mathematics

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A Survey on Valdivia Open Question on Nikodým Sets

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Abstract: Let $\mathcal{A}$ be an algebra of subsets of a set Ω and $ba(\mathcal{A})$ the Banach space of bounded finitely additive scalar-valued measures on $\mathcal{A}$ endowed with the variation norm. A subset $\mathcal{B}$ of $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$ if each countable $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is norm bounded. A subset $\mathcal{B}$ of $\mathcal{A}$ is a Grothendieck set for $ba(\mathcal{A})$ if for each bounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $ba(\mathcal{A})$ the $\mathcal{B}$-pointwise convergence on $ba(\mathcal{A})$ implies its $ba(\mathcal{A})^*$-pointwise convergence on $ba(\mathcal{A})$. A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ is a strong-Nikodým (Grothendieck) set for $ba(\mathcal{A})$ if in each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ there exists $\mathcal{B}_m$ which is a Nikodým (Grothendieck) set for $ba(\mathcal{A})$. The answer of the following open question for an algebra $\mathcal{A}$ of subsets of a set Ω, proposed by Valdivia in 2013, has not yet been found: Is it true that if $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$ then $\mathcal{A}$ is a strong Nikodým set for $ba(\mathcal{A})$? In this paper we surveyed some results related to this Valdivia's open question, as well as the corresponding problem for strong Grothendieck sets. The new Propositions 1 and 3 provide more simplified proofs, particularly in their application to Theorems 1 and 2, which were the main results surveyed. Moreover, the proofs of almost all other propositions are wholly or partially original.

Keywords: Grothendieck set; Nikodým set; strong Grothendieck set; strong Nikodým set; algebra of subsets; bounded scalar measure; σ-algebra; variation norm

MSC: 28A33; 46B25

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Citation: López-Alfonso, S.; López-Pellicer, M.; Moll-López, S.; Sánchez-Ruiz, L.M. A Survey on Valdivia Open Question on Nikodým Sets. *Mathematics* **2022**, *10*, 2660. https://doi.org/10.3390/math10152660

Academic Editor: Rekha Srivastava

Received: 4 July 2022
Accepted: 25 July 2022
Published: 28 July 2022

1. Introduction

Let $\mathcal{B}$ be a subset of an algebra $\mathcal{A}$ of subsets of a set Ω and let $L(\mathcal{B})$ be the real or complex normed space generated by the characteristics functions e_B with $B \in \mathcal{B}$, endowed with the supremum norm, denoted by $\|\cdot\|_\infty$. In what follows, dual means topological dual and the dual of a normed space E endowed with the dual norm is denoted as E^*. In particular, the dual $L(\mathcal{A})^*$ is isometric to the real or complex Banach space $ba(\mathcal{A})$ formed by the bounded finitely additive scalar measures defined on $\mathcal{A}$ provided with the variation norm, denoted by $|\cdot|$. The variation norm is equivalent to the supremum norm, i.e., $\sup\{|\mu(A)| : A \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$. This equivalence follows easily from [1] (Propositions 1 and 2). We identified $L(\mathcal{A})^*$ and $ba(\mathcal{A})$ and then $\mu(e_A) = \mu(A)$, for each $\mu \in L(\mathcal{A})^* = ba(\mathcal{A})$ and each $A \in \mathcal{A}$. We also identified its duals $L(\mathcal{A})^{**}$ and $ba(\mathcal{A})^*$.

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a Nikodým set for the Banach space $ba(\mathcal{A})$ if every $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is a $|\cdot|$-bounded subset of $ba(\mathcal{A})$, where this $|\cdot|$-boundedness is equivalent to the uniform boundedness of M in $\mathcal{A}$. If $\mathcal{A}$ is a σ-algebra, then $\mathcal{A}$ is a Nikodým set for the Banach space $ba(\mathcal{A})$, and this property is the famous Nikodým–Grothendieck uniform boundedness theorem for the scalar bounded additives measures defined on $\mathcal{A}$. This theorem is a good test for uniform boundedness in $ba(\Sigma)$ with many applications in Banach spaces and in Measure theory [2] (Chapter VII, Nikodým-Grothendieck Boundedness Theorem).

A first significant improvement of this theorem was obtained by M. Valdivia in [1] (Theorem 2), who found that if $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is an increasing covering of a σ-algebra $\mathcal{A}$ there exists an $\mathcal{A}_p$ which is a Nikodým set for the Banach space $ba(\Sigma)$. A family $\{\mathcal{A}_\sigma : \sigma \in \mathbb{N}^p, p \in \mathbb{N}\}$ of subsets of $\mathcal{A}$ is defined as an increasing web in $\mathcal{A}$ if $\{\mathcal{A}_{n_1} : n_1 \in \mathbb{N}\}$ is an increasing covering of $\mathcal{A}$ and for each $(n_1, \cdots, n_p) \in \mathbb{N}^p$, $p \in \mathbb{N}$, the countable family of sets $\{\mathcal{A}_{n_1,\cdots,n_p n_{p+1}} : n_{p+1} \in \mathbb{N}\}$ is an increasing covering of $\mathcal{A}_{n_1,\cdots,n_p}$. The family $(\sigma_n : \sigma_n \in \mathbb{N}^n, n \in \mathbb{N})$ is a chain if there exists a sequence of natural numbers $(m_p)_{p=1}^{\infty}$ such that $\sigma_n = (m_1, \cdots, m_n)$, for each $n \in \mathbb{N}$. For each increasing web $\{\mathcal{A}_\sigma : \sigma \in \mathbb{N}^p, p \in \mathbb{N}\}$ in a σ-algebra $\mathcal{A}$ there exists a chain $(\sigma_n : \sigma_n \in \mathbb{N}^n, n \in \mathbb{N})$ such that each $\mathcal{A}_{\sigma_n}$, $n \in \mathbb{N}$, is a Nikodým set for $ba(\mathcal{A})$. This result was obtained in [3] (Theorem 2.7) by means of locally convex topological spaces theory.

Motivated by these results, a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a *strong Nikodým set* for $ba(\mathcal{A})$ if for each increasing covering $\mathcal{B}_n : n \in \mathbb{N}$ of $\mathcal{B}$ there exists a $\mathcal{B}_p$ which is a Nikodým set for $ba(\mathcal{A})$. Moreover, if for each web $\{\mathcal{B}_\sigma : \sigma \in \mathbb{N}^p, p \in \mathbb{N}\}$ in $\mathcal{B}$ there exists a chain $(\sigma_n : n \in \mathbb{N})$ such that each $\mathcal{B}_{\sigma_n}$, $n \in \mathbb{N}$, is a Nikodým set for the space $ba(\mathcal{A})$ then $\mathcal{B}$ is called a *web Nikodým set* for the space $ba(\mathcal{A})$. Clearly, a web Nikodým set implies a strong Nikodým set and a strong Nikodým set implies Nikodým set. For a σ-algebra $\mathcal{A}$ the set $\mathcal{A}$ is a web Nikodým set for $ba(\mathcal{A})$ [3] (Theorem 3.1).

A Banach space E is a Grothendieck space if its dual and bidual, E^* and E^{**}, verify that for every sequence of E^* the E-pointwise convergence to 0 implies its E^{**}-pointwise convergence to 0. The current interest in Grothendieck spaces is motivated by interesting characterizations and several open questions. For instance:

1. A Banach space E is a Grothendieck space if and only if every continuous linear operator $T : E \to c_0$ is weakly compact.
2. Is E reflexive, if E and E^* are Grothendieck spaces?
3. If E is Grothendieck, is E^{**} a Grothendieck space?

It is said that an algebra $\mathcal{A}$ of subsets of a set Ω has the Grothendieck property if the completion $L_\infty(\mathcal{A})$ of $L(\mathcal{A})$ is a Grothendieck space. This is equivalent to the property that the $\mathcal{A}$-pointwise convergence to 0 of a bounded sequence of $L(\mathcal{A})^*$ implies its $L(\mathcal{A})^{**}$-pointwise convergence to 0. This characterization motivates that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is called a *Grothendieck set* for $ba(\mathcal{A})$ if the $\mathcal{B}$-pointwise convergence to 0 of a bounded sequence of $L(\mathcal{A})^* = ba(\mathcal{A})$ implies its $ba(A)^*$-pointwise convergence to 0.

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a *strong Grothendieck set* for $ba(\mathcal{A})$ if for each increasing covering $(\mathcal{B}_n : n \in \mathbb{N})$ of $\mathcal{B}$ there exists a $\mathcal{B}_p$ which is a Grothendieck set for $ba(\mathcal{A})$. The property that for every σ-algebra $\mathcal{A}$ the set $\mathcal{A}$ is a Grothendieck set for $ba(\mathcal{A})$ follows from [4] (Introduction).

2. Strong Nikodým Sets

Let $\mathcal{B}$ be a subset of of an algebra $\mathcal{A}$ of subsets of a set Ω. If $L(\mathcal{B})$ is a dense subset of $L(\mathcal{A})$, then there is a natural isometry between $L(\mathcal{B})^*$ and $L(\mathcal{A})^*$ that enables us to identify $L(\mathcal{B})^*$ and $L(\mathcal{A})^*$.

Proposition 1. *A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a Nikodým set for the Banach space $ba(\mathcal{A})$ if and only if the following two conditions hold:*

1. $\overline{L(\mathcal{B})} = L(\mathcal{A})$.
2. *Every $\mathcal{B}$-pointwise bounded subset M of $L(\mathcal{A})^*$ is a bounded subset of $L(\mathcal{B})^*$.*

Proof. Suppose that $\mathcal{B}$ is a Nikodým set for the Banach space $ba(\mathcal{A})$. As the orthogonal set of $L(\mathcal{B})$, named $L(\mathcal{B})^\perp$, is a bounded linear subspace of $L(\mathcal{A})^*$ it follows that $L(\mathcal{B})^\perp = \{0\}$. Therefore, $\overline{L(\mathcal{B})} = L(\mathcal{B})^{\perp\perp} = \{0\}^\perp = L(\mathcal{A})$. Moreover, if M is a $\mathcal{B}$-pointwise bounded subset of $L(\mathcal{A})^*$ then M is a norm bounded subset of $L(\mathcal{A})^*$. By density $L(\mathcal{A})^* = L(\mathcal{B})^*$, hence M is a norm bounded subset of $L(\mathcal{B})^*$.

Conversely, if conditions (1) and (2) are verified, then from (1) it follows that $L(\mathcal{B})^* = L(\mathcal{A})^*$ and this equality and (2) imply that every $\mathcal{B}$-pointwise bounded subset M of $L(\mathcal{A})^*$ is a norm bounded subset of $L(\mathcal{A})^*$. Hence, $\mathcal{B}$ is a Nikodým set for the Banach space $ba(\mathcal{A})$. $\square$

In the following, we abbreviated the norm bounded by bounded.

Proposition 2. *Suppose that an algebra $\mathcal{A}$ of subsets of a set Ω contains a subset $\mathcal{B}$ that it is a Nikodým set for the Banach space $ba(\mathcal{A})$, and let $\{F_n : n \in \mathbb{N}\}$ be an increasing covering of $L(\mathcal{B})$ formed by closed absolutely convex subsets of $L(\mathcal{B})$. Then, there exists n_0 such that each F_{n_0} is a neighborhood of zero in $L(\mathcal{B})$.*

Proof. As $L(\mathcal{B})$ is dense in the completion $\widehat{L(\mathcal{A})}$ of $L(\mathcal{A})$, then we identify the dual of $\widehat{L(\mathcal{A})}$ with $L(\mathcal{B})^*$, hence $L(\mathcal{B})^* = L(\mathcal{A})^* = \widehat{L(\mathcal{A})}^*$. Let G_n be the closure of F_n in the Banach space $\widehat{L(\mathcal{A})}$, for each $n \in \mathbb{N}$. According to the Baire category theorem, it is enough to prove that the increasing family $\{G_n : n \in \mathbb{N}\}$ covers $\widehat{L(\mathcal{A})}$. If there exists $f \in \widehat{L(\mathcal{A})} \setminus \bigcup \{G_n : n \in \mathbb{N}\}$ then, by the Hahn-Banach theorem, there exists $\mu_n \in L(\mathcal{A})^*$ such that $\mu_n(G_n) = \{0\}$ and $\mu_n(f) = n$, for each $n \in \mathbb{N}$. Then, the set $\{\mu_n : n \in \mathbb{N}\}$ is an unbounded subset of $\widehat{L(\mathcal{A})}^* = L(\mathcal{A})^*$ that is a $\mathcal{B}$-pointwise bounded subset of $L(\mathcal{A})^*$. This contradicts the hypothesis that $\mathcal{B}$ is a Nikodým set for the Banach space $ba(\mathcal{A})(= L(\mathcal{A})^*)$. $\square$

Note that this Proposition 2 is found in [5], where Baire-like spaces were introduced. It also follows from [6] (Proposition 1.2.1).

The next corollary is a particular case of Proposition 2.

Corollary 1. *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be an increasing covering of a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω. If $\mathcal{B}$ is a Nikodým set for the Banach space $ba(\mathcal{A})$, then there exists $m \in \mathbb{N}$ such that the closure in $L(\mathcal{A})$ of the absolutely convex hull of $\{me_B : B \in \mathcal{B}_m\}$ contains $\{e_A : A \in \mathcal{A}\}$. Hence, $\overline{L(\mathcal{B}_m)} = L(\mathcal{A})$.*

Proof. By applying Proposition 2 with F_n equal to the closure in $L(\mathcal{B})$ of the absolutely convex hull of $\{ne_B : B \in \mathcal{B}_n\}$, we deduced that there exists n_0 that the closure in $L(\mathcal{A})$ of the absolutely convex hull of $\{n_0 e_B : B \in \mathcal{B}_{n_0}\}$ is a neighborhood of zero in $L(\mathcal{A})$. Hence, there exists $m > n_0$ such that the closure in $L(\mathcal{A})$ of the absolutely convex hull of $\{me_B : B \in \mathcal{B}_m\}$ contains the closed unit ball of $L(\mathcal{A})$. $\square$

In the next proposition, we consider an algebra $\mathcal{A}$ of subsets of a set Ω that contains a set $\mathcal{B}$ such that $\overline{L(\mathcal{B})} = L(\mathcal{A})$ and $\mathcal{B}$ is not a Nikodým set for the Banach space $ba(\mathcal{A})$. If $\langle E, F \rangle$ is a dual pair of topological vector spaces and A is a subset of E, then the polar set of A in F is the set $A^{\circ} := \{f \in F : |f(a)| \leq 1, \text{ for each } a \in A\}$.

Proposition 3. *Let $\mathcal{B}$ be a subset of an algebra $\mathcal{A}$ such that $\mathcal{B}$ is not a Nikodým set for the Banach space $ba(\mathcal{A})$ and suppose that $\overline{L(\mathcal{B})} = L(\mathcal{A})$. Then, there exists an absolutely convex and weakly* closed $\mathcal{B}$-pointwise bounded subset M in $L(\mathcal{A})^*$ such that for each finite subset $\{Q_i : 1 \leq i \leq p\}$ of $\mathcal{A}$ the set $M \cap \{e_{Q_i} : 1 \leq i \leq p\}^{\circ}$ is unbounded.*

Proof. If $\mathcal{B}$ is a subset of an algebra $\mathcal{A}$ such that $\mathcal{B}$ is not a Nikodým set for the Banach space $ba(\mathcal{A})$ and $\overline{L(\mathcal{B})} = L(\mathcal{A})$ then, by condition (2) in Proposition 1, there exists a $\mathcal{B}$-pointwise bounded subset P of $L(\mathcal{A})^*$ such that P is an unbounded subset of $L(\mathcal{B})^*$. The $\mathcal{B}$-pointwise boundedness of P implies that $L(\mathcal{B}) \subset \cup\{nP^{\circ} : n \in \mathbb{N}\}$, from the norm unboundedness of P it follows that its polar set P° in $L(\mathcal{A})$ does not contain a neighborhood of zero in $L(\mathcal{A})$, and from $\overline{L(\mathcal{B})} = L(\mathcal{A})$ it follows that P° does not contain a neighborhood of zero in $L(\mathcal{B})$.

Let $Q_1 \in \mathcal{A}$. If $e_{Q_1} \notin \cup\{nP^{\circ} : n \in \mathbb{N}\}$ then, the absolutely convex hull $\Gamma\{P^{\circ} \cup e_{Q_1}\}$ does not contain a zero neighborhood of $L(\mathcal{B})$. If $e_{Q_1} \in \cup\{nP^{\circ} : n \in \mathbb{N}\}$ then there exists

$m \in \mathbb{N}$ such that $e_{Q_1} \in mP^{\mathrm{o}}$, hence $\Gamma\{P^{\mathrm{o}} \cup e_{Q_1}\} \subset mP^{\mathrm{o}}$. The fact that P^{o} does not contain a neighborhood of zero in $L(\mathcal{B})$ implies that $\Gamma\{P^{\mathrm{o}} \cup e_{Q_1}\}$ does not contain a neighborhood of $L(\mathcal{B})$. In summary, if $Q_1 \in \mathcal{A}$ then $P_1 := \Gamma\{P^{\mathrm{o}} \cup e_{Q_1}\}$ does not contain a zero neighborhood of $L(\mathcal{B})$.

Analogously, if $Q_2 \in \mathcal{A}$ then, if $e_{Q_2} \notin \cup\{nP_1 : n \in \mathbb{N}\}$, the absolutely convex hull $\Gamma\{P_1 \cup e_{Q_2}\}$ does not contain a neighborhood of $L(\mathcal{B})$. If $e_{Q_2} \in \cup\{nP_1 : n \in \mathbb{N}\}$ then there exists $m \in \mathbb{N}$ such that $e_{Q_2} \in mP_1$, hence $\Gamma\{P_1 \cup e_{Q_2}\} \subset mP_1$. The fact that P_1 does not contain a neighborhood of zero in $L(\mathcal{B})$ implies that $\Gamma\{P_1 \cup e_{Q_2}\}$ does not contain a neighborhood of $L(\mathcal{B})$. We then obtained that if $\{Q_1, Q_2\} \subset \mathcal{A}$ the set $\Gamma\{P_1 \cup e_{Q_2}\} = \Gamma\{P^{\mathrm{o}} \cup e_{Q_1} \cup e_{Q_2}\}$ does not contain a neighborhood of $L(\mathcal{B})$.

Repeating this process, we deduced that for each finite subset $\{Q_i : 1 \le i \le p\}$ of $\mathcal{A}$ the set $\Gamma\{P^{\mathrm{o}} \cup \{e_{Q_i} : 1 \le i \le p\}$ does not contain a neighborhood of $L(\mathcal{A})$. Hence, the polar set of $\Gamma\{P^{\mathrm{o}} \cup \{e_{Q_i} : 1 \le i \le p\}$ in $L(\mathcal{A})^*$, given by the equality

$$\left(\Gamma\{P^{\mathrm{o}} \cup \{e_{Q_i} : 1 \le i \le p\}\right)^{\mathrm{o}} = P^{\mathrm{oo}} \cap \{e_{Q_i} : 1 \le i \le p\}^{\mathrm{o}}$$

is not a bounded subset of $L(\mathcal{A})^*$. Finally, the set $M := P^{\mathrm{oo}}$ verified this proposition. $\square$

The unbounded set $M \bigcap \{e_{Q_i} : 1 \le i \le p\}^{\mathrm{o}}$ obtained in Proposition 3 verifies the equality

$$\sup\{|\mu| : \mu \in M \cap \{e_{Q_i} : 1 \le i \le p\}^{\mathrm{o}}\} = \infty \tag{1}$$

Recall that the variation $|\cdot|$ and supremum norms are equivalent in $L(\mathcal{A})^*$, hence equality (1) is equivalent to the equality

$$\sup\left\{|\mu(B)| : B \in \mathcal{A}, \mu \in M \cap \{e_{Q_i} : 1 \le i \le p\}^{\mathrm{o}}\right\} = \infty. \tag{2}$$

We required the following definition, motivated by (2).

Definition 1. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω and let $A \in \mathcal{A}$. A subset M of $L(\mathcal{A})^*$ is quasi-A-bounded if there exists a finite subset $\{Q_i : 1 \le i \le p\}$ of $\mathcal{A}$ such that*

$$\sup\left\{|\mu(B)| : B \subset A,\ B \in \mathcal{A},\ \mu \in M \cap \{e_{Q_i} : 1 \le i \le p\}^{\mathrm{o}}\right\} < \infty. \tag{3}$$

Clearly, Proposition 3 states that, if $\mathcal{B}$ is a subset of an algebra $\mathcal{A}$ such that $\mathcal{B}$ is not a Nikodým set for the Banach space $ba(\mathcal{A})$ and $\overline{L(\mathcal{B})} = L(\mathcal{A})$, then there exists an absolutely convex and weakly* closed $\mathcal{B}$-pointwise bounded subset M in $L(\mathcal{A})^*$ such that M is non quasi-Ω-bounded.

The proof of the next Lemma follows with the direct application of (3) to each A_i, $1 \le i \le p$.

Lemma 1. *Let us suppose that $\{A_1, A_2, \cdots, A_p\}$ is a subset of an algebra $\mathcal{A}$ of subsets of a set Ω whose union is A. Let M be a subset of $L(\mathcal{A})^*$ such that M is quasi-A_i-bounded for $1 \le i \le p$. Then M is quasi-A-bounded.*

Proposition 4. *Let us suppose that $\mathcal{A}$ is an algebra of subsets of a set Ω, $A \in \mathcal{A}$ and that M is an absolutely convex non quasi-A-bounded subset of $L(\mathcal{A})^*$. Then, for each positive natural number s and for each finite family $\{Q_j : 1 \le j \le r\}$, with $Q_j \in \mathcal{A}$, $1 \le j \le r$, there exists $\mu \in M$ and a subset $A_1 \in \mathcal{A}$, $A_1 \subset A$, such that:*

1. $|\mu(A_1)| > s$.
2. $|\mu(A \backslash A_1)| > s$.
3. $\sum_{1 \le j \le r} |\mu(Q_j)| \le 1$.
4. *M is a non quasi-$(A \backslash A_1)$-bounded.*

Proof. For $C := \{e_A, e_{Q_1}, \cdots, e_{Q_r}\}$

$$\sup\{|\nu(B)| : B \in A, B \in \mathcal{A}, \nu \in rM \cap C^0\} = \infty,$$

hence, there exists $P_1 \subset A$, $P_1 \in \mathcal{A}$, and $\nu_0 \in rM \cap C^0$ such that $|\nu_0(P_1)| > r(1+s)$. Then the measure $\mu = r^{-1}\nu_0 \in M \cap (r^{-1}C^0)$ verifies the following inequalities:

$|\mu(P_1)| > 1 + s$,
$|\mu(A)| \leq r^{-1} < 1$,
$|\mu(Q_j)| \leq r^{-1}$, $1 \leq j \leq r$, hence $\sum_{1 \leq j \leq r} |\mu(Q_j)| \leq rr^{-1} = 1$.

From $|\mu(P_1)| > 1 + s$ and $|\mu(A)| < 1$ it follows that $\mu(A \backslash P_1) > s$.

As M is non quasi-A-bounded and $\{P_1, A \backslash P_1\}$ is a partition of A, then we may obtain the following two cases:

1. M is non quasi-P_1-bounded. Then $A_1 := A \backslash P_1$ verifies that M is non quasi-$A \backslash A_1$-bounded, $|\mu(A_1)| = |\mu(A \backslash P_1)| > s$ and $|\mu(A \backslash A_1)| = |\mu(P_1)| > 1 + s > s$.
2. M is non quasi-$(A \backslash P_1)$-bounded. Then the set $A_1 := P_1$ verifies that M is non quasi-$A \backslash A_1$-bounded, $|\mu(A_1)| = |\mu(P_1)| > 1 + s > s$ and $|\mu(A \backslash A_1)| = |\mu(A \backslash P_1)| > s$.

Hence, the proposition is proved, since $\sum_{1 \leq j \leq r} |\mu(Q_j)| \leq \sum_{1 \leq j \leq r} r^{-1} = rr^{-1} = 1$. □

By applying Proposition 4 m times we directly obtain Proposition 5.

Proposition 5. *Let us suppose that $\mathcal{A}$ is an algebra of subsets of a set Ω, $A \in \mathcal{A}$, and let M be an absolutely convex non quasi-A-bounded subset of $L(\mathcal{A})^*$. Then for a positive natural number m, a positive natural number $s > 0$ and for each finite family $\{Q_j : 1 \leq j \leq r\}$, with $Q_j \in \mathcal{A}$, $1 \leq j \leq r$, there exists $\{\mu_1, \mu_2, \cdots, \mu_m\} \subset M$ and pairwise disjoint subsets $A_i \in \mathcal{A}$, $1 \leq i \leq m$, such that each A_i is a subset of A and for each i, $1 \leq i \leq m$, the following inequalities are verified:*

1. $|\mu_i(A_i)| > s$.
2. $|\mu_i(A \backslash \cup \{A_j : 1 \leq j \leq i\})| > s$.
3. $\sum_{1 \leq j \leq r} |\mu_i(Q_j)| \leq 1$.
4. *M is a non quasi-$(A \backslash \cup \{A_j : 1 \leq j \leq i\})$-bounded.*

In the next proposition we consider the natural numbers n_i, $1 \leq i \leq p$, and an infinite subset I of $\{n \in \mathbb{N} : n > n_p\}$.

Proposition 6. *Let us suppose that $\mathcal{A}$ is an algebra of subsets of a set Ω, $A \in \mathcal{A}$, and suppose that for each $n \in \{n_i : 1 \leq i \leq p\} \cup I$ the set M_n is an absolutely convex non quasi-A-bounded subset of $L(\mathcal{A})^*$. Then, for $s > 0$ and for each finite family $\{Q_j : 1 \leq j \leq r\}$, with $Q_j \in \mathcal{A}$, $1 \leq j \leq r$, there exists $\mu_i \in M_{n_i}$, a pairwise disjoint family of subsets $R_i \in \mathcal{A}$ with $R_i \subset A$, for $1 \leq i \leq p$, and an infinite subset J of I such that for each i, $1 \leq i \leq p$, the following inequalities are verified:*

1. $|\mu_i(R_i)| > s$.
2. $\sum_{1 \leq j \leq r} |\mu_i(Q_j)| \leq 1$.
3. *M_n is a non quasi-$(A \backslash \bigcup \{R_j : 1 \leq j \leq p\})$-bounded, for each $n \in \{n_i : 1 \leq i \leq p\} \cup J$.*

Proof. Applying Proposition 5 with $M = M_{n_1}$ there exists a partition $\{P_i : 1 \leq i \leq p+2\}$ of A, with each $P_i \in \mathcal{A}$, and there exists a subset $\{\lambda_i : 1 \leq i \leq p+1\} \subset M_{n_1}$ such that:

1. $|\lambda_i(P_i)| > s$.
2. $\sum_{1 \leq j \leq r} |\lambda_i(Q_j)| \leq 1$.
3. M_{n_1} is a non quasi-P_{p+2}-bounded.

For $2 \leq i \leq p$ the set M_{n_i} is not quasi-A-bounded, hence, by Lemma 1, there exists i_j with $1 \leq i_j \leq p+2$ such that M_{n_i} is not quasi-P_{i_j}-bounded.

Let $I_w := \{n \in I : M_n$ is not quasi-P_w-bounded$\}$, for $1 \leq w \leq p+2$. By Lemma 1, the infinite set I is equal to $\cup\{I_w : 1 \leq w \leq p+2\}$. Hence, there exists w_0 such that the set

$J_1 := I_{w_0}$ is infinite. The cardinality of the set $K = \{p+2\} \cup \{i_j : 2 \leq j \leq p\} \cup \{w_0\}$ is less that $p+2$. Hence if $l \in \{n \in \mathbb{N} : 1 \leq n \leq p+2\} \backslash K$ and we define $R_1 := P_l$ and $\mu_1 = \lambda_l$, then by construction:

1. $|\mu_1(R_1)| = |\lambda_l(P_l)| > s$.
2. $\sum_{1 \leq j \leq r} |\mu_1(Q_j)| = \sum_{1 \leq j \leq r} |\lambda_l(Q_j)| \leq 1$
3. By construction $p+2 \in K$, hence $l \neq p+2$. Therefore $P_{p+2} \subset A \backslash P_l = A \backslash R_1$. This inclusion and the fact that M_{n_1} is a non quasi-P_{p+2}-bounded imply that M_{n_1} is a non quasi-$(A \backslash R_1)$-bounded. For each $2 \leq i \leq p$ we have that $i_j \in K$, hence we deduce, analogously, that M_{n_i} is non quasi-$(A \backslash R_1)$-bounded.

Repeating p times the previous reasoning, the proposition is obtained. For instance, in the second step we apply Proposition 5 with $M = M_{n_2}$, J_1 and we work with non quasi-$(A \backslash R_1)$-boundedness. □

The Proposition 6 enables to get the elements of $\mathcal{A}$ and the measures of the next proposition.

Proposition 7. *Let us suppose that $\mathcal{A}$ is an algebra of subsets of a set Ω and let $\{M_n : n \in \mathbb{N}\}$ be a sequence of absolutely convex non quasi-Ω-bounded subsets of $L(\mathcal{A})^*$. Then there exists an increasing sequence of natural numbers $(n_i : i \in \mathbb{N})$ and two sequences $(\mathbb{A}_s : s \in \mathbb{N} \backslash \{1\})$ and $(\mathbb{M}_s : s \in \mathbb{N} \backslash \{1\})$ formed by the finite families*

$$\mathbb{A}_s = \left\{ A_{ij} \in A : (i,j) \in \mathbb{N}^2,\ i+j = s \right\}$$

and

$$\mathbb{M}_s = \left\{ \mu_{ij} \in M_{n_i} : (i,j) \in \mathbb{N}^2,\ i+j = s \right\}$$

such that for each $s \in \mathbb{N} \backslash \{1\}$ and each $(i,j) \in \mathbb{N}^2$ with $i+j = s$ it is verified that:

1. $\left|\mu_{ij}(A_{ij})\right| > i+j = s$.
2. $\sum\{\left|\mu_{ij}(A_{hk})\right| : h+k < i+j = s\} \leq 1$.
3. *The sets of the family $\{A_{ij} \in \mathcal{A} : (i,j) \in \mathbb{N}^2\}$ are pairwise disjoints.*

Proof. This proof follows by an inductive process on s.

The first step correspond to $s = 2$. Applying Proposition 6 to $A = \Omega$, $s = 2$, $n_1 = 1$, $I = \{n \in \mathbb{N} : 1 < n\}$ and without a finite family $\{Q_j : 1 \leq j \leq r\}$. Then we get the finite families $\mathbb{A}_2 = \{A_{11} \in \mathcal{A}\}$, $\mathbb{M}_2 = \{\mu_{11} \in M_{n_1}\}$ and an infinite subset J_1 of I such that:

1. $|\mu_{11}(A_{11})| > 2$.
2. M_n is non quasi-$(\Omega \backslash A_{11})$-bounded, for each $n \in \{n_1\} \cup J_1$.

In the second step of this inductive process, Proposition 6 is applied again with $A = \Omega \backslash A_{11}$, $s = 3$, n_1 is the natural number defined in the first step, $n_2 := \min\{n \in J_1\}$, $I = \{n \in J_1 : n_2 < n\}$ and $\{Q_1\} = \{A_{11}\}$. Then, we get the family $\mathbb{A}_3 = \{A_{ij} \in \mathcal{A} : (i,j) \in \mathbb{N}^2,\ i+j = 3\}$, formed by pairwise disjoint subsets of $\Omega \backslash A_{11}$, the family $\mathbb{M}_3 = \{\mu_{ij} \in M_{n_i} : (i,j) \in \mathbb{N}^2, i+j = 3\}$, and an infinite subset J_2 of J_1 such that:

1. $\left|\mu_{ij}(A_{ij})\right| > 3$, for each $(i,j) \in \mathbb{N}^2$, with $i+j = 3$.
2. $\left|\mu_{ij}(A_{hk})\right| \leq 1$, for each $(i,j,h,k) \in \mathbb{N}^4$, with $h+k < i+j = 3$.
3. M_n is non quasi-$\left(\Omega \backslash \cup \{A_{ij} : i+j \leq 3\}\right)$-bounded, for each $n \in \{n_1, n_2\} \cup J_2$.

Clearly, in the third step, Proposition 6 is applied to $A = \Omega \backslash \cup \{A_{ij} : i+j \leq 3\}$, $s = 4$, n_1 and n_2 are the natural numbers defined in the previous steps, $n_3 := \min\{n \in J_2\}$, $I = \{n \in J_2 : n_3 < n\}$, and $\{Q_1, Q_2, Q_3\} = \{A_{11}, A_{12}, A_{21}\}$. Then, we get the family $\mathbb{A}_4 = \{A_{ij} \in \mathcal{A} : (i,j) \in \mathbb{N}^2, i+j = 4\}$, formed by pairwise disjoint subsets of $\Omega \backslash \cup \{A_{ij} : i+j \leq 3\}$, the family $\mathbb{M}_4 = \{\mu_{ij} \in M_{n_i} : (i,j) \in \mathbb{N}^2, i+j = 4\}$, and an infinite subset J_3 of J_2 such that:

1. $|\mu_{ij}(A_{ij})| > 4$, for each $(i,j) \in \mathbb{N}^2$, with $i+j=4$.
2. $\sum\{|\mu_{ij}(A_{hk})| : h+k<4\} \leq 1$, for each $(i,j) \in \mathbb{N}^4$, with $i+j=4$.
3. M_n is non quasi-$(\Omega\backslash \cup \{A_{ij} : i+j \leq 4\})$-bounded, for each $n \in \{n_1, n_2, n_3\} \cup J_3$.

The induction continues in an obvious way. □

Theorem 1. *Let $\mathcal{A}$ be a σ-algebra. Then $\mathcal{A}$ is a strong Nikodým set for the Banach space $ba(\mathcal{A})$.*

Proof. Let us suppose that $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{A}$ such that each $\mathcal{A}_n$ is not a Nikodým set for the Banach space $ba(\mathcal{A})$. Corollary 1 with $\mathcal{B} = \mathcal{A}$ and $\mathcal{B}_n = \mathcal{A}_n$, $n \in \mathbb{N}$, implies that there exists n_0 such that for each $n \geq n_0$ we have $\overline{L(\mathcal{A}_n)} = L(\mathcal{A})$. Hence, we may suppose that $n_0 = 1$. For each $n \in \mathbb{N}$, by Proposition 3 with $\mathcal{B} = \mathcal{A}_n$, there exists in $L(\mathcal{A})^*$ a family $\{M_n : n \in \mathbb{N}\}$ of weak* closed absolutely convex $\mathcal{A}_n$-pointwise bounded subsets that are non quasi-Ω-bounded subsets. Therefore, by Proposition 7, there exists an increasing sequence of natural numbers $(n_i : i \in \mathbb{N})$ such that for each $s \in \mathbb{N}\backslash\{1\}$ there exist two finite families $\mathbb{A}_s = \{A_{ij} \in \mathcal{A} : (i,j) \in \mathbb{N}^2, i+j=s\}$ and $\mathbb{M}_s = \{\mu_{ij} \in M_{n_i} : (i,j) \in \mathbb{N}^2, i+j=s\}$ such that for each $(i,j) \in \mathbb{N}^2$, with $i+j=s$,

1. $|\mu_{ij}(A_{ij})| > i+j = s$,
2. $\sum\{|\mu_{ij}(A_{hk})| : h+k < i+j = s\} \leq 1$, and
3. the sets of families $\{A_{ij} \in \mathcal{A} : (i,j) \in \mathbb{N}^2\}$ are pairwise disjoints.

The sequence $(s : 2 \leq s)$ contains an increasing sub-sequence $(w_s : 2 \leq s)$ that has the following property: For each $\mu_{ij} \in \mathbb{M}_{w_s}$, the variation of μ_{ij} in $\bigcup\{A_{ij} : A_{ij} \in \mathbb{A}_{w_t}, s < t\}$ is less than or equal to 1.

In fact, let $s = 2$. Then $\mathbb{M}_2 = \{\mu_{11}\}$ and we define $w_2 = 2$. Suppose that the variation of the bounded measure μ_{11} is less than or equal to a positive natural number a. Let

$$\begin{aligned}
\mathbb{A}'_{w_2+1} &= \bigcup\{\mathbb{A}_{w_2+1+pa} : p = 0,1,2,\cdots\} \\
\mathbb{A}'_{w_2+2} &= \bigcup\{\mathbb{A}_{w_2+2+pa} : p = 0,1,2,\cdots\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbb{A}'_{w_2+a} &= \bigcup\{\mathbb{A}_{w_2+a+pa} : p = 0,1,2,\cdots\}
\end{aligned}$$

The additivity of the variation implies that there exists a natural number $w_3 \in \{w_2+1, w_2+2, \cdots, w_2+a\}$ such that the variation of μ_{11} in $\mathbb{A}'_{w_3}$ is less than or equal to 1.

Then, $\mathbb{M}_{w_3} = \{\mu_{ij} \in M_{n_i} : (i,j) \in \mathbb{N}^2, i+j = w_3\}$ and we may suppose that the variation of each μ_{ij}, $(i,j) \in \mathbb{N}^2$, $i+j = w_3$, is less than or equal to a positive natural number b. Let now be

$$\begin{aligned}
\mathbb{A}'_{w_3+1} &= \bigcup\left\{\mathbb{A}_{w_3+1+pb(w_3-1)} : p = 0,1,2,\cdots\right\} \\
\mathbb{A}'_{w_3+2} &= \bigcup\left\{\mathbb{A}_{w_3+2+pb(w_3-1)} : p = 0,1,2,\cdots\right\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbb{A}'_{w_3+b(w_3-1)} &= \bigcup\left\{\mathbb{A}_{w_3+b(w_3-1)+pb(w_3-1)} : p = 0,1,2,\cdots\right\}
\end{aligned}$$

and again, by the additivity of variation, there exists $w_4 \in \{w_3+1, w_3+2, \cdots, w_3+b(w_3-1)\}$ such that, for each $(i,j) \in \mathbb{N}^2$, $i+j = w_3$, the variation of each μ_{ij} in $\mathbb{A}'_{w_4}$ is less than or equal to 1.

The induction continues in an very natural way.

Let $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(2,2)$, $(3,1)$, $\cdots$ be $\mathbb{N}^2$ ordered by the diagonal order, $(i_s : s = 2,3,\cdots)$ the sequence of the first components of $\mathbb{N}^2$ ordered by the diagonal order, i.e., $(1, 1, 2, 1, 2, 3, \cdots)$, and let $j_s = w_s - i_s$, $s = 2, 3, \cdots$. The union $\bigcup\{A_{i_s,j_s} : s \in \mathbb{N}, s > 1\} \in \mathcal{A} = \bigcup\{\mathcal{A}_{n_i} : i \in \mathbb{N}\}$, hence there exists $r \in N$ such that

$$\bigcup\{A_{i_s,j_s} : s \in \mathbb{N}, s > 1\} \in \mathcal{A}_{n_r}.$$

By construction:

(1) There exists an increasing sequence $(s_v : v \in \mathbb{N})$ such that $i_{s_v} = n_r$, for each $v \in \mathbb{N}$. Then, $j_{s_v} = w_{s_v} - i_{s_v} = w_{s_v} - n_r$.
(2) $\mu_{i_{s_v},j_{s_v}} = \mu_{n_r,j_{s_v}} \in M_{n_r}$.
(3) The set M_{n_r} is $\mathcal{A}_{n_r}$-pointwise bounded, hence

$$\sup\left\{\left|\mu_{n_r,j_{s_v}}\left(\bigcup\{A_{i_s,j_s} : s \in \mathbb{N}, s > 1\}\right)\right| : v \in \mathbb{N}\right\} < \infty.$$

(4) From

$$\left|\mu_{n_r,j_{s_v}}(A_{n_r,j_{s_v}}\})\right| > n_r + j_{s_v} > v,$$
$$\sum\{\left|\mu_{n_r,j_{s_v}}(A_{i_w,j_w})\right| : w < s_v\} \leq 1$$

and the property that the variation of $\mu_{n_r,j_{s_v}}$ in $\bigcup\{A_{i_w,j_w} : w \in \mathbb{N}, w > s_v\}$ is less than or equal than 1, we obtain the contradiction

$$\sup\left\{\left|\mu_{n_r,j_{s_v}}\left(\bigcup\{A_{i_s,j_s} : s \in \mathbb{N}, s > 1\}\right)\right| : v \in \mathbb{N}\right\} = \infty.$$

□

3. Strong Grothendieck Sets

Recall that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a Grothendieck set for the Banach space $ba(\mathcal{A})$ if for each bounded sequence $(\mu_n, n \in \mathbb{N})$ of $ba(\mathcal{A})$ the $\mathcal{B}$-pointwise convergence of $(\mu_n, n \in \mathbb{N})$ to $\mu \in ba(\mathcal{A})$ implies its weak convergence, i.e.,

$$\lim_{n\to\infty} \varphi(\mu_n) = \varphi(\mu), \forall \varphi \in ba(\mathcal{A})^*.$$

In the definition of the Grothendieck set given in [7] (Definition 1) the sentence "each bounded sequence" is replaced by "each sequence". Both definitions of Grothendieck sets agree when $\mathcal{B}$ is a Nikodým set for the Banach space $ba(\mathcal{A})$, because then each sequence $(\mu_n, n \in \mathbb{N})$ of $ba(\mathcal{A})$ that $\mathcal{B}$-pointwise converges is $\mathcal{B}$-pointwise bounded, hence it is norm bounded. In the introduction, it was considered that the definition given in this paper is the natural extension to a subset $\mathcal{B}$ of the property that verifies an algebra $\mathcal{A}$ when $\mathcal{A}$ is a Grothendieck set for the Banach space $ba(\mathcal{A})$, i.e., the completion of $L(\mathcal{A})$ is a Grothendieck space. Moreover, with the definition of Grothendieck sets given in this survey, the Grothendieck sets possessed the favorable hereditary property considered in Theorem 2. We do not know if this hereditary property holds with the definition given in [7] (Definition 1).

It was defined in the introduction that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a strong Grothendieck set for the Banach space $ba(\mathcal{A})$ if for each increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ there exists n_0 such that $\mathcal{B}_{n_0}$ is a Grothendieck set for the Banach space $ba(\mathcal{A})$.

Theorem 2. *Assume that $\mathcal{A}$ is an algebra of subsets of a set Ω that contains a subset $\mathcal{B}$ which is a Nikodým and a Grothendieck set for $ba(\mathcal{A})$. Then $\mathcal{B}$ is a strong Grothendieck set for $ba(\mathcal{A})$.*

Proof. We need to prove that if $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{B}$, there exists some $p \in \mathbb{N}$ such that $\mathcal{B}_p$ is a Grothendieck set for $ba(\mathcal{A})$.

By Corollary 1, there exists p such that closure in $L(\mathcal{A})$ of the absolutely convex hull of $\{pe_B : B \in \mathcal{B}_p\}$ contains $\{e_A : A \in \mathcal{A}\}$.

Let us check that $\mathcal{B}_p$ is a Grothendieck set for $ba(\mathcal{A})$. So, let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence in $ba(\mathcal{A})$ such that $\lambda_n(M) \to 0$, for each $M \in \mathcal{B}_p$.

Then $\sup_{n\in\mathbb{N}}|\lambda_n| < \infty$ and $\lambda_n(u) \to 0$ for every $u \in \mathrm{abx}\{\chi_A : A \in \mathcal{B}_p\}$. We only need to prove that $\lambda_n(v) \to 0$ for each $v \in \overline{\mathrm{abx}\{\chi_A : A \in \mathcal{B}_p\}}^{\|\cdot\|_\infty}$.

Let $(u_k)_{k=1}^\infty$ be a sequence in $\mathrm{abx}\{\chi_A : A \in \mathcal{B}_p\}$ such that $\|u_k - v\|_\infty \to 0$. Consequently, given $\epsilon > 0$ there exists $k(\epsilon) \in \mathbb{N}$ with

$$\left\|u_{k(\epsilon)} - v\right\|_\infty < \frac{\epsilon}{2(1+\sup_{n\in\mathbb{N}}|\lambda_n|)}.$$

By hypothesis $\lim_{n\to\infty}\lambda_n(u_{k(\epsilon)}) = 0$; hence, there exists $n(\epsilon) \in \mathbb{N}$ such that for $n \geq n(\epsilon)$

$$\left|\lambda_n(u_{k(\epsilon)})\right| < \frac{\epsilon}{2},$$

From the two preceding inequalities, it follows that $\lim_{n\to\infty}\lambda_n(v) = 0$, because for $n \geq n(\epsilon)$ we obtain

$$|\lambda_n(v)| \leq \left|\lambda_n(v - u_{k(\epsilon)})\right| + \left|\lambda_n(u_{k(\epsilon)})\right| \leq$$

$$\leq |\lambda_n|\left\|u_{k(\epsilon)} - v\right\|_\infty + \left|\lambda_n(u_{k(\epsilon)})\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\mathcal{B}_p$ is a Grothendieck set for $L(\mathcal{A})$. □

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω is a Vitali–Hahn–Saks set for $ba(\mathcal{A})$ if for each sequence $(\mu_n, n \in \mathbb{N})$ of $ba(\mathcal{A})$ the $\mathcal{B}$-pointwise convergence of $(\mu_n, n \in \mathbb{N})$ to $\mu \in ba(\mathcal{A})$ implies its weak convergence. It is straightforward to prove that $\mathcal{B}$ is a Vitali–Hahn–Saks set for $ba(\mathcal{A})$ if and only if $\mathcal{B}$ is a Nikodým and Grothendieck set for $ba(\mathcal{A})$.

Corollary 2. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω such that the set $\mathcal{A}$ is a Vitali–Hahn–Saks set for $ba(\mathcal{A})$. The set $\mathcal{A}$ is a strong Grothendieck set for $ba(\mathcal{A})$. In particular, if $\mathcal{A}$ is a σ-algebra of subsets of a set Ω then $\mathcal{A}$ is a strong Grothendieck set for $ba(\mathcal{A})$.*

Proof. If $\mathcal{A}$ is an algebra of subsets of a set Ω such that the set $\mathcal{A}$ is a Vitali–Hahn–Saks set for $ba(\mathcal{A})$ then the set $\mathcal{A}$ is a Nikodým set and a Grothendieck set for $ba(\mathcal{A})$. By Theorem 2 with $\mathcal{B} = \mathcal{A}$ we find that $\mathcal{A}$ is a strong Grothendieck set for $ba(\mathcal{A})$. The particular case follows from the fact that if $\mathcal{A}$ is a σ-algebra of subsets of a set Ω then $\mathcal{A}$ is a Nikodým set and a Grothendieck set for $ba(\mathcal{A})$. □

A positive answer to the aforementioned Valdivia open question would help to extend several theorems on Measure theory on σ-algebras of subsets of a set Ω to algebras of subsets of a set Ω. Applications of Theorems 1 and 2 for σ-algebras, improving Phillips lemma about convergence in $ba(\mathcal{A})$, Nikodým's pointwise convergence theorem in $ca(\mathcal{A})$ and the usual characterization of weak convergence in $ca(\mathcal{A})$, with $ca(\mathcal{A})$ being the linear subspace of $ba(\mathcal{A})$ consisting of the countably additive measures on a σ-algebra $\mathcal{A}$ (see [2] (Chapter 7)), are provided in [7] (Propositions 1, 2, and 3).

In [8] (Section 3), a class $\mathcal{C}$ of rings of subsets was determined, such that for each ring $\mathcal{R}$ of subsets of a set Ω with $\mathcal{R} \in \mathcal{C}$ then the property that the set $\mathcal{R}$ is a Nikodým set for $ba(\mathcal{R})$ implies that the set $\mathcal{R}$ is a strong Nikodým set for $ba(\mathcal{R})$. This result provides a partial positive solution of the still open problem for an algebra $\mathcal{A}$ of subsets of a set Ω, of whether the property that the set $\mathcal{A}$ is a Nikodým set for $ba(\mathcal{A})$ implies that this set $\mathcal{A}$ is also a strong Nikodým set for $ba(\mathcal{A})$.

Let J be the algebra of all Jordan measurable subsets of the finite product $\Omega = \Pi\{[a_i, b_i] : 1 \leq i \leq k\}$ of k real closed intervals. $\mathcal{J}$ is not a σ-algebra and it is proved in [9] (Theorem 2) that the set $\mathcal{J}$ is a strong Nikodým set for $ba(\mathcal{J})$. This result was improved in [10] (Theorem 1) finding that this algebra $\mathcal{J}$ is a web Nikodým set for $ba(\mathcal{J})$.

Let us recall (see [4] (2.3. Definition)) that for an algebra $\mathcal{A}$ of subsets of a set Ω a countable subset $\{\mu_n : n \in \mathbb{N}\}$ of $ba(\mathcal{A})$ is *uniformly exhaustive on* $\mathcal{A}$ if for each countable family $\{A_k : k \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{A}$ we have the following:

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} |\mu_n(A_k)| = 0,$$

and that a subset C of a normed space E is an *uniform bounded deciding set for* E if each subset M of E^* which is pointwise bounded on C is norm bounded ([11]). Interesting relations between Nikodým and Grothendieck properties, uniform exhaustivity, uniform bounded deciding property and the so called Rainwater sets ([12]) are considered in [13].

Let $\mathcal{A}$ be a Boolean algebra and let $K_{\mathcal{A}}$ be the Stone space of $\mathcal{A}$. Recall that by the Stone duality theorem, $\mathcal{A}$ is isomorphic with the algebra $\mathcal{C}$ of clopen subsets of $K_{\mathcal{A}}$ (see [14,15]), and that each scalar finitely additive measure μ with finite variation defined on $\mathcal{A}$ has a unique Borel extension, denoted also by μ, defined in the space $K_{\mathcal{A}}$, preserving the variation of μ. In the Riesz representation theorem the dual space $C(K_{\mathcal{A}})^*$ of the Banach space of continuous scalar functions on $K_{\mathcal{A}}$ is isometrically isomorphic with the space of all finitely additive bounded measures on $\mathcal{A}$.

A complete Boolean algebra $\mathcal{A}$ is a Boolean algebra in which every subset of $\mathcal{A}$ has a supremum. More generally, if κ is a cardinal then a Boolean algebra $\mathcal{A}$ is κ-complete if every subset of $\mathcal{A}$ of cardinality less than κ has a supremum; in particular, a Boolean algebra $\mathcal{A}$ is σ-complete if every countable subset of $\mathcal{A}$ has a supremum.

Nikodým and Grothendieck properties in Boolean algebras are defined in a natural way. For instance, a Boolean algebra $\mathcal{A}$ has the Nikodým property if each sequence $(\mu_n)_{n=1}^{\infty}$ of scalar finitely additive bounded measures such that $\sup_n |\mu_n(a)| < \infty$ for all $a \in \mathcal{A}$ verifies that $\sup_n |\mu_n| < \infty$. In brief, a Boolean algebra $\mathcal{A}$ has the Nikodým property if it verifies the Nikodým-Grothendieck boundedness theorem or, equivalently, if the algebra $\mathcal{C}$ of the clopen subsets of the Stone space $K_{\mathcal{A}}$ of $\mathcal{A}$ is a Nikodým set for the Banach space $ba(\mathcal{C})$. Each σ-complete Boolean algebra $\mathcal{A}$ has the Nikodým property. Grothendieck property for boolean algebras is defined similarly.

In [16], it was proved that in the model obtained by side-by-side product of Sacks forcings, the Boolean algebra of subsets of the first infinite countable ordinal ω that belong to the ground model has the Grothendieck property.

In [17], the authors show that in the model obtained by the side-by-side product of Sacks forcing every σ-complete Boolean algebra from the ground model has the Nikodým property and that there exists a Boolean algebra of cardinality less than the cardinal $\mathfrak{c}$ of the continuum with the Nikodým property. In [18], the existence of Boolean algebras with the Nikodým and Grothendieck properties is established in models verified by a quite wide class of forcing notions.

Finally, in [19] the author shows that if κ is a cardinal such that $\kappa^{\mathbb{N}}$ has cofinality κ and the cofinality of the Lebesgue null ideal is at most κ then there is a Boolean algebra of cardinality κ with the Nikodým property. In particular, this shows that there exist, consistently, algebras with the Nikodým property that are of cardinality less than $\mathfrak{c}$. Lower bounds for the minimum cardinality of a Boolean algebra with the Nikodým property were also obtained.

Author Contributions: Conceptualization, S.L.-A. and M.L.-P.; validation, S.L.-A., M.L.-P., S.M.-L. and L.M.S.-R.; investigation, S.L.-A., M.L.-P., S.M.-L. and L.M.S.-R.; writing—original draft preparation, S.L.-A., M.L.-P. and S.M.-L.; writing—review and editing, S.M.-L. and L.M.S.-R.; visualization, S.L.-A., M.L.-P., S.M.-L. and L.M.S.-R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by grant PGC2018-094431-B-I00 of Ministry of Science, Innovation and Universities of Spain for the second named author.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank Professor Dr. Manuel Valdivia (1928–2014) all his excellent mathematical research work and, in particular, his 2013 paper *On Nikodým boundedness property*, (see [9]), which gave many suggestions for this paper. Also we wish to thank the reviewers for their helpful advice and suggestions. The corresponding author is very grateful to Professor Dr. Rekha Srivastava, Guest Editor of the Special Issue "Mathematical Analysis and Analytic Number Theory 2022" in Mathematics, for her invitation to write this invited feature paper.

Conflicts of Interest: The authors declare no conflict of interest.

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