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Fundamental constraints on broadband passive acoustic treatments in unidimensional scattering problems

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In a passive lossy acoustical system, sum rules derived from passivity explicitly relate the broadband response to the spatial dimension of the system, which provide important design criteria as well. In this work, the theory of Herglotz function is applied systematically to derive sum rules for unidimensional scattering problems relying on passive acoustic treatments which are generally made of rigid, motionless and subwavelength structures saturated by air. The rigid-boundary reflection, soft-boundary reflection and transmission problems are analysed. The derived sum rules are applied for guiding the designs of passive absorbers and mufflers: the required minimum space is directly predicted from the target (i.e. the desired absorption or transmission-loss spectra) without any specific design. Besides, it is possible to break this type of sum rules and fundamental constraints in particular cases.

This property, if well used, could result in ultra-compact absorbers working at long wavelength up to infinity.

1. Introduction

Passivity is an inherent property satisfied by many physical systems which do not produce energy by themselves. In particular, for a continuous, causal, linear and time-translational invariant system, passivity may lead to sum rules and impose fundamental constraints on the system. Generally speaking, sum rules are integral identities of the form $\int_0^\infty \omega^{-n} A(\omega) d\omega = C$, where ω refers to the angular frequency, $A(\omega)$ is the considered response function, n is an integer included in the weighting factor ω^{-n} and the constant C is determined by the static and dynamic properties of the system. Particularly, for the sum rules derived from passivity, $A(\omega) \geq 0$, since the definition of a passive system is closely related to non-negative valued functions [1–4]. Then, the constant $C \geq 0$. And moreover, the integration within a finite frequency range provides a lower bound of C , which indicates a fundamental constraint satisfied by the passive system.

In order to derive sum rules and fundamental constraints of a given system, the Kramers–Kronig relations [5,6], the Bode gain-phase relation (or modified Kramers–Kronig relation) [7–9] and the superconvergence theorem [3,10–12] are commonly adopted. Examples can be found in network theory [13,14], optics [15], electromagnetism [16], acoustics [17–19], nuclear physics [20], feedback control systems and filters [21], etc. Note however that additional assumptions are usually made on the considered systems [2,15]. For instance, the response function is assumed to be a rational function in order to apply the Cauchy integral formula [2]. To consider more general cases with non-rational response functions for instance, one can build on the mathematical framework developed by Bernland *et al.* [2]. This approach relies on the properties of Herglotz functions [15,22] and has been used for various applications in electromagnetism, considering both lossless and lossy systems [3,23–26].

In contrast to electromagnetism, viscothermal losses usually play significant roles in the designs of passive treatments when we focus on acoustical systems in air, e.g. broadband absorbers [27–31], silencers [32,33], meta-diffusers [34,35], etc. See [36–40] and the references therein for other applications. Moreover, in lossy acoustical systems, the static/dynamic limits of the system response (and the aforementioned integral constant C) can be explicitly expressed by relevant spatial dimensions (i.e. total length or volume) as well as lumped structural parameters (i.e. porosity, tortuosity, etc.) of the system. Note that although these lumped parameters are unknowns without a specific design, their ranges and bounds are well known, e.g. $(0, 1]$ and $[1, \infty)$ for the porosity and the tortuosity, respectively [41]. Thus, this feature of lossy acoustical systems allows *a priori* estimates on the required spatial dimensions for any given target spectrum, which is of primary importance for the design of subwavelength passive treatments: whenever some information of the relevant response function (i.e. the design target) is known or assumed within finite frequency bands, sum rules of the system can be directly used to evaluate the lower bound of the required treatment size.

For most of the commonly used acoustical treatments, the system response functions are not rational functions, such as the acoustic impedance of a quarter-wavelength resonator, which is a cotangent function of frequency, and the radiation impedance of a baffled circular piston or orifice, which is related to Bessel functions [42]. Thus, in this work, we revisit the generalized theory based on the Herglotz functions [2] to derive and analyse the sum rules and fundamental constraints for lossy acoustical systems in unidimensional scattering problems. In §2, the mathematical framework of the theory of Herglotz function is recalled from [2]. The transfer matrix modelling of the considered scattering problems is described in §3a. The sum rules and fundamental constraints are then derived and discussed for a rigid-boundary reflection problem (§3b), a soft-boundary reflection problem (§3c) and a transmission problem (§3d). In §4, two specific examples are given to explain how to evaluate the required minimum treatment size by using these sum rules. Conclusions are drawn in §5. Further validation and

parametric studies of the derived sum rules are also provided in the electronic supplementary material.

2. Sum rules and fundamental constraints for a passive unidimensional system

For a continuous, causal, linear and time-translational invariant system, the output of the system $Y(t)$ can be expressed as a convolution between the input $X(t)$ and Green's function $G(t)$:

$$Y(t) = \int_0^{\infty} G(\tau)X(t - \tau) d\tau, \quad (2.1)$$

where τ is the retarded time. Due to the causality of the system, $G(\tau) = 0$ for $\tau < 0$. After a time-domain Fourier transform, the convolution reduces to a product in the frequency domain that provides the definition of the frequency response function

$$\tilde{G}(\omega) = \frac{\tilde{Y}(\omega)}{\tilde{X}(\omega)}. \quad (2.2)$$

Note that in this work, we use the following Fourier transformation pair:

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt \quad \text{and} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega. \quad (2.3)$$

With the response function well defined, two equivalent definitions of the passivity of a system are generally considered, namely the admittance/impedance passivity and the scattering passivity [1–4]. For unidimensional scattering problems, the passivity of the system ensures that the flux of acoustic energy into the surface remains non-negative. When the surface admittance $\eta(\omega)$, or the impedance $\zeta(\omega) = 1/\eta(\omega)$ is the relevant frequency response function, the sign of the resistive part of the response function is therefore fixed (either non-negative or non-positive) depending on the definitions of η and ζ . Based on the non-negative-resistive-part definition, the admittance/impedance passivity implies that

$$\text{Re}[\eta(\omega)] \geq 0 \quad \text{and} \quad \text{Re}[\zeta(\omega)] \geq 0. \quad (2.4)$$

When enforcing the scattering passivity of the system, the reflection and transmission coefficients, $R(\omega)$ and $T(\omega)$, should be such that $0 \leq |R(\omega)|^2 + |T(\omega)|^2 \leq 1$, i.e. $-\ln[|R(\omega)|^2 + |T(\omega)|^2] \geq 0$, or separately, $0 \leq |R(\omega)| \leq 1$, and $0 \leq |T(\omega)| \leq 1$, so that scattering passivity results in

$$-\ln |R(\omega)| \geq 0 \quad \text{and} \quad -\ln |T(\omega)| \geq 0. \quad (2.5)$$

These relations state that the energy output of the system cannot exceed the input energy.

Herglotz functions are particularly useful to analyse the response of a passive system, since they are directly related to non-negative valued functions. By definition, a Herglotz function $H(\omega)$ is a holomorphic function in the upper half complex ω plane that also satisfies $\text{Im}[H(\omega)] \geq 0$ for $\text{Im}(\omega) > 0$ [2,22]. Based on the different descriptions of passivity mentioned above, two types of Herglotz functions can be introduced for the unidimensional scattering problem. For the admittance/impedance passivity equation (2.4), we define

$$H_1(\omega) = i\eta(\omega) \quad (2.6a)$$

or

$$H_1(\omega) = i\zeta(\omega). \quad (2.6b)$$

Analogously, for the scattering passivity in equation (2.5), the Herglotz functions can be defined as follows:

$$H_2(\omega) = -i \log[R(\omega)B(\omega)] \quad (2.7a)$$

or

$$H_2(\omega) = -i \log[T(\omega)B(\omega)], \quad (2.7b)$$

where $\log(\cdot)$ is the complex logarithm, in contrast to the natural logarithm $\ln(\cdot)$, which stands for real variables. $B(\omega)$ is a Blaschke product introduced to remove the zeros of $R(\omega)$ or $T(\omega)$ (i.e. the singularities of the corresponding complex logarithm) from the upper half complex ω plane [2,16–18]. The Blaschke product is defined as follows:

$$B(\omega) = \prod_n \frac{1 - \omega/\omega_n^*}{1 - \omega/\omega_n}, \quad (2.8)$$

where ω_n is the n th zero of $R(\omega)$ or $T(\omega)$ in equations (2.7) with $\text{Im}(\omega_n) > 0$ and the star denotes the complex conjugate. Note that $-B(\omega)$ can be used instead of $B(\omega)$ when the soft-boundary reflection problem is considered. Also note that we will use different Herglotz functions for each passivity condition, i.e. either the form (a) or (b) in equations (2.6) and (2.7), depending on their static and dynamic limits.

When the input/output functions are real-valued in the time domain, the Herglotz functions introduced in equations (2.6) and (2.7) are symmetric, i.e. $H(\omega) = -H^*(-\omega^*)$ is satisfied. From the representation theorem [2], the asymptotic expansions of a symmetric Herglotz function in the static limit ($\omega \hat{\rightarrow} 0$) and the dynamic limit ($\omega \hat{\rightarrow} \infty$) can be written as

$$H(\omega) = \frac{a_{-1}}{\omega} + a_1\omega + \dots + a_{2N_0-1}\omega^{2N_0-1} + o(\omega^{2N_0-1}), \quad \text{when } \omega \hat{\rightarrow} 0 \quad (2.9a)$$

and

$$H(\omega) = b_1\omega + \frac{b_{-1}}{\omega} + \dots + \frac{b_{1-2N_\infty}}{\omega^{2N_\infty-1}} + o\left(\frac{1}{\omega^{2N_\infty-1}}\right), \quad \text{when } \omega \hat{\rightarrow} \infty, \quad (2.9b)$$

where a_n, b_n are real coefficients, and N_0, N_∞ are non-negative integers. Moreover, the expansions should be stopped whenever a term that does not satisfy the forms of equations (2.9) appears (e.g. $\log \omega, \sqrt{\omega}$, constant or any even-order term of ω). Note that the notation $\hat{\rightarrow}$ denotes the limit in the Stoltz domain $\theta \leq \arg \omega \leq \pi - \theta$, for any $0 < \theta \leq \pi/2$, in the complex ω plane rather than on the real axis. As shown in [2], the coefficients a_n and b_n in equations (2.9) can then be used to construct a series of integral identities or sum rules

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \frac{2}{\pi} \int_\varepsilon^{\varepsilon^{-1}} \frac{\text{Im}[H(\omega + i\delta)]}{\omega^{2n}} d\omega = a_{2n-1} - b_{2n-1}, \quad (2.10)$$

where $n = 1 - N_\infty, \dots, N_0$. Note that not every symmetric Herglotz function does admit a sum rule. A trivial example is that $H(\omega) = iC$, where $C > 0$. It is easy to validate that $H(\omega)$ is a symmetric Herglotz function; however, the leading terms of its asymptotic expansions are a_0 and b_0 terms, with which the general formula (2.10) is not applicable.

In each case considered in this work, the $n = 1$ sum rule is allowed. We have thus

$$\frac{2}{\pi} \int_0^\infty \frac{\text{Im}[H(\omega)]}{\omega^2} d\omega = a_1 - b_1, \quad (2.11)$$

in which the integral from zero to infinity is a shorthand notation for the limits in equation (2.10). In the following sections, we will show that the diverging term $b_1\omega$ in the dynamic limit does not exist in most cases, i.e. $b_1 = 0$, so that the $n = 1$ sum rule only involves the static limit of the system. The only exception is the sum rule for $T(\omega)$ in the transmission problem. In this case, b_1 , which represents the group delay [43] induced by the material layer, does not vanish. Hence, this particular sum rule is influenced by both the static and dynamic limits of the system.

3. Fundamental constraints for unidimensional scattering problems

(a) Transfer matrix modelling of the system

We consider several unidimensional scattering problems in which an incident plane wave is scattered by an effective material with thickness L as illustrated in figure 1. We assume that this material consists of rigid, motionless and subwavelength elements, e.g. tubes, cavities, perforations, porous materials, etc. Note that these elements are not restricted to be merely inside

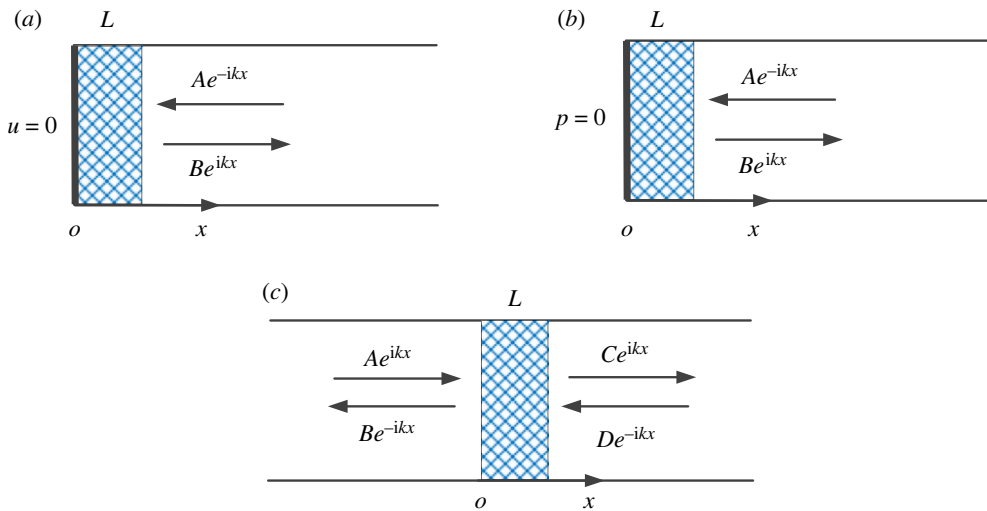


Figure 1. The unidimensional scattering problem of a plane wave by a mirror-symmetric effective material: (a) rigid-boundary reflection problem, (b) soft-boundary reflection problem and (c) transmission problem. (Online version in colour.)

the waveguide. Side branch elements could also be accounted for in this effective layer through the homogenization theory [39,40]. All the elements as well as the waveguide are saturated by air with density ρ_0 and adiabatic sound speed c_0 . The losses within the system are induced by viscothermal boundary layers near the no-slip and isothermal boundaries.

We start from the case in which the effective material is mirror symmetric, so that the acoustic behaviour of the material can be modelled as an equivalent fluid layer with frequency-dependent effective parameters [39–41]. This configuration is subsequently generalized to account for multilayer structures that are locally mirror symmetric (each sublayer is mirror symmetric).

Under the above assumptions, the state vector at $x=L$ can be related to that at $x=0$ through the transfer matrix \mathbf{T} of the single-layer system [39,40]:

$$\begin{bmatrix} P(L) \\ U(L) \end{bmatrix} = \mathbf{T} \begin{bmatrix} P(0) \\ U(0) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} P(0) \\ U(0) \end{bmatrix}, \quad (3.1)$$

where $P = p/K_0$, $U = u/c_0$ refer to the dimensionless forms of acoustic pressure and particle velocity: the acoustic pressure p and particle velocity u are normalized by the adiabatic bulk modulus $K_0 = \rho_0 c_0^2$ and the sound speed c_0 of air, respectively. The elements of the transfer matrix are

$$t_{11} = t_{22} = \cos(k_e L), \quad (3.2a)$$

$$t_{12} = i \frac{z_e}{z_0} \sin(k_e L) \quad (3.2b)$$

and

$$t_{21} = i \frac{z_0}{z_e} \sin(k_e L), \quad (3.2c)$$

in which $z_0 = \rho_0 c_0$ refers to the characteristic impedance of air. The effective parameters of the material are frequency-dependent, including its density ρ_e , sound speed c_e , characteristic impedance $z_e = \rho_e c_e$ and acoustic wavenumber $k_e = \omega/c_e$. The static and dynamic limits of these parameters can be rigorously evaluated [41]; details are provided in appendix A. Either for the unidimensional reflection problems or the unidimensional transmission problem, the acoustical response functions of the system can be expressed in terms of the elements of the transfer matrix.

(b) Rigid-boundary reflection problem

We first consider the reflection problem shown in figure 1a where a rigid boundary ($u = 0$) is imposed at one end of the material ($x = 0$). In this case, the relevant response functions are the surface admittance and the reflection coefficient at $x = L$. They are given by

$$\eta(\omega) = -\frac{t_{21}}{t_{11}} = -i \frac{z_0}{z_e} \tan(k_e L) \quad (3.3a)$$

and

$$R(\omega) = \frac{t_{11} + t_{21}}{t_{11} - t_{21}} = \frac{z_e + iz_0 \tan(k_e L)}{z_e - iz_0 \tan(k_e L)}. \quad (3.3b)$$

With the choice of Herglotz functions H_1 and H_2 defined in equation (2.6a) and equation (2.7a), together with the Blaschke product B from equation (2.8), the following $n = 1$ sum rules (as shown in equation (2.11)) can be derived for η and R

$$\frac{2c_0}{\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{1}{\omega^2} \text{Re}[\eta(\omega)] d\omega = L \quad (3.4a)$$

and

$$\frac{c_0}{\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq L. \quad (3.4b)$$

The first sum rule (3.4a) implies that when the thickness of a passive treatment is fixed, the resistance cannot be an arbitrarily chosen function of frequency. The second sum rule (3.4b) is an inequality because the static limit of the Blaschke product is dropped in the expansion of the Herglotz function (i.e. in the a_1 term), as explained in appendix B. Besides, equation (3.4b) can be rearranged as a sum rule for the weighted absorption spectrum, due to the relation $\alpha(\omega) = 1 - |R(\omega)|^2$

$$\frac{c_0}{2\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq L. \quad (3.5)$$

The sum rules (3.4a)–(3.5) are clearly controlled by the effective bulk modulus of the material $K_e = \rho_e c_e^2$, which is expected since the low-frequency response of the system is monopolar (see [44–46] and ch. 5 of [40]). The static limit of K_e can be derived as (see appendix A, or p. 56 in [41])

$$\frac{K_e(0)}{K_0} = \frac{1}{\gamma\phi}, \quad (3.6)$$

where $\gamma = 1.4$ is the adiabatic index of air and ϕ is defined as the volume ratio between the saturating air and the whole layer. When all the elements of the material are fixed inside the waveguide, ϕ is the porosity of the material and $\phi \leq 1$ is well satisfied. In the opposite, when side-branch elements are also included, ϕ could be larger than unity, because the volume of the the effective layer could be less than that of the material. The inequality (3.5) is in accordance with the results given in [18,27], which provide a design criterion for the minimum total length of a broadband absorber. However, a difference exists in the evaluation of $K_e(0)$. In [18], the system is modelled as a Lorentz oscillator, so that the losses are independent of frequency. As a result, the static limit of the effective bulk modulus tends to the adiabatic limit, which leads to $K_e(0)/K_0 = 1/\phi$ (shown in [18]). Conversely, the viscothermal losses in the system are considered here by introducing frequency-dependent effective parameters in the transfer matrix. As a consequence, the effective bulk modulus tends to the isothermal limit, which leads to the additional factor γ in equation (3.6).

From a practical point of view, the sum rule (3.5) can be used to provide guidelines for the design of acoustic treatments. For instance, if the aim is to achieve an absorption target $\alpha(\omega)$ then equation (3.5) provides a lower bound for the quantity $LK_0/K_e(0)$. Since the porosity of the treatment can be estimated before a specific design, it is then possible to identify the minimum length L_{\min} of the treatment to meet this absorption target. An example is provided in §4. Alternatively, for a given treatment with thickness L and static limit $K_e(0)$, equation (3.5) indicates the levels of the absorption spectrum that can be achieved with this treatment. Furthermore, the

importance of the $1/\omega^2$ weighting terms in equation (3.5) should be emphasized. It implies that absorbing lower frequencies will require increasingly larger treatments. In other words, if the target absorption α is close to one at low frequencies, L_{\min} will be particularly large.

The above sum rules are readily extended to the case of a multilayer composite material. In the static limit, the effective bulk modulus of the stratified medium can be obtained from those of the individual layers via a first-order homogenization scheme [39,40]. This analysis directly results in the following sum rules:

$$\frac{2c_0}{\pi} \int_0^\infty \frac{1}{\omega^2} \text{Re}[\eta(\omega)] d\omega = \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right], \quad (3.7a)$$

$$\frac{c_0}{\pi} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right] \quad (3.7b)$$

and

$$\frac{c_0}{2\pi} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right], \quad (3.7c)$$

in which $K_{e,i}$, L_i are the effective bulk modulus and length of the i th layer, respectively. With equation (3.6), the right-hand sides of equations (3.7) can be further evaluated as

$$\sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right] \leq \gamma \phi_{\max} L, \quad (3.8)$$

where ϕ_{\max} is the maximum air filling ratio among all the layers, and $L = \sum_i L_i$ is the total length of the material.

(c) Soft-boundary reflection problem

We now consider the reflection problem depicted in figure 1b where a pressure-release boundary condition ($p = 0$) is used at $x = 0$. The relevant response functions are then

$$\zeta(\omega) = -\frac{t_{12}}{t_{22}} = -i \frac{z_e}{z_0} \tan(k_e L) \quad \text{or} \quad \eta(\omega) = -\frac{t_{22}}{t_{12}} = i \frac{z_0}{z_e} \cot(k_e L) \quad (3.9a)$$

and

$$R(\omega) = \frac{t_{12} + t_{22}}{t_{12} - t_{22}} = \frac{z_e - iz_0 \cot(k_e L)}{z_e + iz_0 \cot(k_e L)}. \quad (3.9b)$$

In contrast to the rigid-boundary reflection problem, the static limits of ζ , η and R depend on the normalized steady flow resistance of the layer (see appendix B for the derivation), defined by

$$\xi = \frac{L\sigma}{z_0}, \quad (3.10)$$

where σ is the steady flow resistivity and $L\sigma$ measures the pressure loss caused by a low-speed steady flow across the layer. When in-series elements within the waveguide are considered, the steady flow resistivity can be expressed by $\sigma = (C_1 \mu_0)/(r_H^2 \phi)$. The positive real constant C_1 is determined by the geometry of the elements of the material, e.g. for a circular tube $C_1 = 8$ while for a two-dimensional slit $C_1 = 3$ (see pp. 58–59 in [41]). The dynamic viscosity of air is denoted μ_0 and $r_H = 2\bar{A}_c/\bar{P}_c$ is a typical length scale of the elements. \bar{A}_c and \bar{P}_c represent the averaged values of the area and perimeter of the cross sections of all the elements of the material, respectively. When all these have identical cross sections, r_H is one-half of the hydraulic diameter (e.g. r_H is the radius for a circular cross section) and is then the half-width of a two-dimensional slit.

The $n = 1$ sum rule (2.11) is applicable only when the layer is either non-resistive ($\xi \rightarrow 0$) or highly resistive ($\xi \rightarrow \infty$). Apart from these two cases, the Herglotz functions H_1 and H_2 approach to constants in the static and dynamic limits (i.e. a_0 and b_0 are the leading order terms in the asymptotic expansions as shown in appendix B), and thus, no sum rule is available. However, it is still worth mentioning that, in the special case that the steady flow resistance of the layer matches the characteristic impedance of air (i.e. $\xi = 1$), the static limit of the reflection coefficient

vanishes, i.e. $R(0) = 0$ (see the asymptotic expansion of $R(\omega)$ in appendix B). In other words, the specific length $L = z_0/\sigma$ leads to perfect absorption in the static limit for any given material. This feature of soft-boundary reflection problem enables designs of space-saving absorbers in the ultra-low-frequency range. In [47], a specific design has been proposed and validated experimentally.

In the next two subsections, we focus on the two distinct cases $\xi \rightarrow 0$ and $\xi \rightarrow \infty$ in which sum rules of the system are available. When the layer is non-resistive ($\xi \rightarrow 0$), the static limit $R(0) = -1$, and the response of the system is of dipolar nature [40,44–46]. In the opposite, when the layer is highly resistive ($\xi \rightarrow \infty$), $R(0) = 1$ and a monopolar-type response is recovered.

(i) Non-resistive material: $\xi \rightarrow 0$

First, we consider a non-resistive material. In contrast to the rigid-boundary reflection problem, the impedance ζ is used to construct the Herglotz function H_1 in equation (2.6b) because $\zeta(0) = O(\omega)$ in the static limit, which provides the a_1 term for the $n = 1$ sum rule, whereas $\eta(0)$ blows up. The Herglotz function H_2 is built from equation (2.7a) with the reflection coefficient R and the Blaschke product $-B$ instead of B . These Herglotz functions H_1 and H_2 lead to the following sum rules:

$$\frac{2c_0}{\pi} \frac{\rho_0}{\text{Re}[\rho_e(0)]} \int_0^\infty \frac{1}{\omega^2} \text{Re}[\zeta(\omega)] d\omega = L \quad (3.11a)$$

and

$$\frac{c_0}{\pi} \frac{\rho_0}{\text{Re}[\rho_e(0)]} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq L. \quad (3.11b)$$

Moreover, equation (3.11b) can be rearranged to obtain a sum rule for the absorption coefficient

$$\frac{c_0}{2\pi} \frac{\rho_0}{\text{Re}[\rho_e(0)]} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq L. \quad (3.12)$$

Due to the dipolar nature of the system, the real part of the effective density, $\text{Re}[\rho_e]$, controls the above sum rules. Its static limit is given by

$$\frac{\text{Re}[\rho_e(0)]}{\rho_0} = \frac{C_2}{\phi}, \quad (3.13)$$

where the positive real constant C_2 depends on the cross-sectional geometry of the components of the material. When a material composed of in-series elements along the waveguide is considered, C_2 varies between 1.2 and 1.44 for commonly used shapes (see p. 64 in [41]).

The sum rules equations (3.11) and (3.12) can be generalized to a multilayer material through the homogenization theory [39,40], which yields

$$\frac{2c_0}{\pi} \int_0^\infty \frac{1}{\omega^2} \text{Re}[\zeta(\omega)] d\omega = \sum_i \left\{ \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} L_i \right\}, \quad (3.14a)$$

$$\frac{c_0}{\pi} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq \sum_i \left\{ \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} L_i \right\} \quad (3.14b)$$

and

$$\frac{c_0}{2\pi} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq \sum_i \left\{ \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} L_i \right\}, \quad (3.14c)$$

where $\rho_{e,i}$ is the effective density of the i -th layer with length L_i . With equation (3.13), the right-hand sides of equations (3.14) can be written as

$$\sum_i \left\{ \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} L_i \right\} \leq L \left(\frac{C_2}{\phi} \right)_{\max}, \quad (3.15)$$

which indicates that, to involve the total length $L = \sum_i L_i$ in the sum rules (3.14) explicitly, the maximum value of C_2/ϕ among all the layers should be evaluated.

(ii) Highly resistive material: $\xi \rightarrow \infty$

Second, a highly resistive material is considered. Similarly to the reflection problem with a rigid wall, the surface admittance η and reflection coefficient R are the relevant response functions. Correspondingly, the Herglotz functions H_1 and H_2 from equation (2.6a) and equation (2.7a) with Blaschke product B in equation (2.8) are employed to yield the following $n = 1$ sum rules:

$$\frac{6c_0}{\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{1}{\omega^2} \operatorname{Re}[\eta(\omega)] d\omega = L \quad (3.16a)$$

and

$$\frac{3c_0}{\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq L. \quad (3.16b)$$

The corresponding sum rule for the absorption coefficient is

$$\frac{3c_0}{2\pi} \frac{K_e(0)}{K_0} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq L. \quad (3.17)$$

Note that the coefficients on the left-hand sides of equations (3.16) and (3.17) are three times larger than those in the sum rules in equations (3.4) and (3.5). This factor three arises from the first-order terms in ω in the low-frequency expansions of $\tan(k_e L)/z_e$ in the rigid-boundary reflection problem and of $\cot(k_e L)/z_e$ in the soft-boundary reflection problem, respectively (see equations (3.3) and (3.9)). They are thus intrinsically related to the boundary condition, either rigid wall or pressure release boundary conditions.

For the multilayer case, the above sum rules can be generalized to

$$\frac{6c_0}{\pi} \int_0^\infty \frac{1}{\omega^2} \operatorname{Re}[\eta(\omega)] d\omega = \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right], \quad (3.18a)$$

$$\frac{3c_0}{\pi} \int_0^\infty \frac{-1}{\omega^2} \ln|R(\omega)| d\omega \leq \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right] \quad (3.18b)$$

and

$$\frac{3c_0}{2\pi} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq \sum_i \left[\frac{K_0}{K_{e,i}(0)} L_i \right], \quad (3.18c)$$

where $K_{e,i}$ is the effective bulk modulus of the i th layer with length L_i . Similarly to the rigid-boundary reflection problem, the right-hand sides of equations (3.18) could be evaluated by equation (3.8).

(d) Transmission problem

For a non-reciprocal and asymmetric system, the reflection and transmission coefficients in the transmission problem are given by

$$R^-(\omega) = \frac{-t_{11} - t_{12} + t_{21} + t_{22}}{t_{11} - t_{12} - t_{21} + t_{22}} = \frac{i \tan(k_e L)[(z_0/z_e) - (z_e/z_0)]}{2 - i \tan(k_e L)[(z_0/z_e) + (z_e/z_0)]}, \quad (3.19a)$$

$$T^-(\omega) = \frac{2e^{-ik_0 L}(t_{11}t_{22} - t_{12}t_{21})}{t_{11} - t_{12} - t_{21} + t_{22}} = \frac{2e^{-ik_0 L}}{2 \cos(k_e L) - i \sin(k_e L)[(z_0/z_e) + (z_e/z_0)]}, \quad (3.19b)$$

$$R^+(\omega) = \frac{t_{11} - t_{12} + t_{21} - t_{22}}{t_{11} - t_{12} - t_{21} + t_{22}} = \frac{i \tan(k_e L)[(z_0/z_e) - (z_e/z_0)]}{2 - i \tan(k_e L)[(z_0/z_e) + (z_e/z_0)]} \quad (3.19c)$$

$$\text{and} \quad T^+(\omega) = \frac{2e^{-ik_0 L}}{t_{11} - t_{12} - t_{21} + t_{22}} = \frac{2e^{-ik_0 L}}{2 \cos(k_e L) - i \sin(k_e L)[(z_0/z_e) + (z_e/z_0)]}, \quad (3.19d)$$

in which the superscripts $-$ and $+$ refer to the surface $x = 0$ and $x = L$, respectively. In the case shown in figure 1c, we consider a reciprocal system, $t_{11}t_{22} - t_{12}t_{21} = 1$ and thus $T^-(\omega) = T^+(\omega) \equiv T(\omega)$. Moreover, for a mirror symmetric single layer, $R^-(\omega) = R^+(\omega) \equiv R(\omega)$, i.e. $t_{11} = t_{22}$. Similarly to the soft-boundary reflection problem, the static limits of both $T(\omega)$ and $R(\omega)$ depend on the steady flow resistance ξ .

When the material is non-resistive ($\xi \rightarrow 0$), $T(0) = 1$ and $R(0) = 0$. As a result, the $n = 1$ sum rule (2.11) is available only for Herglotz functions constructed from the transmission coefficient $T(\omega)$. By contrast, when the material is highly resistive ($\xi \rightarrow \infty$), $T(0) = 0$ and $R(0) = 1$. Consequently, the sum rules with the reflection coefficient $R(\omega)$ can be derived from equation (2.11). Similarly to the soft-boundary reflection problem, in other cases when ξ possesses the moderate values, the Herglotz functions H_1 and H_2 do not lead to sum rules. However, in the special case where $\xi = 2$, $R(0) = T(0) = 1/2$, so that the absorption coefficient, defined by $\alpha = 1 - |R|^2 - |T|^2$, reaches its maximum value in the static limit, i.e. $\alpha = 1/2$. Note that the maximum absorption coefficient of a point scatterer in the transmission problem is $1/2$ at any frequency. Proofs are given in [44,46].

(i) Non-resistive material: $\xi \rightarrow 0$

We first consider the case of a non-resistive material, $\xi \rightarrow 0$. By an analogy with the surface admittance in the reflection problem, a bilinear transformation, which maps the unit disc to the closed upper half plane (see p. 131 in [48]), is used to construct the Herglotz function H_1 with the transmission coefficient

$$H_1(\omega) = i \frac{1 - T(\omega)}{1 + T(\omega)}. \quad (3.20)$$

The other Herglotz function, H_2 , is derived from equation (2.7*b*), with the Blaschke product B in equation (2.8). It follows that the corresponding sum rules are

$$\frac{8c_0}{\pi} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\}^{-1} \int_0^\infty \frac{1}{\omega^2} \text{Re} \left[\frac{1 - T(\omega)}{1 + T(\omega)} \right] d\omega = L \quad (3.21a)$$

and

$$\frac{4c_0}{\pi} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \frac{c_0}{c_\infty} \right\}^{-1} \int_0^\infty \frac{-1}{\omega^2} \ln|T(\omega)| d\omega \leq L. \quad (3.21b)$$

In the static limit, the layer of composite material reduces to a point scatterer and thus the transmission problem can be interpreted as a linear combination of a monopolar- and a dipolar-type reflection problems [40,44–46,49]. As a consequence, both the effective bulk modulus and the effective density are involved in equations (3.21). Note that the dynamic behaviour of the system contributes to the second sum rule (3.21*b*). c_∞ refers to the dynamic limit of the effective sound speed, and can be evaluated by

$$\frac{c_\infty}{c_0} = \frac{1}{\sqrt{\tau_\infty}}, \quad (3.22)$$

where $\tau_\infty \geq 1$ is the tortuosity of the material (see p. 67 of [41]). According to equations (3.6) and (3.13),

$$\frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} = \gamma\phi + \frac{C_2}{\phi}. \quad (3.23)$$

For any $\phi \geq 0$, it can be found that $\gamma\phi + C_2/\phi \geq 2\sqrt{\gamma C_2}$. In [50] and p. 82 of [41], C_2 is alternatively denoted as the static tortuosity, τ_0 , of the material. Moreover, according to [51], $\tau_0 \geq \tau_\infty$. Therefore, $\gamma\phi + C_2/\phi \geq 2\sqrt{\gamma\tau_\infty} > 2\sqrt{\tau_\infty}$, and it follows that

$$\frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} > 2 \frac{c_0}{c_\infty} > 2, \quad (3.24)$$

which ensures that the left-hand sides of equations (3.21) are positive. When the tortuosity τ_∞ is hard to evaluate before a specific design, $\tau_\infty = 1$ could be used, which results in a lower bound of L when applying the sum rule (3.21*b*).

To obtain the constraints on the transmission loss and the absorption coefficient, the relations $\text{TL}(\omega) = -20 \log_{10}|T(\omega)|$ and $1 - \alpha(\omega) = |T(\omega)|^2 + |R(\omega)|^2 \geq |T(\omega)|^2$ can be combined with

equation (3.21b) to write

$$\frac{c_0 \ln 10}{5\pi} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \frac{c_0}{c_\infty} \right\}^{-1} \int_0^\infty \frac{\text{TL}(\omega)}{\omega^2} d\omega \leq L \quad (3.25a)$$

and

$$\frac{2c_0}{\pi} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \frac{c_0}{c_\infty} \right\}^{-1} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq L. \quad (3.25b)$$

We then consider a multilayer material. The reciprocity of the system still holds, so that $T^-(\omega) = T^+(\omega)$, but this system can then become asymmetric and consequently $R^-(\omega)$ and $R^+(\omega)$ might be different. In the low-frequency regime, the asymmetric part can be modelled by introducing the Willis coupling constant [52–54], when the homogenization method is applied. The coupling constant appears as a second-order term of ω , and thus, the system falls back to a symmetric one in the static limit. However, the dynamic limit has to be derived from the transfer matrix of the entire system. These considerations provide a generalization of the sum rules in equations (3.21) and (3.25)

$$\frac{8c_0}{\pi} \int_0^\infty \frac{1}{\omega^2} \text{Re} \left[\frac{1 - T(\omega)}{1 + T(\omega)} \right] d\omega = \sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \right\}, \quad (3.26a)$$

$$\frac{4c_0}{\pi} \int_0^\infty \frac{-1}{\omega^2} \ln |T(\omega)| d\omega \leq \sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \frac{c_0}{c_{\infty,i}} \right\}, \quad (3.26b)$$

$$\frac{c_0 \ln 10}{5\pi} \int_0^\infty \frac{\text{TL}(\omega)}{\omega^2} d\omega \leq \sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \frac{c_0}{c_{\infty,i}} \right\} \quad (3.26c)$$

and
$$\frac{2c_0}{\pi} \int_0^\infty \frac{-1}{\omega^2} \ln[1 - \alpha(\omega)] d\omega \leq \sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \frac{c_0}{c_{\infty,i}} \right\}, \quad (3.26d)$$

in which $K_{e,i}$, $\rho_{e,i}$ are the effective bulk modulus and density, respectively, $c_{\infty,i}$ denotes the dynamic limit of the effective sound speed of the i th layer with length L_i . Note that

$$\sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \right\} \leq L \left(\gamma\phi + \frac{C_2}{\phi} - 2 \right)_{\max} \quad (3.27)$$

and

$$\sum_i L_i \left\{ \frac{K_0}{K_{e,i}(0)} + \frac{\text{Re}[\rho_{e,i}(0)]}{\rho_0} - 2 \frac{c_0}{c_{\infty,i}} \right\} \leq L \left(\gamma\phi + \frac{C_2}{\phi} - 2\sqrt{\tau_\infty} \right)_{\max}, \quad (3.28)$$

where $L = \sum_i L_i$ is the total length of the material.

(ii) Highly resistive material: $\xi \rightarrow \infty$

For the transmission problem with a highly resistive material, the relevant response functions are the surface admittance

$$\eta^\pm(\omega) = \frac{1 - R^\pm(\omega)}{1 + R^\pm(\omega)}, \quad (3.29)$$

and the reflection coefficients $R^\pm(\omega)$. Since the material is considered mirror symmetric, both η and R are identical on either side. It follows that the Herglotz functions H_1 and H_2 are defined in equations (2.6a) and (2.7a) with the Blaschke product in equation (2.8).

It is found that the sum rules in this situation take exactly the same form as in equations (3.16) and (3.17) in the soft-boundary reflection problem. The details of the asymptotic expansions of H_1 and H_2 are available in appendix B. A multilayer generalization directly results in sum rules expressed by equations (3.18), considering that the multilayer material still has the mirror symmetry in the static limit because the Willis material falls back to be symmetric.

4. Applications of the sum rules

The derived sum rules relate the target spectrum of the relevant response functions to the static/dynamic limits of the equivalent parameters as well as the total length of the layer (or the length of each layer). Thus, the required minimum total length of the treatment can be predicted before any specific design provided these limits are well predicted by *a priori* estimates on the porosity and the tortuosity. Two specific examples are given in this section to show the practical applications of the sum rules.

The first example deals with the sum rule of a broadband absorber in the rigid-boundary reflection problem: when a target absorption spectrum is provided, the sum rule can be used to evaluate the required minimum length of the absorber. Apart from the rigid-boundary reflection problem, all the sum rules derived in the soft-boundary reflection problem and the transmission problem depend on the steady flow resistance $\xi = L\sigma/z_0$. Therefore, an estimation of ξ should be made to select the suitable sum rule. After the prediction of the minimum length, the value of ξ should be examined. However, some cases exist for which ξ can be easily fixed. These cases are of great importance in practical applications as well. The second example of this section focuses on the transmission problem of a ring-shaped muffler in parallel of the circular waveguide with grazing incident wave. When the viscothermal losses in the main duct can be neglected compared with those induced by the muffler, this scenario provides a case for which the effective layer is perfectly non-resistive, i.e. $\xi = 0$. It is shown that, when a target transmission loss spectrum is provided, the required minimum volume of the muffler can be predicted by the sum rule.

It should be noted that due to the weighting factor $1/\omega^2$ in all the derived sum rules, the low-frequency spectrum of the target response function has a dramatic effect on the predicted total length of the system. However, without a specific design, there is usually a lack of details on the target function near the static limit. In both of the following examples, we fix this problem by interpolating the target spectra, in which the basis functions are properly chosen to preserve the asymptotic behaviours of the corresponding target functions in the static limit.

(a) Minimum length of a broadband absorber for a target absorption spectrum in the rigid-boundary reflection problem

As illustrated in figure 1a, we assume that the layer of material is composed of in-series elements or laminates fixed within the waveguide. According to the sum rule (3.5), the minimum length can be predicted by

$$L_{\min} = \frac{-c_0}{4\pi^2\gamma\phi} \int_{f_1}^{f_2} \frac{\ln[1 - \alpha_T(f)]}{f^2} df, \quad (4.1)$$

where α_T is the target spectrum, given within the frequency range $[f_1, f_2]$. From equation (4.1), it is clear that the integral blows up if we choose $\alpha_T = 1$ from f_1 to f_2 . This implies that, in principle, broadband-perfect absorption cannot be realized by an absorber with finite length. In fact, if perfect absorption is achieved in a finite frequency range, $\alpha = 1$ is guaranteed at any frequency. Provided that the reflection coefficient $R(\omega)$ is a constant zero within a finite interval, then $R(\omega)$ equals the same constant in the whole complex ω plane, due to the Cauchy–Riemann conditions [48,55] satisfied by real and imaginary parts of a holomorphic function. However, this does not prevent perfect absorption occurring at discrete frequencies. Moreover, when perfect absorption is achieved, α should be less than unity in the bands between these discrete frequencies [28].

In practical applications, the target spectrum is usually presented in octave bands or 1/3 octave bands. By contrast, a simplified target spectrum is employed here. We assume that the absorption coefficient is larger than 0.3, 0.6 and 0.9 at 100 Hz, 500 Hz and 2500 Hz, respectively. According to equation (4.1), it seems that the most efficient absorber (which has the minimum length) is achieved when α_T is exactly zero outside the considered frequency band. However, this idealized target spectrum is not physically accessible. Particularly, the absorption coefficient possesses the quadratic nature in the low-frequency range [27], i.e. $\alpha \sim (k_0L)^2$. Thus, in this case, we assume

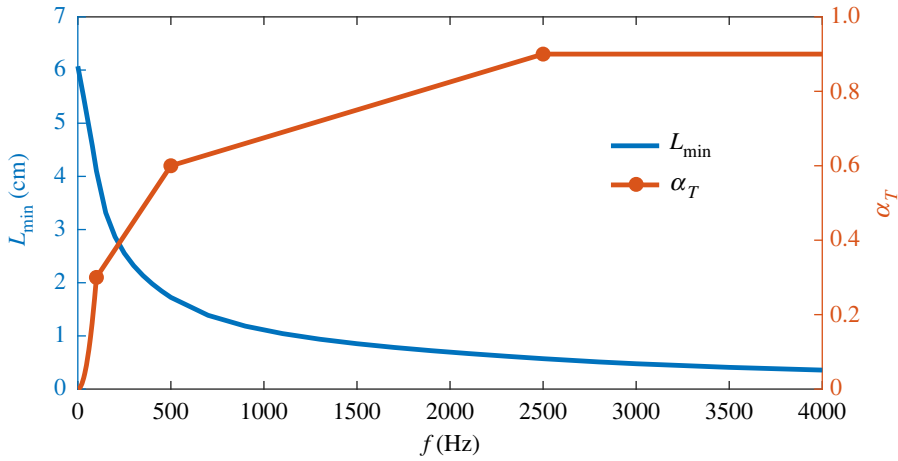


Figure 2. Target absorption spectrum α_T and the minimum length L_{\min} evaluated from the contribution of α_T within the frequency range $[f, \infty)$. (Online version in colour.)

$\alpha_T = 30(f/10^3)^2$ when $f \leq 100$ Hz. For a higher frequency range, the target spectrum α_T is derived from interpolations

$$\alpha_T(f) = \begin{cases} 30 \left(\frac{f}{10^3}\right)^2 & f \leq 100 \text{ Hz} \\ 0.75 \left(\frac{f}{10^3}\right) + 0.225 & 100 \text{ Hz} \leq f \leq 500 \text{ Hz} \\ 0.15 \left(\frac{f}{10^3}\right) + 0.525 & 500 \text{ Hz} \leq f \leq 2500 \text{ Hz} \\ 0.9 & f \geq 2500 \text{ Hz} \end{cases} \quad (4.2)$$

With this target α_T , the minimum length is predicted from equation (4.1), which results in $L_{\min} = 6.1$ cm. The effect of the low-frequency spectrum of α_T on L_{\min} is further studied through calculation of equation (4.1) within the frequency band $[f, \infty)$. The results are illustrated in figure 2. It is found that, although α_T is less than 0.3 below 100 Hz, the low-frequency spectrum still has a significant effect on L_{\min} . When we start the sum-rule integration at $f = 100$ Hz, L_{\min} reduces to 4.1 cm, which is 30% lower than that predicted by the full spectrum. By contrast, the high-frequency spectrum (although α_T is close to one) has much less effect on L_{\min} . If we set $\alpha_T = 0$ for $f \geq 2500$ Hz, the predicted L_{\min} is merely 10% less than 6.1 cm.

In the above calculations, the porosity $\phi = 1$ is used. For commonly used porous materials, ϕ is very close to unity as shown in [56]. For other in-series structures used within the waveguide, when ϕ is difficult to evaluate, $\phi = 1$ can be used as well to predict the lower bound of L_{\min} .

(b) Minimum volume of a ring-shaped muffler for a target transmission loss spectrum with grazing incident wave

We consider the transmission problem of a ring-shaped muffler with length L in the axial direction and thickness H in the radial direction, as illustrated in figure 3. This muffler is set in parallel of a circular waveguide whose radius is R_d . It is modelled as an equivalent material, with a rigid outer boundary at $r = R_d + H$. The muffler is assumed to be locally reacting so that there is only unidimensional radial wave within the equivalent material. Then the viscothermal losses within the material can be modelled by complex-valued effective parameters: the effective density and bulk modulus of the material are denoted by ρ_m and K_m , respectively. By contrast, losses in the main waveguide are neglected.

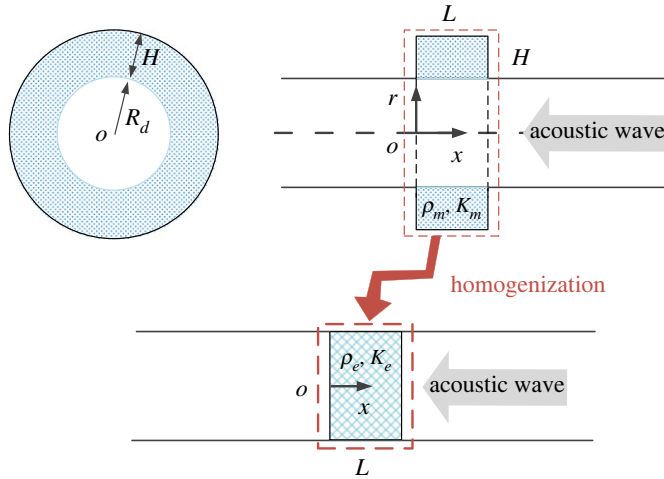


Figure 3. Effective layer introduced by a ring-shaped muffler in parallel of a circular waveguide. The muffler is filled with a uniform material whose effective density and bulk modulus are ρ_m and K_m , respectively. Via the homogenization method, the acoustic performance of the muffler can be well described by the layer with effective parameters ρ_e and K_e . (Online version in colour.)

Under these assumptions, the effect of the muffler on the wave propagation in the waveguide can be well described by introducing an effective layer with the same axial length L . Then the static and dynamic limits of the effective parameters of the layer could be evaluated and the sum rules in the unidimensional transmission problem can be directly applied. Via the zeroth-order homogenization scheme [39,40,54], the static limit of the effective density of the layer ($\rho_e(0)$) reduces to that of the medium within the waveguide, whereas the static limit of the effective bulk modulus ($K_e(0)$) can be expressed by that of each component weighted by the volume. It follows that

$$\frac{\rho_e(0)}{\rho_0} = 1 \quad (4.3a)$$

and

$$\frac{K_e(0)}{K_0} = 1 + \frac{V}{V_e} \frac{K_m(0)}{K_0} = 1 + \frac{V}{V_e} \frac{1}{\gamma \phi_m}, \quad (4.3b)$$

where $V_e = \pi R_d^2 L$ is the volume of the effective layer, $V = \pi(2R_d H + H^2)L$ is the volume of the ring-shaped material and ϕ_m is the porosity of the material. Another derivation of the acoustical response of the system via the transfer matrix method is provided in the electronic supplementary material. Because the imaginary part of the effective density is proportional to the steady flow resistivity in the static limit, equation (4.3a) implies that the effective layer is perfectly non-resistive ($\xi = 0$). Therefore, the sum rule (3.25a) can be directly used to evaluate the minimum length of the effective layer with the target transmission loss spectrum TL_T given. Note that under a grazing incident wave, c_∞ (or τ_∞) of the effective layer is usually difficult to evaluate. Here, $c_\infty = c_0$ (i.e. $\tau_\infty = 1$) is used as an approximation. With $\rho_e(0)$, $K_e(0)$ and c_∞ provided, the sum rule (3.25a) can be rewritten as

$$V_{\min} = \frac{c_0 R_d^2 \ln 10}{10\pi \gamma \phi_m} \int_{f_1}^{f_2} \frac{TL_T(f)}{f^2} df, \quad (4.4)$$

where $V_{\min} = \min[\pi(2R_d H + H^2)L]$ denotes the minimum volume of the ring-shaped muffler.

We assume that the target transmission loss TL_T is larger than 2 dB, 12 dB, 8 dB and 2 dB at 250 Hz, 500 Hz, 1000 Hz and 2000 Hz, respectively. The radius of the circular waveguide R_d is assumed to be 5 cm. Similarly to the previous example, the TL_T spectrum under 250 Hz is

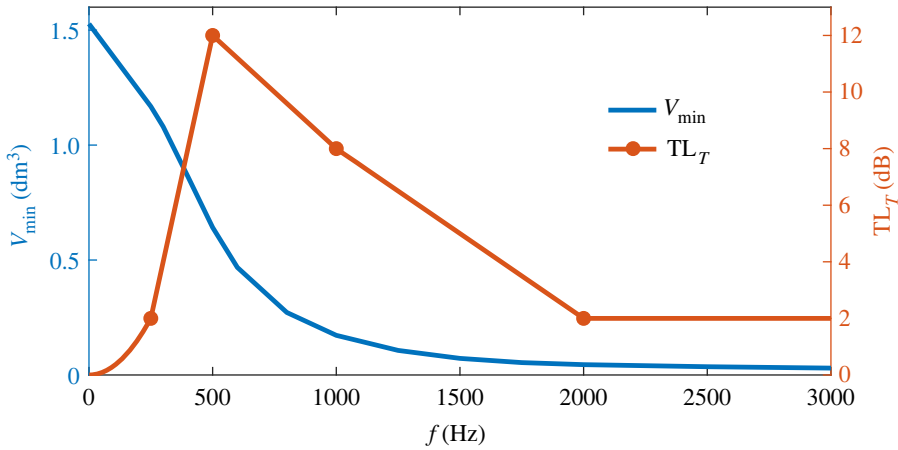


Figure 4. Target transmission loss spectrum TL_T and the minimum volume V_{\min} evaluated from the contribution of TL_T within the frequency range $[f, \infty)$. (Online version in colour.)

important for the prediction of the minimum volume. Since $|T(\omega)|^2 = 1 - A\omega^2 + o(\omega^2)$ at low frequency, where A is a positive coefficient, the transmission loss also possesses the quadratic dependence on frequency near the static limit. Thus, the target spectrum TL_T is expressed by the following function:

$$TL_T(f) = \begin{cases} 32 \left(\frac{f}{10^3}\right)^2 & f \leq 250 \text{ Hz} \\ 40 \left(\frac{f}{10^3}\right) - 8 & 250 \text{ Hz} \leq f \leq 500 \text{ Hz} \\ -8 \left(\frac{f}{10^3}\right) + 16 & 500 \text{ Hz} \leq f \leq 1000 \text{ Hz} \\ -6 \left(\frac{f}{10^3}\right) + 14 & 1000 \text{ Hz} \leq f \leq 2000 \text{ Hz} \\ 2 & f \geq 2000 \text{ Hz} \end{cases} \quad (4.5)$$

With TL_T provided, the effect of the given spectrum on the prediction of V_{\min} can be analysed. Results of parametric studies are illustrated in figure 4, where $V_{\min}(f)$ is calculated by equation (4.4) within the frequency band $[f, \infty)$, and $\phi_m = 1$. If TL_T is approximated by zero when $f \leq 250$ Hz, the underestimation of V_{\min} is nearly 25%. In the opposite, the high-frequency spectrum of TL_T (e.g. $f > 1000$ Hz) does not contribute much to V_{\min} , as one would expect.

5. Conclusion

With the sum rules in lossy acoustical systems, the minimum size of a passive metamaterial is readily predicted from its target broadband response, which is of primary importance for guiding the design of subwavelength acoustical treatments.

In this work, the theory of Herglotz function is revisited and applied systematically to derive sum rules for unidimensional acoustical scattering problems, including the rigid/soft-boundary reflection and the transmission problems. The derivations in all the cases involve the following standard steps: (a) Identify non-negative valued functions (i.e. Herglotz functions) from the relevant response functions, considering either admittance/impedance passivity or scattering passivity of the system. (b) Analyse the asymptotic behaviours of these Herglotz functions in the static and dynamic limits. (c) Derive the sum rules from the coefficients of the asymptotic expansions.

In the rigid-boundary reflection problem, the recently derived constraint for the absorption coefficient in [18] is recovered. Moreover, a new constraint for the surface admittance is obtained.

In the soft-boundary reflection and the transmission problems, the asymptotic behaviours of Herglotz functions depend on the steady flow resistance of the effective material, i.e. $\xi = L\sigma/z_0$. Consequently, sum rules in these two problems exist merely when $\xi \rightarrow 0$ or $\xi \rightarrow \infty$, corresponding to cases in which the material is either non-resistive or highly resistive, respectively. Besides, in special cases where $\xi = 1$ or 2 in the soft-boundary reflection or the transmission problem, the absorption coefficients can reach their maximum static-limit values, 1 or $1/2$, respectively, although the considered Herglotz functions do not admit sum rules.

Data accessibility. The data are provided in the electronic supplementary material [58].

Authors' contributions. Y.M.: conceptualization, investigation, methodology, validation, writing—original draft, writing—review and editing; V.R.-G.: conceptualization, investigation, methodology, validation, writing—review and editing; G.G.: conceptualization, investigation, methodology, validation, writing—review and editing; J.-P.G.: conceptualization, funding acquisition, investigation, methodology, project administration, validation, writing—review and editing; C.B.: validation, writing—review and editing; S.G.: validation, writing—review and editing; P.S.: conceptualization, validation, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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Appendix A. Static/dynamic limits of the effective parameters

The losses of the system arise from viscothermal effects near the no-slip and isothermal boundaries. Here, we fix the discussion on an in-series material inside the waveguide. When side-branch structures are included, the asymptotic expansions of the effective parameters of the material preserve the same dependence on ω , but the coefficients should be interpreted differently.

For in-series material with porosity ϕ , the asymptotic expansions can be directly evaluated [41]. In the static limit $\omega \rightarrow 0$,

$$\frac{\rho_e}{\rho_0} \rightarrow \frac{1}{\phi} \left(i \frac{C_1}{S_H^2} + C_2 \right) + o(1) = i \frac{\sigma}{\rho_0 \omega} + \frac{C_2}{\phi} + o(1), \quad (\text{A } 1a)$$

$$\frac{c_e}{c_0} \rightarrow (1 - i)C_3 S_H + o(\sqrt{\omega}), \quad (\text{A } 1b)$$

$$\frac{z_e}{z_0} \rightarrow \frac{(1 + i)C_1 C_3}{\phi S_H} + o\left(\frac{1}{\sqrt{\omega}}\right) \quad (\text{A } 1c)$$

and
$$\frac{K_e}{K_0} \rightarrow \frac{2C_1 C_3^2}{\phi} + o(1) = \frac{1}{\phi \gamma} + o(1), \quad (\text{A } 1d)$$

where C_1 , C_2 and C_3 are positive real constants (see pp. 55–57 of [41]), $S_H = r_H \sqrt{\omega/\nu_0}$ is the shear number of the cross section [57], $\nu_0 = \mu_0/\rho_0$ is the kinematic viscosity of air. Note that, from equation (A 1a), the static limit of the density has a diverging imaginary part $\sim 1/\omega$. The real part tends to a constant and is written $C_2/\phi \equiv \text{Re}[\rho_e(0)]/\rho_0$. According to equation (A 1d), the static limit of the bulk modulus is real-valued and is written $2C_1 C_3^2/\phi = 1/(\phi \gamma) \equiv K_e(0)/K_0$.

Furthermore, in the dynamic limit $\omega \rightarrow \infty$,

$$\frac{\rho_e}{\rho_0} \rightarrow \frac{\tau_\infty}{\phi} \left[1 + (1 + i) \frac{D_1}{S_H} \right] + o\left(\frac{1}{\sqrt{\omega}}\right), \quad (\text{A } 2a)$$

$$\frac{c_e}{c_0} \rightarrow \frac{1}{\sqrt{\tau_\infty}} \left[1 - (1 + i) \frac{D_2}{S_H} \right] + o\left(\frac{1}{\sqrt{\omega}}\right), \quad (\text{A } 2b)$$

$$\frac{z_e}{z_0} \rightarrow \frac{\sqrt{\tau_\infty}}{\phi} \left[1 + \frac{(1 + i)(D_1 - D_2)}{S_H} \right] + o\left(\frac{1}{\sqrt{\omega}}\right) \quad (\text{A } 2c)$$

and
$$\frac{K_e}{K_0} \rightarrow \frac{1}{\phi} \left[1 + \frac{(1 + i)(D_1 - 2D_2)}{S_H} \right] + o\left(\frac{1}{\sqrt{\omega}}\right), \quad (\text{A } 2d)$$

where D_1 and D_2 are positive real constants (see p. 71 of [41]), τ_∞ is the dynamic tortuosity. It is found that the dynamic limits of all these effective parameters are positive constants. Here, we denote $1/\sqrt{\tau_\infty} \equiv c_\infty/c_0$ and $\sqrt{\tau_\infty}/\phi \equiv z_\infty/z_0$.

Appendix B. Asymptotic performances of the Herglotz functions

(a) Rigid-boundary reflection problem

In the reflection problem with a rigid boundary, the relevant response functions are the surface admittance $\eta(\omega)$ and the reflection coefficient $R(\omega)$. Their asymptotic behaviours are given by

$$\eta(\omega) = \begin{cases} -\frac{i\omega L}{c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \rightarrow 0 \\ \frac{z_0}{z_\infty} + o(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 1)$$

and

$$R(\omega) = \begin{cases} 1 + \frac{2i\omega L}{c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \rightarrow 0 \\ \frac{z_\infty - z_0}{z_\infty + z_0} + o(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 2)$$

The following two Herglotz functions are introduced: $H_1(\omega) = i\eta(\omega)$ and $H_2(\omega) = -i \log[R(\omega)B(\omega)]$. Their asymptotic expansions are written as

$$H_1(\omega) = \begin{cases} \frac{\omega L}{c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \rightarrow 0 \\ O(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 3)$$

and

$$H_2(\omega) = \begin{cases} \omega \left[\frac{2L}{c_0} \frac{K_0}{K_e(0)} + \sum_n \text{Im} \left(\frac{1}{\omega_n} \right) \right] + o(\omega) \leq \frac{2\omega L}{c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \rightarrow 0 \\ O(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 4)$$

Note that the inequality in equation (B 4) is derived with $\text{Im}(\omega_n) > 0$. From the above expansions, it is found that, for H_1 :

$$a_1 = \frac{L}{c_0} \frac{K_0}{K_e(0)} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 5)$$

And for H_2 :

$$a_1 \leq \frac{L}{c_0} \frac{K_0}{K_e(0)} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 6)$$

(b) Soft-boundary reflection problem

In the soft-boundary reflection problem, the relevant response functions include the impedance $\zeta(\omega)$, the admittance $\eta(\omega)$ and the reflection coefficient $R(\omega)$. The asymptotic behaviours of these functions are as follows:

$$\zeta(\omega) = \begin{cases} \xi - \frac{i\omega L}{c_0} \left\{ \frac{\text{Re}[\rho_e(0)]}{\rho_0} - \frac{\xi^2 K_0}{3K_e(0)} \right\} + o(\omega), & \omega \rightarrow 0 \\ \frac{z_\infty}{z_0} + o(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 7)$$

$$\eta(\omega) = \begin{cases} \frac{1}{\xi} - \frac{i\omega L}{c_0} \left\{ \frac{K_0}{3K_e(0)} - \frac{\text{Re}[\rho_e(0)]}{\xi^2 \rho_0} \right\} + o(\omega), & \omega \rightarrow 0 \\ \frac{z_0}{z_\infty} + o(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 8)$$

and

$$R(\omega) = \begin{cases} \frac{\xi-1}{\xi+1} + \frac{2i\omega L}{c_0} \frac{(\xi^2/3) - (K_e(0)/K_0)(\text{Re}[\rho_e(0)]/\rho_0)}{(\xi+1)^2 (K_e(0)/K_0)} + o(\omega), & \omega \rightarrow 0 \\ \frac{z_\infty - z_0}{z_\infty + z_0} + o(1), & \omega \rightarrow \infty \end{cases} \quad (\text{B } 9)$$

in which $\xi = C_1 L v_0 / (\phi c_0 r_H^2) = L\sigma/z_0$.

(i) Non-resistive material: $\xi \rightarrow 0$

When $\xi \rightarrow 0$, the Herglotz functions $H_1(\omega) = i\zeta(\omega)$ and $H_2(\omega) = -i \log[-R(\omega)B(\omega)]$ are considered. Their asymptotic expansions are

$$H_1(\omega) = \begin{cases} \frac{\omega L}{c_0} \frac{\text{Re}[\rho_e(0)]}{\rho_0} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B } 10)$$

and

$$H_2(\omega) = \begin{cases} \omega \left\{ \frac{2L}{c_0} \frac{\text{Re}[\rho_e(0)]}{\rho_0} + \sum_n \text{Im} \left(\frac{1}{\omega_n} \right) \right\} + o(\omega) \leq \frac{2\omega L}{c_0} \frac{\text{Re}[\rho_e(0)]}{\rho_0} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases}. \quad (\text{B } 11)$$

It follows that, for H_1 :

$$a_1 = \frac{L}{c_0} \frac{\text{Re}[\rho_e(0)]}{\rho_0} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 12)$$

And for H_2 :

$$a_1 \leq \frac{2L}{c_0} \frac{\text{Re}[\rho_e(0)]}{\rho_0} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 13)$$

(ii) Highly resistive material: $\xi \rightarrow \infty$

When $\xi \rightarrow \infty$, the Herglotz functions $H_1(\omega) = i\eta(\omega)$, and $H_2(\omega) = -i \log[R(\omega)B(\omega)]$ are used. It follows that

$$H_1(\omega) = \begin{cases} \frac{\omega L}{3c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B } 14)$$

and

$$H_2(\omega) = \begin{cases} \omega \left[\frac{2L}{3c_0} \frac{K_0}{K_e(0)} + \sum_n \text{Im} \left(\frac{1}{\omega_n} \right) \right] + o(\omega) \leq \frac{2\omega L}{3c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases}. \quad (\text{B } 15)$$

Therefore, for H_1 :

$$a_1 = \frac{L}{3c_0} \frac{K_0}{K_e(0)} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 16)$$

And for H_2 :

$$a_1 \leq \frac{2L}{3c_0} \frac{K_0}{K_e(0)} \quad \text{and} \quad b_1 = 0. \quad (\text{B } 17)$$

(c) Transmission problem

In the transmission problem, $R^-(\omega) = R^+(\omega) \equiv R(\omega)$ and $T^-(\omega) = T^+(\omega) \equiv T(\omega)$ when we consider a symmetric single-layer material. The asymptotic behaviours of $R(\omega)$ and $T(\omega)$ are provided by

$$R(\omega) = \begin{cases} \frac{\xi}{\xi+2} + \frac{2i\omega L}{c_0} \frac{(\xi^2/3)+\xi+1-(K_e(0)/K_0)(\text{Re}[\rho_e(0)]/\rho_0)}{(\xi+2)^2(K_e(0)/K_0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ \frac{z_\infty - z_0}{z_\infty + z_0} + o(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B } 18)$$

and

$$T(\omega) = \begin{cases} \frac{2}{\xi+2} + \frac{2i\omega L}{c_0} \frac{1+\xi[(\xi/6)-(K_e(0)/K_0)+1]+(K_e(0)/K_0)(\text{Re}[\rho_e(0)]/\rho_0)-2}{(\xi+2)^2(K_e(0)/K_0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ o(1), & \omega \hat{\rightarrow} \infty \end{cases}. \quad (\text{B } 19)$$

(i) Non-resistive material: $\xi \rightarrow 0$

Consider a non-resistive material with $\xi \rightarrow 0$. The Herglotz functions $H_1(\omega) = i[1 - T(\omega)]/[1 + T(\omega)]$ and $H_2(\omega) = -i \log[T(\omega)B(\omega)]$ are used. Then the asymptotic expansions provide

$$H_1(\omega) = \begin{cases} \frac{\omega L}{4c_0} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B 20})$$

and

$$H_2(\omega) = \begin{cases} \omega \left\{ \frac{L}{2c_0} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\} + \sum_n \text{Im} \left(\frac{1}{\omega_n} \right) \right\} + o(\omega) \\ \leq \omega \left\{ \frac{L}{2c_0} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\} \right\} + o(\omega), & \omega \hat{\rightarrow} 0 \\ \frac{\omega L}{c_0} \left(\frac{c_\infty}{c_0} - 1 \right) + o(\omega), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B 21})$$

It can be found that, for H_1 :

$$a_1 = \frac{L}{4c_0} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\} \quad \text{and} \quad b_1 = 0. \quad (\text{B 22})$$

And for H_2 :

$$a_1 \leq \frac{L}{2c_0} \left\{ \frac{K_0}{K_e(0)} + \frac{\text{Re}[\rho_e(0)]}{\rho_0} - 2 \right\} \quad \text{and} \quad b_1 = \frac{L}{c_0} \left(\frac{c_\infty}{c_0} - 1 \right). \quad (\text{B 23})$$

(ii) Highly resistive material: $\xi \rightarrow \infty$

When the material is highly resistive ($\xi \rightarrow \infty$), Herglotz functions $H_1(\omega) = i\eta(\omega) = i[1 - R(\omega)]/[1 + R(\omega)]$ and $H_2(\omega) = -i \log[R(\omega)B(\omega)]$ are used. Their asymptotic expansions are

$$H_1(\omega) = \begin{cases} \frac{\omega L}{3c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B 24})$$

and

$$H_2(\omega) = \begin{cases} \omega \left[\frac{2L}{3c_0} \frac{K_0}{K_e(0)} + \sum_n \text{Im} \left(\frac{1}{\omega_n} \right) \right] + o(\omega) \leq \frac{2\omega L}{3c_0} \frac{K_0}{K_e(0)} + o(\omega), & \omega \hat{\rightarrow} 0 \\ O(1), & \omega \hat{\rightarrow} \infty \end{cases} \quad (\text{B 25})$$

For H_1 :

$$a_1 = \frac{L}{3c_0} \frac{K_0}{K_e(0)} \quad \text{and} \quad b_1 = 0. \quad (\text{B 26})$$

And for H_2 :

$$a_1 \leq \frac{2L}{3c_0} \frac{K_0}{K_e(0)} + o(\omega) \quad \text{and} \quad b_1 = 0. \quad (\text{B 27})$$

The asymptotic behaviours of H_1 and H_2 are exactly those in the soft-boundary reflection problem with $\xi \rightarrow \infty$.

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