes with every subgroup of the other factor. In particular, this situation holds when both factors are normal in the group. Some results about normal products of supersoluble groups were extended to mutually permutable products in [5], for instance, a mutually permutable product $G = AB$ of two supersoluble groups A and B is supersoluble whenever G' is nilpotent or one of the factors is nilpotent. They also showed that totally permutable products (that is, every subgroup of each factor permutes with every subgroup of the other factor) of supersoluble groups are supersoluble. Of course, central products and direct products are instances of totally permutable products. Mutually and totally permutable products have been considered as well in [2, 11, 15, 16, 19, 20].

On the other hand, Kegel [26] proved that all subgroups of a group G which permute with all the Sylow subgroups of G are subnormal. We call these subgroups S -permutable. This motivates the definition of the class of *PST-groups* or groups in which every subnormal subgroup is *S-permutable*. Agrawal [1] obtained a characterisation of soluble PST-groups as the groups G in which the nilpotent residual L is an abelian normal Hall subgroup of G and all elements of G induce power automorphisms in L . Some interesting subclasses of the class of all PST-groups are the class of all PT-groups (groups in which permutability is a transitive relation, or in which every subnormal subgroup is permutable) and the class of all T-groups (groups in which normality is a transitive relation). These classes of groups have been studied by several authors (for instance, [3, 4, 7, 8, 10, 12, 13, 14, 17, 18, 19, 20, 21, 23, 29, 33]).

As a consequence of the theorem of Agrawal [1], soluble PST-groups are supersoluble. Robinson [29] showed that, in the general finite universe, PST-groups have all their chief factors simple, or, as he says, they are *SC-groups*. The classification of finite simple groups and the truth of the Schreier conjecture yields the following description of SC-groups:

Theorem 1 ([29, Proposition 2.4]). *A group G is an SC-group if and only if*

there is a perfect normal subgroup D such that G/D is supersoluble, $D/Z(D)$ is a direct product of G -invariant simple groups, and $Z(D)$ is supersolubly embedded in G (i.e., there is a G -admissible series of $Z(D)$ with cyclic factors).

The relation between totally and mutually permutable products and SC-groups has been investigated in [9, 11, 15, 19, 20]. For instance:

Theorem 2 ([11, Theorems 2 and 3]). *Assume that G is the mutually permutable product of its subgroups A and B . Then:*

1. *If G is an SC-group, then A and B are SC-groups.*
2. *If A and B are SC-groups, then $G/\text{Core}_G(A \cap B)$ is an SC-group.*

Now let us pay attention to the class \mathcal{Y} of all groups G in which for every subgroup H and all primes q dividing the index $|G : H|$ there exists a subgroup K of G such that H is contained in K and $|K : H| = q$. The class \mathcal{Y} has been studied in Chapter 1 and Section 6.1 of [31] and becomes a generalisation of the class of groups satisfying Lagrange's theorem. These groups can be characterised as follows:

Theorem 3. *A group G is a \mathcal{Y} -group if, and only if, the nilpotent residual L of G is a Hall subgroup of G and for all subgroups H of L , $G = LN_G(H)$.*

In [6], it is proved that the class of soluble PST-groups coincides with the class of \mathcal{Y} -groups with abelian nilpotent residual.

The theory of finite groups has benefited from the local techniques. Given a group theoretical property \mathcal{A} , we are interested in finding another weaker property \mathcal{A}_p , depending on a prime p , such that a group satisfies \mathcal{A} if and only if it satisfies \mathcal{A}_p for all primes p . For instance, p -solubility (p a prime) becomes a good "localisation" of solubility. Local techniques turn out very useful in the study of PST-groups and other related classes. For example, in [21] and [28] the authors have presented some interesting local characterisations of soluble T-groups. A local characterisation of soluble PT-groups appears in [17]. In [3, 12, 13], local characterisations of soluble PST-groups are studied. Let us recall some of these properties:

Definition 4. Let p be a prime number.

1. A p -soluble group G satisfies PST_p when every p' -perfect subnormal subgroup of G permutes with every Hall p' -subgroup of G (see [3]).
2. A group G satisfies \mathcal{U}_p^* when G is p -supersoluble and all p -chief factors of G are G -isomorphic when regarded as G -modules (see [3]).

3. A group G satisfies \mathcal{Y}_p when for every pair of p -subgroups H and K such that $H \leq K$, H is S -permutable in $N_G(K)$ ([13]).

It is shown in [3] and [13] that for p -soluble groups, all three properties are equivalent and so soluble PST-groups are exactly the groups satisfying PST_p for all primes p . Other local properties for PST-groups in the general finite universe appear in [8]:

Definition 5. Let p denote a prime number. A group G is said to satisfy N_p when every non-abelian chief factor of G of order divisible by p is simple and for each normal subgroup N of G , p' -elements of G induce power automorphisms in $O_p(G/N)$.

The paper [8] characterises PST-groups as the groups satisfying N_p for all primes p . If we fix a prime p , it is rather clear that a p -soluble group G satisfying property N_p has all p -chief factors G -isomorphic when regarded as G -modules by conjugation. Hence G is a PST_p -group. Conversely, Assume that G is a p -soluble PST_p -group. Consider a normal subgroup N of G and take a subgroup L/N of $O_p(G/N)$. By [3, Lemma 2], G/N is a PST_p -group as well. Then L/N is a subnormal p' -perfect subgroup of G/N , and so L/N permutes with all Hall p' -subgroups of G/N . Let H be a Hall p' -subgroup of G . Then HN/N is a Hall p' -subgroup of G/N and L/N is a subnormal Sylow p -subgroup of $(L/N)(HN/N)$. In particular, L/N is normalised by HN/N . This implies that all elements of H normalise L . It follows that G is an N_p -group.

Therefore we have:

Lemma 6. *Let p be a prime number. If a group G is p -soluble, then G satisfies N_p if and only if G satisfies PST_p .*

The local method has also been successfully applied to the study of \mathcal{Y} -groups in [6] with the definition of the property \mathcal{Z}_p (p a prime):

Definition 7. We say that G satisfies \mathcal{Z}_p when for every p -subgroup X of G and for every power q^m of a prime q dividing $|G : X O_{p'}(G)|$, there exists a subgroup K of G containing $X O_{p'}(G)$ such that $|K : X O_{p'}(G)| = q^m$.

In [6, Theorem 13], it is proved that property \mathcal{Z}_p is equivalent to the following one:

Theorem 8. *Let G be a group and let p be a prime. Then G satisfies \mathcal{Z}_p if and only if G satisfies either of the following conditions:*

1. G is p -nilpotent, or

2. $G(p)/O_{p'}(G(p))$ is a Sylow p -subgroup of $G/O_{p'}(G(p))$ and for every p -subgroup H of $G(p)$, we have that $G = N_G(H)G(p)$.

Here $X(p)$ denotes the p -nilpotent residual of a group X , that is, the smallest normal subgroup N of X such that X/N is p -nilpotent.

Theorem 9 ([6, Theorem 15]). *A soluble group satisfies \mathcal{Y} if and only if it satisfies \mathcal{Z}_p for all primes p .*

In this paper we prove some results on mutually permutable products whose factors belong to some of the above classes. We start with a localisation of SC-groups.

Definition 10. Let p be a prime number. A group G is said to be an SC_p -group whenever every chief factor of G whose order is divisible by p is simple.

It is clear that G is an SC-group (i.e., all its chief factors are simple) if and only if G is an SC_p -group for all primes p . In what follows, p will denote a fixed prime number. The proofs of Theorem 2 can be adapted to prove:

Lemma 11. *Assume that G is a mutually permutable product of its subgroups A and B .*

1. *If G is an SC_p -group, then A and B are SC_p -groups.*
2. *If A and B are SC_p -groups, then $G/\text{Core}_G(A \cap B)$ is an SC_p -group.*

Mutually permutable products of SC_p -groups and p -soluble \mathcal{Z}_p -groups are the object of the next result:

Theorem 12. *Let $G = AB$ be a mutually permutable product of its subgroups A and B . Assume that A is an SC_p -group and that B is a p -soluble \mathcal{Z}_p -group. Then G is an SC_p -group.*

The following corollaries follow immediately from Theorem 12:

Corollary 13. *If G is a mutually permutable product of an SC-group A and a \mathcal{Y} -group B , then G is an SC-group. In particular, if G is a mutually permutable product of a supersoluble group A and a \mathcal{Y} -group B , then G is supersoluble.*

Let \mathfrak{X} be a class of groups. A class of groups \mathfrak{F} is called the *Fitting core* of \mathfrak{X} provided that whenever if $A \in \mathfrak{X}$ and $B \in \mathfrak{F}$, and A and B are normal subgroups of a group G , then $AB \in \mathfrak{X}$ (see [18]). From Corollary 2 of [18]

it follows that the class of soluble PST-groups belongs to the Fitting core of the formation of supersoluble groups. In fact from Corollary 13 we obtain a more general statement, mainly: the class \mathcal{Y} is contained in the Fitting core of both the formation of supersoluble groups and hence the formation of SC-groups.

Corollary 14. *If G is a mutually permutable product of two p -soluble \mathcal{Z}_p -groups, then G is p -supersoluble.*

Corollary 15. *If G is a mutually permutable product of two \mathcal{Y} -groups, then G is supersoluble.*

Corollary 15 admits the following generalisation:

Theorem 16. *Let $G = G_1G_2 \cdots G_r$ be a group such that G_1, G_2, \dots, G_r are pairwise mutually permutable subgroups of G . If all G_i are \mathcal{Y} -groups, then G is supersoluble.*

We do not know whether a local version of Theorem 16 is true, namely, if all G_i are \mathcal{Z}_p -groups, then G is p -supersoluble.

In [20, Theorem 5], the following result is proved:

Theorem 17. *Let $G = AB$ be a mutually permutable product of the subgroups A and B . If G is a PST-group, then A is a PST-group.*

We present in this paper a local version of Theorem 17, from which it follows immediately:

Theorem 18. *Let G be a mutually permutable product of its subgroups A and B . If G is a SC-group and satisfies N_p , then A satisfies N_p .*

2 Proofs

Proof of Theorem 12. Assume that G is a counterexample of least order to the result. Since the class of SC_p -groups is a formation, then G has a unique minimal normal subgroup N . By Lemma 11, N is a non-cyclic p -subgroup contained in $A \cap B$. The minimal choice of G implies that, G/N is an SC_p -group.

Set $C = C_G(N)$. Then G/C is a mutually permutable product of its subgroups AC/C and BC/C . Since A is an SC_p -group, N has an A -composition series with cyclic factors. By [22, IV, 6.9], it follows that AC/C is p -supersoluble. The same argument shows that BC/C is p -supersoluble. Since G/C is p -soluble by [19, Corollary 2] and an SC_p -group by minimality of G ,

we conclude that G/C is p -supersoluble. Hence its derived subgroup $G'C/C$ is p -nilpotent by a result of Shemetkov [30]; in particular, G'/N is p -nilpotent.

Assume that N is contained in the Frattini subgroup $\Phi(G)$ of G . By [24, 6.6.3], G' is p -nilpotent. Since N is the unique minimal normal of G and is a p -group, it follows that G' is a p -group. Therefore G has a normal Sylow p -subgroup G . This implies that G is p -soluble. But since A and B are p -soluble SC_p -groups, A and B are p -supersoluble. Consequently, G is p -supersoluble by [19, Corollary 5]. This contradicts the choice of G .

Hence N is not contained in $\Phi(G)$. Then G is a primitive group and N is a self-centralising minimal normal subgroup of G . Moreover G/N is p -supersoluble. This implies that A and B are p -soluble. Since A and B are SC_p -groups, we have that A and B are p -supersoluble. By [19, Theorem 4], $K = \text{O}_{p',p}(G) = \text{O}_p(G) = N$ satisfies that G/K is a p' -group. Hence N is a normal Sylow p -subgroup of G . In particular, G is p -soluble and so A and B are p -supersoluble.

By hypothesis, B is a \mathcal{Z}_p -group. Then Theorem 8 shows that either B is p -nilpotent, or $B(p)/\text{O}_{p'}(B(p))$ is a Sylow p -subgroup of $B/\text{O}_{p'}(B(p))$ and for every p -subgroup H of $B(p)$, we have that $B = B(p)N_B(H)$, where $B(p)$ denotes the p -nilpotent residual of B . In the first case, G is p -supersoluble by [19, Corollary 5]. In the second case, $\text{O}_{p'}(B(p)) = 1$ because $\text{C}_G(N) = N$. Hence $B(p)$ is a p -group and so $B(p) = N$. Let N_1 be a minimal normal subgroup of A contained in N ; N_1 is cyclic because A is p -supersoluble. Since N_1 is normalised by N and by a Hall p' -subgroup of B , we have that N_1 is normalised by B . It follows that N_1 is normal in G . This contradiction proves the theorem. \square

The next result is needed in the proof of Theorem 16.

Lemma 19. *Let G be a \mathcal{Y} -group with an abelian normal Sylow p -subgroup P for a prime p . Then every subgroup of P is normal in G .*

Proof. Applying Theorem 3, the nilpotent residual N of G is a nilpotent Hall subgroup of G . Therefore there exists a nilpotent subgroup K of G such that $G = NK$ and $\gcd(|N|, |K|) = 1$ by [22, A, 11.3]. Assume that P is a subgroup of N and let H be a subgroup of P . Then $G = NN_G(H)$ by Theorem 3. Since N is nilpotent and H is normal in P , it follows that N normalises H and so $G = N_G(H)$.

Suppose that P is a subgroup of K . Then, if H is a subgroup of P , we have that H is normalised by K . Since H is subnormal in G , it follows that H is a subnormal Sylow p -subgroup of HN . This implies that N normalises H and hence H is normal in $G = NK$. \square

Proof of Theorem 16. Assume that the theorem is false and let G be a counterexample with $|G| + |G_1| + |G_2| + \cdots + |G_r|$ minimal. We shall show that this supposition leads to a contradiction. Clearly $r > 2$ by Corollary 15. Besides, the hypotheses of the theorem are inherited by every epimorphic image of G . The minimal choice of G implies that every proper quotient of G is supersoluble. Since the class of all supersoluble groups is a saturated formation, we have that $\text{Soc}(G)$ is a supplemented minimal normal subgroup of G such that $G/\text{Soc}(G)$ is supersoluble.

Let p denote the largest prime dividing $|G|$. Then there exists $l \in \{1, 2, \dots, r\}$ such that G_l contains a non-trivial Sylow p -subgroup, P_l say. Let $i \in \{1, \dots, r\}$, $i \neq l$. Then $G_l G_i$ is supersoluble by Corollary 15. Hence $G_l G_i$ has a normal Sylow p -subgroup. In particular, P_l is subnormal in $G_l G_i$. Applying [27, 7.7.1], P_l is subnormal in G and so $O_p(G) \neq 1$. This implies that $\text{Soc}(G) = F(G) = O_p(G)$ is an abelian minimal normal subgroup of G . Since $G/O_p(G)$ is supersoluble, it follows that G has a normal Sylow p -subgroup. Consequently $P = F(G)$ is the unique Sylow p -subgroup of G .

Fix an index $i \in \{1, \dots, r\}$, $i \neq l$. Then $P \cap G_i$ is a normal Sylow p -subgroup of G_i and so $G_i = (P \cap G_i)L_i$ for each Hall p' -subgroup L_i of G_i . Moreover, the product $G_l G_i$ is supersoluble and mutually permutable. Consequently $G_l L_i$ is a supersoluble subgroup of G . This means that $P \cap G_l$ is normalised by L_i . Since $P \cap G_l$ is normal in G_l , we have that $P \cap G_l$ is normalised by a Hall p' -subgroup K of G . Since P is abelian and $G = PK$, we can conclude that $P \cap G_l$ is normal in G and $P \cap G_l = P$.

Let K_l be the subgroup generated by all G_i 's, $i \neq l$. Assume that p divides $|K_l|$. Then, since K_l is supersoluble, by the minimal choice of G , we have that K_l contains a normal subgroup L of order p . By Lemma 19, L is also normalised by G_l . Consequently L is normal in $G = K_l G_l$, contrary to our supposition. Consequently K_l is a p' -group. On the other hand, consider an index $j \in \{1, \dots, r\}$, $j \neq l$ and the subgroup Z generated by all G_m 's, $m \neq j$. The minimal choice of G implies that Z is supersoluble. Besides, P is a subgroup of Z . Let N be a minimal normal subgroup of Z contained in P . Then $|N| = p$ and NG_m is a subgroup of G as $N \leq G_l$. Since NG_m is supersoluble and G_m is a p' -group, it follows that N is normalised by G_m . Consequently N is normal in G and G is supersoluble. This final contradiction proves the theorem. \square

Our proof of Theorem 18 depends on the following:

Lemma 20. *Let N be a normal subgroup of G such that G/N satisfies N_p . If either N is non-abelian and simple or N is a p' -group, then G satisfies N_p .*

Proof. We proceed by induction on $|G|$. The hypotheses imply that every chief factor of G whose order is divisible by p is simple. Clearly we may assume $N \neq 1$. Let g be a p' -element of G and let H be a subgroup of $O_p(G)$. Then HN/N is a subgroup of $O_p(G/N)$ and so gN normalises HN/N . Hence $HN = H^gN$. Since N is either non-abelian simple or N is a p' -group, $H^g = O_p(H^gN) = O_p(HN) = H$. Therefore g normalises H : This means that p' -elements of G induce power automorphisms in $O_p(G)$.

Let T be a minimal normal subgroup of G . If $T = N$, then G/T satisfies N_p . Suppose that $T \neq N$ and T is contained in N . Then N/T is a non-trivial normal subgroup of G/T and N/T is a p' -group. By induction, G/T satisfies N_p . Assume that T is not contained in N . Then NT/T is a non-trivial normal subgroup of G/T which is G -isomorphic to N . Besides $(G/T)/(NT/T) \cong G/NT$ satisfies N_p . The induction hypothesis implies that G/T satisfies N_p .

Now if R is a non-trivial normal subgroup of G , there exists a minimal normal subgroup T of G such that $T \leq R$. Since G/T satisfies N_p , we conclude that G/R satisfies N_p . The induction argument is therefore complete. \square

Proof of Theorem 18. Suppose, for a contradiction, that the result is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G . Then G/N is the mutually permutable product of AN/N and BN/N . In addition, G/N satisfies N_p . The minimal choice of G implies that $AN/N \cong A/(A \cap N)$ satisfies N_p . In particular, $A \cap N \neq 1$. Applying [19, Lemma 1], we conclude that N is contained in A . Since A does not satisfy property N_p , it follows that N is of order p . In particular, $O_{p'}(G) = 1$ and $\text{Soc}(G)$ is contained in A . Since G is an SC-group, we can apply Theorem 1 to conclude that the soluble residual D of G satisfies that $D/Z(D)$ is a direct product of minimal normal subgroups of $G/Z(D)$. Moreover, if S is the soluble radical of G , we have that $C_G(D) = S$ by [29, Lemma 2.6]. This implies that $Z(D)$ is contained in $Z(S)$.

On the other hand, S is an N_p -group. Hence, by Lemma 6, S is a PST_p -group. Moreover $O_{p'}(S) \leq O_{p'}(G) = 1$. Applying [12, Theorem A], we conclude that either S is a p -group or the p -nilpotent residual $S(p)$ of S is an abelian Sylow p -subgroup of S on which S induces a group of power automorphisms. In any case, $O_p(G)$ is the Sylow p -subgroup of S . Applying [11, Corollary 1], the soluble residual A^\ominus of A is a normal subgroup of G .

Assume that S is a p -group. Since A is not an N_p -group, there exists a normal subgroup K of A and a subgroup H/K of $O_p(A/K)$ such that H/K is not S -permutable in A/K . Let us choose H of the smallest possible order. It is rather clear that H is subnormal in A . Hence the soluble residual H^\ominus of H is subnormal in A as well, and so is in G because H^\ominus is contained in A^\ominus . Since $D/Z(D)$ is a direct product of non-abelian minimal normal subgroups

of $G/Z(D)$, which are simple, and $H^\mathfrak{S}Z(D)/Z(D)$ is subnormal in $D/Z(D)$, it follows that $H^\mathfrak{S}Z(D)$ is normal in G . Since $(H^\mathfrak{S}Z(D))' = (H^\mathfrak{S})' = H^\mathfrak{S}$, we conclude that $H^\mathfrak{S}$ is normal in G . If $H^\mathfrak{S} \neq 1$, then $A/H^\mathfrak{S}$ satisfies N_p by minimality of G . It implies that A/K satisfies N_p as $H^\mathfrak{S}$ is a subgroup of K . This contradicts our choice of the pair (H, K) . Consequently $H^\mathfrak{S} = 1$ and H is soluble. In this case, H is contained in the soluble radical of A , which is a subgroup of S by [20, Theorem 4]. Since G has N_p and $H \leq O_p(G)$, it follows that H is S -permutable in G and so is in A . This contradiction implies that S is not a p -group. Then $O_p(G) = S(p)$ is the abelian Sylow p -subgroup of S and it does not contain central chief factors by [22, IV, 5.18] and [22, V, 3.2, 4.2]. In particular, $Z(D) = Z(S) = 1$. Since all minimal normal subgroups of G have order p , it follows that $D = 1$. Hence G is soluble and $G = S$. Applying [3, Corollary 2] and Lemma 6, A satisfies N_p . This final contradiction completes the proof of the theorem. \square

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