INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

WILEY

INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

Intl. Trans. in Op. Res. 30 (2023) 2571–2594
DOI: 10.1111/itor.13178

Geometric compromise programming: application in portfolio selection

Francisco Salas-Molina*, David Pla-Santamaria, Maria Luisa Vercher-Ferrandiz and Ana Garcia-Bernabeu

Universitat Politècnica de València, Ferrándiz y Carbonell s/n, Alcoy (Alicante) 03801, Spain
E-mail: frasamo@upv.es [Salas-Molina]; dplasan@upv.es [Pla-Santamaria]; maverfer@upv.es [Vercher-Ferrandiz]; angarber@upv.es [Garcia-Bernabeu]

Received 2 September 2021; received in revised form 24 June 2022; accepted 2 July 2022

Abstract

Compromise programming (CP) aims to find solutions by minimising distances to an ideal point with maximum achievement which is usually infeasible. A common assumption in CP is that it is highly unlikely that the optimum decision will lie out of the bounds of the compromise set given by metrics $p = 1$ and $p = \infty$ of the Minkowski distance function. This assumption excludes the use of multiplicative functions as a measure of achievement. We propose geometric CP (GCP) to provide alternative solutions based on multiplicative functions to overcome this limitation. This methodology is an extension of CP that allows to incorporate the principle of limited compensability. An additional interesting feature of GCP is that, under reasonable assumptions, characterises extreme seekers' behaviour with non-concave utility functions (expressing no preference for any of the extremes). We discuss the practical implications of our approach and present three numerical illustrations in the context portfolio selection.

Keywords: multiple-objective programming; multiplicative utility functions; extreme seekers; compromise set

1. Introduction

The increasing complexity of financial decision making highlights the need of using efficient quantitative analysis techniques (Zopounidis and Doumpos, 2002). In portfolio selection, new lines of research have been developed to extend the classical mean-variance model by Markowitz (1952). One of these techniques is compromise programming (CP). As pointed out in the literature review on multi-criteria decision aid techniques in the portfolio optimisation problem conducted by Aouni et al. (2018), the goal programming technique is the most widely used technique followed by CP. In order to elicit the best compromise solutions, CP relies on the minimisation of a parametric family

*Corresponding author.

[Correction added on August 31, 2022, after first online publication: The copyright line was changed.]

of additive distance functions with respect to an ideal point as a surrogate for utility (Yu, 1973; Zeleny, 1973; Yu, 1985).

The pioneering applications of CP for portfolio selection are due to Ballestero and Romero (1996). Since then, several works have extended the CP approach in this area of research (Ballestero and Pla-Santamaria, 2003, 2004; Bilbao-Terol et al., 2014; Pla-Santamaria et al., 2021). In addition, Parra et al. (2005) and Bilbao-Terol et al. (2006) proposed a new CP model for portfolio selection including the imprecision and subjectivity inherent to some data. Gharakhani and Sadjadi (2013) also combined fuzzy logic and CP to integrate the investor's view about future asset returns.

The portfolio selection problem is usually represented as a bi-criteria optimisation problem in which profitability is measured by expected returns and risk is measured by portfolio variance of returns. Some authors argue that these two factors cannot capture all relevant information in investment decisions and suggest alternative candidates for a third criterion such as liquidity, dividends, number of securities in a portfolio and social responsibility. Li and Xu (2013) and Qi et al. (2017) considered three criteria: returns, risk and liquidity. Gong et al. (2021) proposed a cloud theory based multi-objective portfolio selection model which incorporates four objectives, mean, variance, skewness and liquidity. When additional criteria are incorporated to the portfolio selection, complexity increases and the need for some principles to motivate the selection of the best solutions becomes more important. In this paper, we discuss those principles.

The concept of compromise set, given by metrics $p = 1$ and $p = \infty$ of the Minkowski distance function, is central to CP. It is assumed that it is highly unlikely that the optimum decision will lie out of the bounds of the compromise set (Ballestero and Romero, 1998). However, this assumption excludes the incorporation of the principle of limited compensability by means of multiplicative functions. Compensability is defined as the possibility of offsetting the shortfall in some criteria with a superior performance in other criteria (Garcia-Bernabeu et al., 2020). Maximum compensability is achieved when using metric $p = 1$ and no compensability (or maximum balance) is achieved when using $p = \infty$. We here propose geometric CP (GCP) as an extension of CP by means of multiplicative utility functions as a measure of achievement.

One of the main properties derived from the use of the Minkowski family of distance functions in CP is the uniqueness of solutions. There is only one solution that minimises the distance to the ideal point. However, this feature prevents undifferentiated extreme seekers from using CP to characterise their decision-making process. Note that extreme solutions can be attained by setting objectives weights appropriately. However, this way of action implies that a subset of objectives outweighs all other objectives and we here focus on extreme seekers with no preference for any of the objectives. As a result, one of the main goals of this paper is to provide a new class compromise solutions for undifferentiated extreme seekers. Undifferentiated extreme seekers get very little utility from balanced solutions (with a similar amount for any of the outcomes) and, at the same time, there is no a particular preference for any of the outcome extremes. All extremes are alternative optimal. Conventional theory usually ignores extremes even though maximum deviations from average behaviour are central to some high impact works in statistics (Gumbel, 1958) and economics (Stigler and Becker, 1977; Schelling, 1978) and other social sciences (Barthold and Hochman, 1988).

Within the context of CP, we here explore the use multiplicative functions and their relation to the more conventional additive functions. To this end, we first consider the case in which the usual family of parametric distance functions takes parameter $p = 0$. It can be shown that the Minkowski function with $p = 0$ of a q-dimensional vector of non-negative normalised criteria differences with respect to an ideal point is equivalent to the geometric mean of order q of these set of distances. In a CP context, q is the number of criteria under consideration. From a set of multiplicative utility functions, we analytically show the conditions under what the best compromise solutions are connected to the extremes of the compromise set. As a theoretical result, we find that maximum geometric distances from the anti-ideal point (with minimum independent achievement) yield solutions that are connected to those obtained by minimising the maximum distances to the ideal of alternative solutions (maximin principle). This connection is established by means of a theorem stating that the condition is that weights for alternative criteria must be equal to the partial derivatives of the non-dominated frontier in a normalised space of criteria. We illustrate our approach in the context of portfolio selection.

The main results of this work imply some kind of duality. On the one hand, the particular value of parameter p in CP distance functions has been interpreted as a measure of balance of solutions (Ballestero, 2007) and also as a representation of ethical principles (Romero, 2001; González-Pachón and Romero, 2011, 2016). In social choice theory, ethical principles refer to criteria that serve as a basic justification for policy evaluation. In GCP, the interpretation of p must be subjected to the direction of optimisation. In order to provide the appropriate semantics, recall that $p = 1$ represents the Benthamite principle of maximum efficiency and $p = \infty$ represents the Rawlsian principle of maximum fairness as proposed in González-Pachón and Romero (2016). With $p = 0$, we are able to represent the principle of maximum imbalance that characterises the behaviour of extreme seekers when minimising distances to the ideal. For instance, fund managers trying to focus on particular features of their funds and product managers aiming to differentiate their products by focusing on unusual features within the product space (Tirole, 1988). On the other hand, we are also able to represent the principle of limited compensability with $p = 0$ when maximising distances from the anti-ideal point. As a result, we here argue that GCP embodies a novel and interesting area of research due to its duality.

The structure of this paper is as follows. In Section 2, we provide basic background on the use of parametric distance functions in CP. Next, in Section 3, we describe novel theoretical results for three different cases of GCP. In Section 4, we discuss the practical implications of GCP and present two numerical illustrations in portfolio selection. Finally, in Section 5, we give some concluding remarks and highlight natural extensions of this work.

2. Parametric distance functions in compromise programming

Due to the usual conflict among criteria, CP relies on the concept of distance to an ideal point where all criteria are simultaneously optimised. This point is usually infeasible but it plays a key role as a reference because Zeleny's axiom of choice states that alternatives that are closer to the ideal point are preferred to those that are further (Zeleny, 1973). Given a feasible set $\mathcal{X}$ and solution $\boldsymbol{x} \in \mathcal{X}$ in a maximisation context, the ideal and anti-ideal (or nadir) points are defined as follows (Zeleny, 1973; Ehrgott, 2005):

Definition 1. *Point* $\boldsymbol{z}_I = (z_1^I, \ldots, z_j^I, \ldots, z_q^I)$ *given by* $z_j^I := max_{\boldsymbol{x}\in\mathcal{X}}\ z_j(\boldsymbol{x})$ *is called the ideal point of the multi-criteria problem* $\max_{x\in\mathcal{X}}\ (z_1(\boldsymbol{x}), \ldots, z_j(\boldsymbol{x}), \ldots, z_q(\boldsymbol{x}))$*.*

Definition 2. *Point* $\boldsymbol{z}_N = (z_1^N, \ldots, z_j^N, \ldots, z_q^N)$ *given by* $z_j^N := min_{\boldsymbol{x}\in\mathcal{X}}\ z_j(\boldsymbol{x})$ *is called the anti-ideal point of the multi-criteria problem* $max_{\boldsymbol{x}\in\mathcal{X}}\ (z_1(\boldsymbol{x}), \ldots, z_j(\boldsymbol{x}), \ldots, z_q(\boldsymbol{x}))$*.*

Using $\boldsymbol{z}_I$ as a reference point, we develop a CP model as a method to evaluate any pair of feasible alternatives, denoted by $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, by means of the next preference relations (Ballestero and Romero, 1998):

$$\boldsymbol{z}_1 \succ \boldsymbol{z}_2 \iff \mathcal{L}(\boldsymbol{z}_1) < \mathcal{L}(\boldsymbol{z}_2) \tag{1}$$

$$\boldsymbol{z}_2 \succ \boldsymbol{z}_1 \iff \mathcal{L}(\boldsymbol{z}_2) < \mathcal{L}(\boldsymbol{z}_1) \tag{2}$$

$$\boldsymbol{z}_1 \sim \boldsymbol{z}_2 \iff \mathcal{L}(\boldsymbol{z}_1) = \mathcal{L}(\boldsymbol{z}_2), \tag{3}$$

where $\succ$ means 'is preferred to', $\sim$ means 'is indifferent to' and $\mathcal{L}(\boldsymbol{z})$ is a distance function between any feasible solution $\boldsymbol{z}$ and ideal point $\boldsymbol{z}_I$.

In order to avoid meaningless comparison among criteria, all elements of vector $\boldsymbol{z}$ must be normalised. By assuming that we want to maximise all criteria, we can use the following normalisation for each criterion $z_j \in \boldsymbol{z}$:

$$z_j(\boldsymbol{x}) = \frac{g_j(\boldsymbol{x}) - g_{j,min}}{g_{j,max} - g_{j,min}}, \tag{4}$$

where $g_{j,max}$ and $g_{j,min}$ are, respectively, the maximum and minimum value achievements as measured by objective function $g_j(\boldsymbol{x})$. As a result, the ideal point in a bi-dimensional criteria space is (1,1) and the anti-ideal point is (0,0). As a result, Zeleny–Yu utility (U_{ZY}) for each point $\boldsymbol{z}$ in a q-dimensional criteria space can be expressed as the difference of a sufficiently large number M and the Minkowski distance of order p from any point $\boldsymbol{z}$ to the ideal point (Ballestero, 2007):

$$U_{\mathrm{ZY}} = M - \mathcal{L}_p(\boldsymbol{z}, \boldsymbol{w}) = M - \left[\sum_{j=1}^{q} w_j^p (1 - z_j)^p.\right]^{1/p}, \tag{5}$$

where $\boldsymbol{z}$ and $\boldsymbol{w}$ are q-dimensional vectors with criteria achievements and weights attached to each criterion, respectively.

In CP, metric p in Equation (5) is a topological metric belonging to the closed interval $[1, \infty]$. The extremes of the previous interval define the compromise set as the subset of the non-dominated frontier between points:

- L_1 with minimum Manhattan distance ($p = 1$) to the ideal point.
- L_∞ with minimum Chebyshev distance ($p = \infty$) to the ideal point.

Figure 1 shows the most relevant points in CP bi-criteria space and represents the compromise set as the part of the non-dominated frontier $T(z_1, z_2) = K$ between points L_1 and L_∞. Given weights $\boldsymbol{w}$, we say that point $\boldsymbol{z}$ belongs to compromise set S when its parametric distance $\mathcal{L}_p(\boldsymbol{z}, \boldsymbol{w})$ to the ideal point is minimum. Formally, $S = \{\boldsymbol{z} \in \mathbb{R}^q | \min_{\boldsymbol{z}} \mathcal{L}_p(\boldsymbol{z}, \boldsymbol{w}), p \in [1, \infty]\}$.

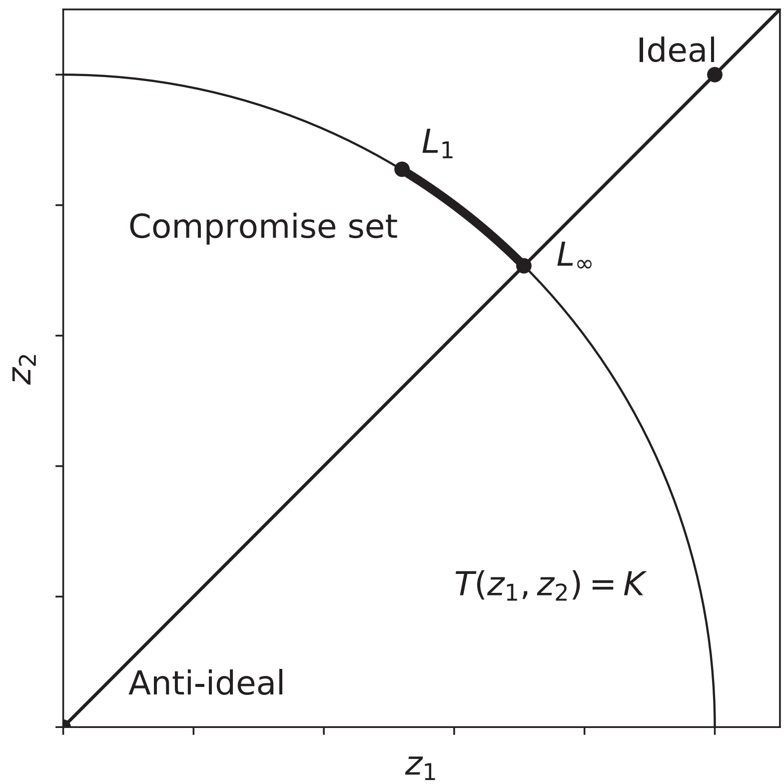

Fig. 1. Most relevant points in CP and the compromise set (bolded).

The value of metric p can also be interpreted as a measure of balance of solutions. As described in Ballestero (2007) in a portfolio selection context, $p = 1$ tends to yield corner or imbalanced solutions, and larger values of p, in the limit $p = \infty$, yield more balanced solutions. In addition, metric p is also as a representation of ethical principles (González-Pachón and Romero, 2016). Metric $p = 1$ represents the Benthamite (Bentham, 1789) principle of maximum efficiency (full compensability among criteria) and $p = \infty$ represents the Rawlsian principle of maximum fairness (Rawls, 1973). However, we next extend the analysis in CP by considering alternative values of metric p that lead to multiplicative functions.

3. Compromise programming with multiplicative functions

We here explore the case $p = 0$ in the Minkowski distance function as a way to find a new class of solutions in CP. This approach leads to consider multiplicative functions as a measure of utility in CP such as the geometric mean of criteria values or Cobb–Douglas functions. As a result, we call this technique GCP. This extension of CP allows us to characterise extreme seeking behaviour (Stigler and Becker, 1977; Schelling, 1978; Barthold and Hochman, 1988) and to incorporate the concept of limited compensability in decision-making analysis. As mentioned in the introduction, the concept of compensability is defined as the possibility of offsetting low achievement in some criteria with a superior performance in other criteria (Munda, 2005; Garcia-Bernabeu et al., 2020). Maximum compensability is achieved when using metric $p = 1$ and no compensability (or maximum balance) is achieved when using $p = \infty$.

It can be shown that the parametric distance function $\mathcal{L}_p(\boldsymbol{z})$ with $p = 0$ of a vector $\boldsymbol{z}$ of positive numbers is the geometric mean of order q:

$$\mathcal{L}_0(\boldsymbol{z}) = \lim_{p \to 0} \left[\sum_{i=1}^{q} z_i^p \right]^{(1/p)} = \left[\prod_{i=1}^{q} z_i \right]^{(1/q)}. \tag{6}$$

When considering vector of weights $\boldsymbol{w}$ in Zeleny–Yu utility in Equation (5), we find two different approaches in the literature. In CP, we usually find weights raised to metric p (see, e.g., Ballestero and Romero, 1996; Salas-Molina et al., 2019). This course of action results in the following weighted multiplicative function as shown in the Appendix:

$$\lim_{p \to 0} \left[\sum_{j=1}^{q} w_j^p z_j^p \right]^{(1/p)} = \left[\prod_{j=1}^{q} w_j z_j \right]^{(1/q)}. \tag{7}$$

On the other hand, we find other approaches in which individual weights are not affected by exponent p (see, e.g., Merigó and Gil-Lafuente, 2008; Xian et al., 2016). This approach leads to the following weighted multiplicative function as shown in the Appendix:

$$\lim_{p \to 0} \left[\sum_{j=1}^{q} w_j z_j^p \right]^{(1/p)} = \prod_{j=1}^{q} z_j^{w_j}. \tag{8}$$

From Equations (7) and (8), we next follow two different approaches: (i) minimisation of distances to the ideal point in Section 3.1; (ii) maximisation of distances to the anti-ideal point in Sections 3.2 and 3.3.

3.1. A new class of compromise solutions for extreme seekers by minimising geometric distances to the ideal

Intuitively, one can expect that minimising distances to the ideal point results in similar solutions to maximising distances to the anti-ideal. In general, this intuition may be appropriate for additive distance functions in traditional CP but not for multiplicative distances in GCP. Geometric utility is dual. Let us denote the first geometric utility function by U_{G1} and, similar to Zeleny–Yu utility U_{ZY}, consider the minimisation of the geometric distance to the ideal point derived from Equation (7) within a normalised criteria space as follows:

$$U_{G1} = M - \mathcal{L}_0(\boldsymbol{z}) = M - \left[\prod_{j=1}^{q} w_j(1 - z_j) \right]^{1/q}. \tag{9}$$

Clearly, U_{G1} reaches a maximum when $z_j = 1$ for any of the j criteria under consideration because z_j is restricted to the interval [0, 1] after normalisation. In other words, maximum utility is achieved when any of the criteria achievements takes an extreme value. While extreme solutions may seem quite undesired at first glance within a context of multiple-criteria decision making, it is actually not unrealistic, at least for extreme seekers. To illustrate this statement, let us consider the following scenario described in Steuer (1986). A man would like to collect both stamps and coins. The more he has of each, the better. However, with regard to his collections, he must live within a limited budget and other restrictions. He is an extreme seeker. That is, he gets very little gratification from doing things halfway. Within this situation, it is conceivable that a larger stamp collection and a larger coin collection could be alternative optimal. To move forward, it seems reasonable to propose formal definitions of the degree of balance and extreme seekers in a maximisation context.

Definition 3. *Given a feasible set $\mathcal{X}$ and solution $\boldsymbol{x} \in \mathcal{X}$ in a multi-criteria maximisation problem $\max_{x \in \mathcal{X}} (z_1(\boldsymbol{x}), \ldots, z_j(\boldsymbol{x}), \ldots, z_q(\boldsymbol{x}))$, the degree of balance of a solution in a normalised criteria space $\boldsymbol{z} = (z_1, \ldots, z_j, \ldots, z_q)$ is a real function $b(\boldsymbol{z})$ that reaches its global maximum when $z_1 = \cdots = z_j = \cdots = z_q$.*

An example of a balance function that fits this definition when $z_j : \mathcal{X} \rightarrow [0, 1]$ is

$$b(\boldsymbol{z}) = 1 - \text{var}(\boldsymbol{z}), \tag{10}$$

where operator var is the variance of the elements of vector of achievements $\boldsymbol{z}$. Another example of a balance function can be constructed using the Gini coefficient (Gini, 1912; Ceriani and Verme, 2012). Note that the balance function from Definition 3 simultaneously reaches its global maximum both at the ideal point and the anti-ideal point because, by definition, $z_1 = \cdots = z_j = \cdots = z_q$ both at the ideal and the anti-ideal point.

Definition 4. *An extreme seeker is a decision maker with the following preference relations for solutions in a multi-criteria optimisation problem:*

$$\boldsymbol{z}_1 \succ \boldsymbol{z}_2 \iff b(\boldsymbol{z}_1) < b(\boldsymbol{z}_2) \tag{11}$$

$$\boldsymbol{z}_1 \sim \boldsymbol{z}_2 \iff b(\boldsymbol{z}_1) = b(\boldsymbol{z}_2). \tag{12}$$

As a result, criterion vectors with well-balanced solutions (smaller and similar stamp and coin collections) might be much less desirable because little satisfaction is drawn from either collection. This scenario leads to consider a non-concave utility function with respect to the ideal point as represented in Fig. 2. The solid line represents the frontier of the feasible criterion vector set Z, and the dashed line is a non-concave utility function describing an undifferentiated extreme seeker behaviour. Points z_1 and z_3 are alternative optimal and point z_2 is not. As a result, a utility function that describes well the behaviour of an extreme seeker as the stamps–coins collector in our example

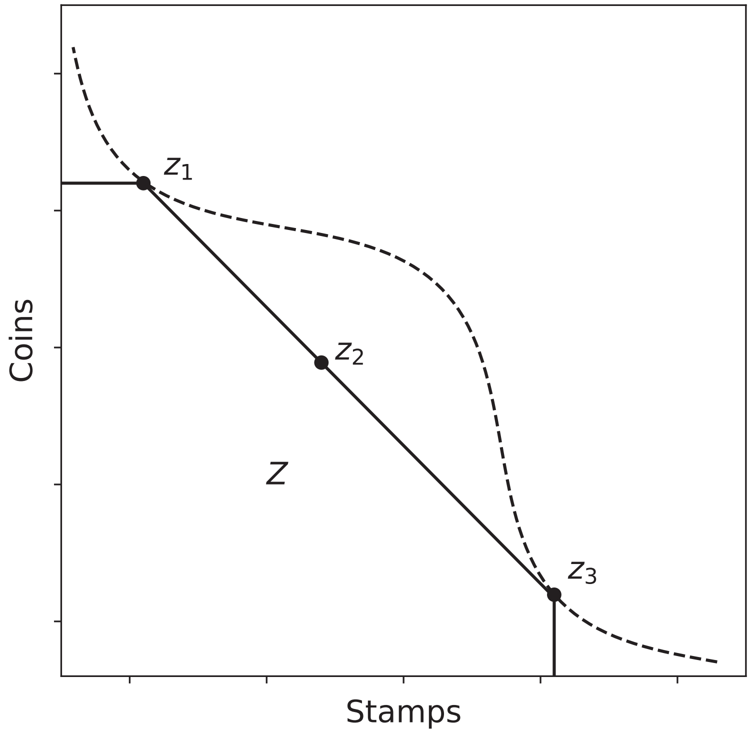

Fig. 2. Extreme seekers criteria space.

is the geometric utility encoded in Equation (9). In order to provide the appropriate semantics to these results, recall that large values of p (in the limit, $p = \infty$) yield balanced solutions. Smaller values of p tend to yield corner or imbalanced solutions (in the other limit, $p = 1$) (Ballestero, 2007). Finally, if we move one step further (beyond the compromise set) by setting $p = 0$, we get exactly the extreme values of the non-dominated frontier as the optimal solutions.

In order to illustrate our point, consider the following scenario in the context of portfolio selection. Steuer et al. (2005) suggested a list of alternative objectives to be maximised:

1. max $\{z_1 =$ portfolio return$\}$;
2. max $\{z_2 =$ dividends$\}$;
3. max $\{z_3 =$ amount invested in R&D$\}$;
4. max $\{z_4 =$ social responsibility$\}$;
5. max $\{z_5 =$ liquidity$\}$;
6. max $\{z_6 =$ portfolio return over that of a benchmark$\}$;
7. max $\{z_7 = -$deviations from asset allocation percentages$\}$;
8. max $\{z_8 = -$number of securities in portfolio$\}$;
9. max $\{z_9 = -$turnover (i.e., costs of adjustment)$\}$;
10. max $\{z_{10} = -$maximum investment proportion weight$\}$;
11. max $\{z_{11} = -$amount of short selling$\}$;
12. max $\{z_{12} = -$number of securities sold short$\}$.

It seems reasonable to assume that among all possible investors, there is a set of extreme seeking investors that are alternatively interested in maximising z_1 and z_3. Many other combinations of objectives would presumably characterise the behaviour of an extreme seeker.

3.2. *Maximising geometric distances to the anti-ideal: a theorem connecting geometric utility and Zeleny–Yu utility*

Within a normalised bi-criteria space derived from the application of max-min normalisation as described in Equation (4), the ideal point is (1,1) and the anti-ideal point is (0,0). Maximising geometric distances to the anti-ideal point as a measure of utility seems to be an additional suitable strategy to find compromise solutions. Let us consider the following a second geometric utility function U_{G2}:

$$U_{G2} = \left[\prod_{j=1}^{q} w_j z_j \right]^{1/q}. \tag{13}$$

By means of the following lemma, we show that weights in utility U_{G2} have no effect for optimisation purposes.

Lemma 1. *Weights w_j have no influence in the maximisation of geometric utility U_{G2} in Equation (13) subject to non-dominated frontier $T(z_1, \ldots, z_q) = K$.*

Proof. For optimisation purposes, we can remove exponent $1/q$ in Equation (13) without affecting the optimal solutions. Let us form the Lagrangean:

$$\prod_{j=1}^{q} w_j z_j + \lambda(K - T(z_1, \ldots, z_q)). \tag{14}$$

The first-order conditions are

$$\begin{aligned}
w_1 \prod_{j \neq 1}^{q} w_j z_j - \lambda T_1 &= 0 \\
w_2 \prod_{j \neq 2}^{q} w_j z_j - \lambda T_2 &= 0 \\
&\vdots \\
w_q \prod_{j \neq q}^{q} w_j z_j - \lambda T_q &= 0 \\
T(z_1, \ldots, z_q) &= K,
\end{aligned} \tag{15}$$

where $T_1, T_2, \ldots, T_q$ are the partial derivatives of T with respect to $z_1, z_2, \ldots, z_q$ respectively. By dividing equations in system (15) in pairs with the exception of the last one and rearranging terms we obtain

$$\begin{aligned}
\frac{(w_1 w_2 \ldots w_q)\cdot(z_2 z_3 \ldots z_q)}{(w_1 w_2 \ldots w_q)\cdot(z_1 z_3 \ldots z_q)} &= \frac{T_1}{T_2}\\
\frac{(w_1 w_2 \ldots w_q)\cdot(z_1 z_3 \ldots z_q)}{(w_1 w_2 \ldots w_q)\cdot(z_1 z_2 \ldots z_q)} &= \frac{T_2}{T_3}\\
&\vdots\\
\frac{(w_1 w_2 \ldots w_q)\cdot(z_1 z_3 \ldots z_{q-2} z_q)}{(w_1 w_2 \ldots w_q)\cdot(z_1 z_2 \ldots z_{q-1})} &= \frac{T_{q-1}}{T_q}.
\end{aligned} \tag{16}$$

Cancelling weights w_j and combining all equations in system (16), we find that optimal solutions do not depend on weights because

$$T_1 z_1 = T_2 z_2 = \cdots = T_q z_q. \tag{17}$$

□

Furthermore, there is a close relationship between geometric utility U_{G2} in Equation (13) and Zeleny–Yu utility U_{ZY} in Equation (5). It is shown elsewhere (Ballestero and Romero, 1998) that Zeleny–Yu utility U_{ZY} for $p = \infty$ is maximised when the following chain of equations holds:

$$w_1 z_1 = w_2 z_2 = \cdots = w_q z_q. \tag{18}$$

In other words, the L_∞ solution represents a point where weighted deviations to the ideal point are equal. By comparing conditions in Equations (17) and (18), the following result is direct.

Theorem 1. *Zeleny–Yu utility U_{ZY} in Equation (5) for $p = \infty$ and geometric utility U_{G2} in Equation (13), both subject to non-dominated frontier $T(z_1, \ldots, z_q) = K$, lead to the same optimal solutions when weights w_j in U_{ZY} are equal to partial derivatives T_j.*

Proof. Immediate from Lemma 1 and Equation (18). □

A situation in which weights for each criterion are equal to partial derivatives of the non-dominated frontier with respect to each criterion is not as unrealistic as it may seem at first glance. To show this aspect, let us consider the following common two cases in a bi-dimensional criteria space:

- Linear non-dominated frontier described by $z_1 + z_2 = 1$. In this case, T_1 and T_2 are equal to 1. As a result, a neutral decision maker with no particular preference for any of the two criteria $(w_1, w_2 = 1)$ will find no difference between optimising U_{ZY} for $p = \infty$ and geometric utility U_{G2}. In both cases, the optimal solutions are $z_1^* = 0.5$ and $z_2^* = 0.5$.
- Quadratic non-dominated frontier described by $z_1^2 + z_2^2 = 1$. In the quadratic case, $T_1 = 2z_1$ and $T_2 = 2z_2$. Similar to the linear case, a neutral decision maker with no particular preference for

any of the two criteria ($w_1, w_2 = 1$) will find no difference between optimising U_{ZY} for $p = \infty$ and geometric utility U_{G2}. In both cases, the optimal solution is $z_1^* = \sqrt{1/2}$ and $z_2^* = \sqrt{1/2}$.

Moreover, we know that Zeleny–Yu utility U_{ZY} for $p = 1$ is maximised when the following chain of equations holds (Ballestero and Romero, 1998):

$$\frac{w_1}{T_1} = \frac{w_2}{T_2} = \cdots = \frac{w_q}{T_q}. \tag{19}$$

By relying on Theorem 1 and Equation (19), we derive the following theoretical result.

Theorem 2. *If the compromise set in non-dominated frontier $T(z_1, \ldots, z_q) = K$ derived from Zeleny–Yu utility U_{ZY} in Equation (5) for $p = 1$ and $p = \infty$ is a single point, then this point attains maximum geometric utility U_{G2} in Equation (13).*

Proof. If the compromise set is a single point, then Equations (18) and (19) hold. By dividing equations in (18) into equations in (19), we obtain the set of equations in (17). Then, it follows that this point attains maximum geometric utility U_{G2} in Equation (13) as Lemma 1 shows. □

Theorem 2 implies that if one point within the non-dominated frontier $T(z_1, \ldots, z_q) = K$ attains maximum geometric utility U_{G2} and coincides with any of the bounds of the compromise set L_1 and L_∞, then the compromise set is a single point.

3.3. Cobb–Douglas multiplicative utility

If optimal solutions for geometric utility U_{G2} do not depend on weights, one may ask how can a decision maker consider weights within a context of multiplicative functions. One possible answer to this question is Cobb–Douglas utility. Note that optimisation problems with monomial objective functions can be efficiently solved by means of geometric programming and interior-point algorithms (Duffin, 1970; Boyd et al., 2007). However, we next follow an analytical approach for completeness and illustrative purposes.

Consider the following Cobb–Douglas utility function derived from Equation (8) by setting $p = 0$ when weights in the Zeleny–Yu utility are not raised to exponent p:

$$U_{\text{CD}} = \prod_{j=1}^{q} c z_j^{w_j}, \tag{20}$$

with weights satisfying condition $\sum_{j=1}^{q} w_j = q/2$ as described in Ballestero (2007). Constant c in Equation (20) is irrelevant for optimisation purposes. To solve the maximisation of U_{CD} subject to $T(z_1, \ldots, z_q) = K$, consider the Lagrangean:

$$\prod_{j=1}^{q} z_j^{w_j} + \lambda(K - T(z_1, \ldots, z_q)), \tag{21}$$

with the following first-order conditions:

$$
\begin{aligned}
&w_1 z_1^{w_1-1}\prod_{j\neq 1}^{q} z_j^{w_j} - \lambda T_1 = 0\\
&w_2 z_2^{w_2-1}\prod_{j\neq 2}^{q} z_j^{w_j} - \lambda T_2 = 0\\
&\qquad\vdots\\
&w_q z_q^{w_q-1}\prod_{j\neq q}^{q} z_j^{w_j} - \lambda T_q = 0\\
&T(z_1,\ldots,z_q) = K.
\end{aligned}
\tag{22}
$$

Again, by dividing equations in system (22) in pairs with the exception of the last one, we obtain

$$
\begin{aligned}
\frac{w_1 z_1^{w_1-1} z_2^{w_2}\ldots z_q^{w_q}}{z_1^{w_1} w_2 z_2^{w_2-1}\ldots z_q^{w_q}} &= \frac{T_1}{T_2}\\
\frac{z_1^{w_1} w_2 z_2^{w_2-1}\ldots z_q^{w_q}}{z_1^{w_1} z_2^{w_2} w_3 z_3^{w_3-1}\ldots z_q^{w_q}} &= \frac{T_2}{T_3}\\
&\vdots\\
\frac{z_1^{w_1} z_2^{w_2}\ldots w_{q-1} z_{q-1}^{w_{(q-1)}-1} z_q^{w_q}}{z_1^{w_1} z_2^{w_2}\ldots w_q z_q^{w_q-1}} &= \frac{T_{q-1}}{T_q}.
\end{aligned}
\tag{23}
$$

Cancelling terms and combining all equations in system (23), we obtain the following chain of equations:

$$\frac{w_1 z_1}{T_1} = \frac{w_2}{z_2 T_2} = \cdots = \frac{w_q}{z_q T_q}. \tag{24}$$

$$\frac{w_2 z_2}{T_2} = \frac{w_3}{z_3 T_3} = \cdots = \frac{w_q}{z_q T_q}. \tag{25}$$

$$\frac{w_{q-1} z_{q-1}}{T_{q-1}} = \frac{w_q}{z_q T_q}. \tag{26}$$

Then, we conclude that weights are relevant to find optimal solutions in the case of the Cobb–Douglas multiplicative utility as a geometric utility function. Again, we can consider the cases of a linear and a quadratic non-dominated frontier in combination with Cobb–Douglas utility:

- Linear non-dominated frontier described by $z_1 + z_2 = 1$. In this case, T_1 and T_2 are equal to 1. Then, the optimal solution must satisfy $z_1 z_2 = w_2/w_1$ and $z_1 z_2 = w_1/w_2$, which ultimately leads to $w_1 = w_2$.
- Quadratic non-dominated frontier described by $z_1^2 + z_2^2 = 1$. In the quadratic case, $T_1 = 2z_1$ and $T_2 = 2z_2$. Then, the optimal solutions are

$$z_1 z_2 = \frac{w_2 2z_2}{w_1 2z_1} \rightarrow z_1 = \sqrt{\frac{w_2}{w_1}} \tag{27}$$

$$z_1^2 + z_2^2 = 1 \rightarrow z_2 = \sqrt{1 - \frac{w_2}{w_1}}. \tag{28}$$

3.4. Theoretical insights

The implications derived from the previous results are manifold. In contrast to what we observe in CP with additive distance functions, we find a duality between the minimisation of geometric distances to the ideal in U_{G1} and the maximisation of the same geometric distances to the anti-ideal in U_{G2}. In the first case, we describe the behaviour of extreme seekers and, in the second case, we integrate the principle of limited compensability in multiple-criteria decision making. More precisely, we find that minimum geometric distances to the ideal point (with maximum independent achievement) yield extreme solutions. On the contrary, maximum geometric distances from the anti-ideal point (with minimum independent achievement) yield balanced solutions in which weights attached to different criteria are not relevant.

This duality implies a redefinition of Zeleny's axiom of choice to include not only minimum distances to the ideal but also maximum distances to the anti-ideal point as follows:

$$z_1 \succ z_2 \iff \mathcal{U}(z_1) > \mathcal{U}(z_2) \tag{29}$$

$$z_1 \sim z_2 \iff \mathcal{U}(z_1) = \mathcal{U}(z_2), \tag{30}$$

where $\mathcal{U}$ is some utility function based on any possible distance function.

Furthermore, we analytically show the conditions under which the best geometric compromise solutions are within the bounds of the compromise set. Theorem 1 proves that Zeleny–Yu utility U_{ZY} in Equation (5) for $p = \infty$ and geometric utility U_{G2} in Equation (13), both subject to non-dominated frontier $T(z_1, \ldots, z_q) = K$, lead to the same optimal solutions when weights w_j in U_{ZY} are equal to partial derivatives T_j. In other words, the best geometric compromise solutions are within the bounds of the compromise set, at least, when the equality of weights and partial derivatives of the non-dominated frontier occur. In addition, Theorem 2 shows the relation between the compromise set and geometric utility U_{G2}. This connection leads us to propose the definition of a geometric compromise set between points L_0 and L_1 in the non-dominated frontier:

- L_1 with minimum Manhattan distance to the ideal point.
- L_0 with maximum geometric distance to the anti-ideal point.

Table 1
Alternative utility functions and their corresponding ethical principle

Utility function	Distance function	Optimisation direction	Ethical or social principle
U_{ZY1}	$\mathcal{L}_1$	Min to ideal	Maximum efficiency
U_{ZY2}	$\mathcal{L}_2$	Min to ideal	Minimum deviation
$U_{ZY\infty}$	$\mathcal{L}_\infty$	Min to ideal	Maximum fairness
U_{G1}	$\mathcal{L}_0$	Min to ideal	Extreme seekers
U_{G2}	$\mathcal{L}_0$	Max to anti-ideal	Limited compensability
U_{CD}	$\mathcal{L}_0$	Max to anti-ideal	Weighted compensability

Note that we do not claim that L_0 always belongs to the $L_1 - L_\infty$ conventional definition of the compromise set. On the contrary, we argue that an $L_1 - L_0$ extension of the compromise set to integrate geometric utility is possible and useful.

Finally, we mentioned in the introduction that different values of topological metric p lead to different distance functions as a representation of desired ethical or social principles in multiple-criteria decision making. A summary of utility functions, optimisation directions and principles integrated in the different CP approaches considered in this paper are summarised in Table 1. The first block of utility functions corresponds to the conventional Zeleny–Yu utility U_{ZY} in which metric p can take values between 1 and ∞. As proposed in Romero (2001) and González-Pachón and Romero (2016), $p = 1$ corresponds to the utilitarian principle and $p = \infty$ corresponds to the fairness principle. This correspondence between metrics and principles implies that, at least in theory, there is a social principle that is represented by any value of p greater than 1 and below ∞. An example of this intermediate principle is $p = 2$ representing solutions with minimum deviation (in terms of the Euclidean distance) with respect to the ideal point (or any other given reference).

The second block of utility functions in Table 1 includes the geometric distance functions proposed in this paper by setting metric $p = 0$. In this case, we differentiate between the minimisation of geometric distances to the ideal characterising extreme seeking behaviour and the maximisation of geometric distances to the anti-ideal integrating the principle of limited and weighted compensability. The main difference between geometric utility U_{G2} and Cobb–Douglas utility U_{CD} derives from the particular treatment of weights in the initial definition of the Minkowski distance function within Zeleny–Yu utility before setting $p = 0$.

4. An application in portfolio selection

In this section, we discuss the relevance of our GCP approach in the context of portfolio selection. We first illustrate the concept of compensability by means of a numerical example. Second, we describe an empirical case study to validate our approach using real data. Next, we study the impact of risk preferences in both additive and geometric portfolio selection. Finally, we further elaborate on the practical implications of GCP.

4.1. A numerical illustration of compensability in portfolio selection

In this section, we compare most of the utility functions in Table 1 by means of a numerical example derived from the results reported by Ballestero and Pla-Santamaria (2003). As a starting point, we

Table 2
Alternative utility functions in portfolio selection (best values in bold)

i	Return	Safety	U_{ZY1}	U_{ZY2}	U_{G1}	U_{G2}	ΔU_{ZY1}	ΔU_{ZY2}	ΔU_{G1}	ΔU_{G2}
1	0.0000	1.0000	0.0000	0.0000	**1.0000**	0.0000				
2	0.0588	0.9981	0.0569	0.0588	0.9577	0.2423	0.0569	0.0588	0.0423	0.2423
3	0.1176	0.9942	0.1118	0.1176	0.9285	0.3419	0.0549	0.0588	0.0293	0.0997
4	0.1765	0.9841	0.1606	0.1763	0.8856	0.4168	0.0488	0.0588	0.0429	0.0748
5	0.2353	0.9770	0.2123	0.2350	0.8674	0.4795	0.0517	0.0586	0.0182	0.0627
6	0.2941	0.9639	0.2580	0.2932	0.8404	0.5324	0.0457	0.0582	0.0270	0.0530
7	0.3529	0.9471	0.3000	0.3507	0.8150	0.5781	0.0420	0.0576	0.0254	0.0457
8	0.4118	0.9280	0.3398	0.4074	0.7942	0.6182	0.0398	0.0567	0.0208	0.0401
9	0.4706	0.9032	0.3738	0.4618	0.7736	0.6520	0.0340	0.0544	0.0206	0.0338
10	0.5294	0.8717	0.4011	0.5122	0.7543	0.6793	0.0273	0.0504	0.0193	0.0274
11	0.5882	0.8328	0.4210	0.5556	0.7376	0.6999	0.0199	0.0433	0.0167	0.0206
12	0.6471	0.7858	0.4329	0.5872	0.7251	0.7131	0.0119	0.0316	0.0125	0.0132
13	0.7059	0.7278	**0.4337**	**0.5993**	0.7171	**0.7168**	0.0008	0.0121	0.0080	0.0037
14	0.7647	0.6544	0.4191	0.5819	0.7148	0.7074	0.0146	0.0174	0.0022	0.0094
15	0.8235	0.5661	0.3896	0.5316	0.7233	0.6828	0.0295	0.0503	0.0084	0.0246
16	0.8824	0.4501	0.3325	0.4377	0.7457	0.6302	0.0571	0.0939	0.0224	0.0526
17	0.9412	0.2629	0.2041	0.2606	0.7918	0.4974	0.1284	0.1771	0.0461	0.1328
18	1.0000	0.0000	0.0000	0.0000	**1.0000**	0.0000	0.2041	0.2606	0.2082	0.4974

use the data required to construct an efficient frontier in a normalised return–safety space. Each pair of return–safety elements in the second and third columns of Table 2 corresponds to a particular portfolio and the goal of the analyst is to select one of them by relying on some utility function.

Within the traditional CP context, Manhattan ($\mathcal{L}_1$) and Euclidean ($\mathcal{L}_2$) distances to the ideal point (1,1) are commonly used as alternative surrogates for Zeleny–Yu utility functions. To this end, we compute U_{ZY1} and U_{ZY2} for each pair of return–safety values using Equation (5) with $M = 1$, $q = 2$, weights equal to 1 and parameter $p = 1$ and $p = 2$, respectively. Moreover, we compute U_{G1} and U_{G2} as GCP utility functions using Equations (9) and (13) respectively, with $M = 1$, $q = 2$ and weights equal to 1. The results in Table 2 show that the best values for utilities U_{ZY1}, U_{ZY2} and U_{G2} are obtained for portfolio 13. On the contrary, U_{G1} reaches its maximum in both extremes of the efficient frontier characterising the extreme seekers' behaviour. Note that a limitation in using U_{G1} as a measure of utility is that its optimisation is equivalent to searching for the maximum achievement of a single-criterion disregarding the achievement of the rest. This drastic approach cancels out the idea of simultaneous multi-objective optimisation. However, recall that any of these single-criterion optimisations are equally desirable for an extreme seeker.

We also pay special attention to compensability. According to Munda (2005), compensability refers to the possibility of offsetting a disadvantage on some criteria by a sufficiently large advantage on another criterion. Complete compensability implies that an excellent performance in one criterion can justify a bad performance in the other criteria. We here follow the approach of exploring compensability by computing the difference in utility between two near solutions using the following expression:

$$\Delta U_{ij} = |U_{ij} - U_{(i-1)j}|, \quad \forall i > 1. \tag{31}$$

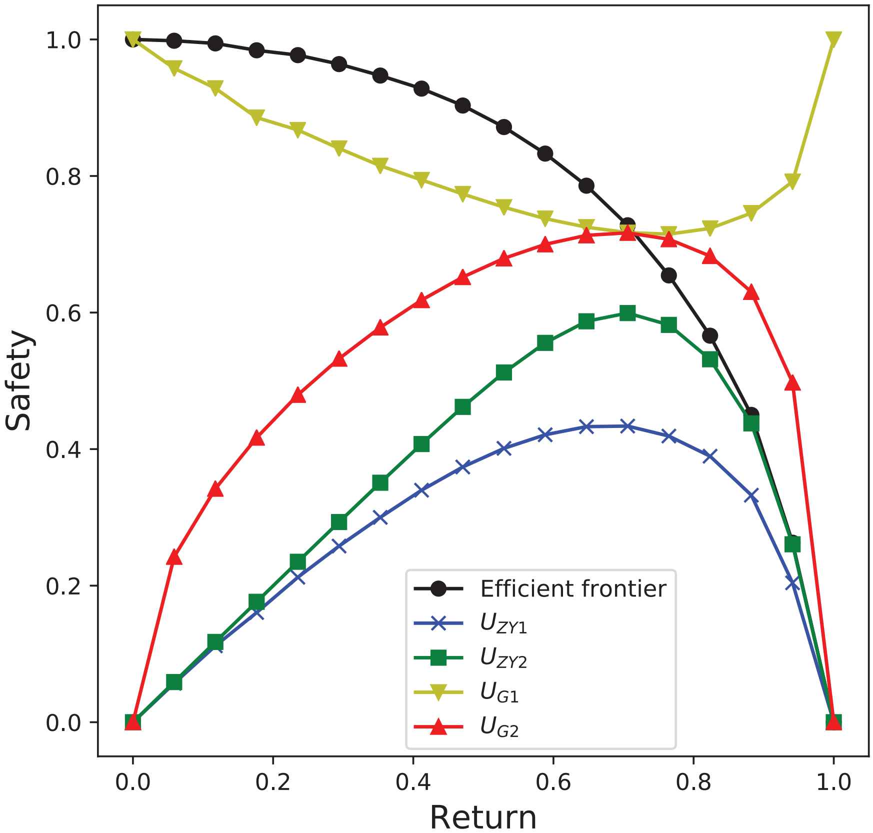

Fig. 3. Efficient frontier and different utility functions.

In our portfolio selection context, a concurrent excellent performance in profits (safety) and a bad performance in safety (profits) is observed in the extremes of the efficient frontier ($i = 1$ and $i = 18$). If solutions near to the extremes present a similar utility, the compensability is high because the cost in terms of utility of going from a near-to-the-extreme solution to an extreme solution is low. On the contrary, if solutions near to the extremes present a much larger utility, the compensability is limited because the cost in terms of utility of going from a near-to-the-extreme solution to an extreme solution is high. This fact can also be observed in Fig. 3. The dotted line represents the efficient frontier in a normalised profit-safety space. The rest of the marked lines depict the values for each to the four alternative utility functions. Clearly, the use of utility function U_{G2} implies limited compensability because the change in utility of going from a near-to-the-extreme solution to an extreme solution is higher than in the cases of U_{YZ1} y U_{YZ2}.

4.2. Empirical case study

To validate our GCP proposal using real data, we next solve a portfolio selection problem following the classical mean-variance formulation by Markowitz (1952), but using U_{G2} in Equation (13) as a measure of the achievements for an investor concerned with the limited compensability principle (see Table 1). From Lemma 1, we know that weights in utility U_{G2} have no effect for optimisation purposes. As a result, our hypothetical investor is neutral with respect to return and risk.

In order to obtain the best portfolios, we use a data set with weekly returns in the Spanish Stock Exchange comprising 35 alternative assets that were part of the IBEX35 index from 2014 to 2019. The major benefit of using this period of time is that it is a good representation of a sideways trend in the market with an average weekly return of −0.001%. Interested readers can obtain the data themselves at http://es.finance.yahoo.com.

The first step in our case study is the derivation of the efficient frontier with minimum variance portfolios for 50 target annualised returns (E_0) ranging from 0 to 0.24, this last value corresponding to the maximum return obtained by any of the assets in this period. To this end, we solve 50 instances of the following quadratic program varying E_0:

$$\min \ \boldsymbol{x}^T V \boldsymbol{x} \tag{32}$$

subject to

$$\boldsymbol{\mu}^T \boldsymbol{x} = E_0 \tag{33}$$

$$\sum_{j=1}^{35} x_j = 1, \tag{34}$$

where $\boldsymbol{x}$ is a 35 × 1 column vector of weights, V is the 35 × 35 covariance matrix of returns and $\boldsymbol{\mu}$ is the 35 × 1 column vector of average returns. In order to solve the previous optimisation problems, we use the open-source library SciPy in Python (Version 3.6.4), Jupyter Notebooks (Kluyver et al., 2016) and the SLSQP (sequential least squares programming) method originally implemented by Kraft (1988).

From the set of non-dominated pairs of risk and return obtained after the optimisation, we derive the efficient frontier within a normalised Return (θ_1)-Safety (θ_2) bi-criteria space after the application of the following max–min normalisation indexes:

$$\theta_1 = \frac{E - E_{min}}{E_{max} - E_{min}} \tag{35}$$

$$\theta_2 = \frac{S_{max} - S}{S_{max} - S_{min}}, \tag{36}$$

where E is the return of any portfolio in the efficient frontier and E_{min} and E_{max} are, respectively, the minimum and the maximum returns of portfolios in the efficient frontier. Similarly, S is the standard deviation of any portfolio in the efficient frontier and S_{min} and S_{max} are, respectively, the minimum and the maximum standard deviations of portfolios in the efficient frontier.

Once derived, the efficient frontier in the normalised Return (θ_1)-Safety (θ_2) space, investors concerned with limited compensability are able to make a final decision by selecting the portfolio that maximises the geometric distance to the anti-ideal point (0,0) using U_{G2} in Equation (13) as shown in Fig. 4. Note that by using normalised index θ_2 described in Equation (36), we change the perspective from minimising risk to maximising safety.

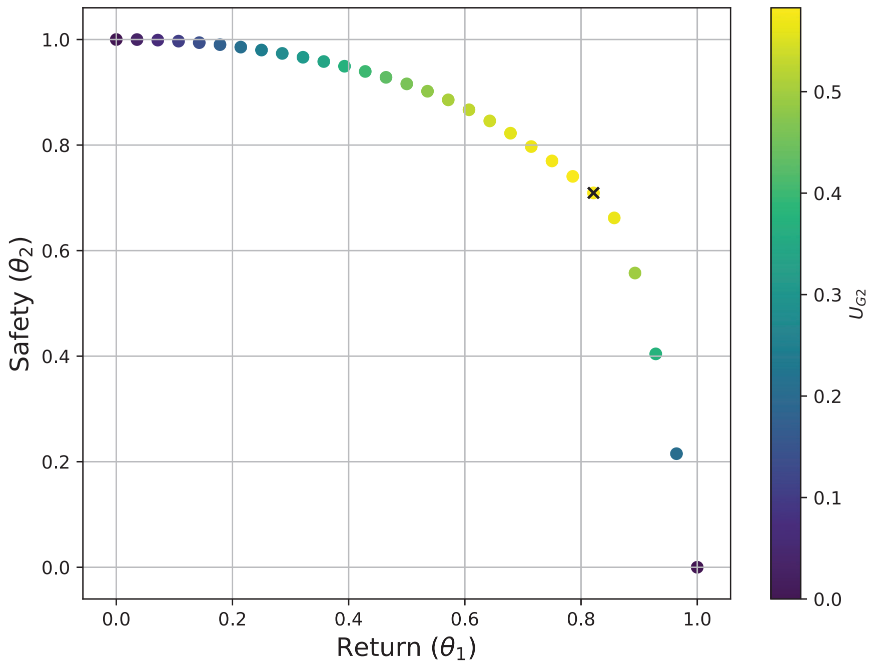

Fig. 4. Normalised efficient frontier and U_{G2}. Portfolio with maximum U_{G2} marked with 'x'.

4.3. The impact of risk preferences in additive and geometric portfolio selection

From Lemma 1, we know that weights have no impact in the maximisation of U_{G2}. An alternative geometric utility function that incorporates weights to reflect the preference of investors in a portfolio selection context is Cobb–Douglas utility, denoted by U_{CD}. Indeed, U_{G2} in Equation (5) is equivalent for optimisation purposes to U_{CD} in Equation (20) when weights w_j are set to one. In this section, we evaluate the impact that different risk preferences expressed in terms of weights assigned to returns and safety pairs from Table 2. Each pair of returns and safety represents a candidate portfolio from the efficient frontier to be ultimately selected according to the preferences of investors. As a result, an interesting research question is establishing the sensitiveness of the final portfolio choice to changes in the risk profile of different investors when using alternative additive and geometric utility functions.

In Table 3, we summarise the best portfolios from Table 2 according to additive and geometric utility functions for different risk preferences. We consider nine different risk profiles ranging from extremely conservative investors to extremely risky investors. Each risk profile is expressed in terms of weights w_1 and w_2 that add up to 1. The closer the value of w_1 to 1, the riskier the investor because portfolios with high returns and low safety are preferred to those with low returns and high safety. The final selection of the best portfolio not only depends on the risk preferences but also on the utility function used to evaluate the available options. In Table 3, we consider several types of utility functions: (1) $U_{\mathrm{ZY}1}$ representing the principle of maximum efficiency; (2) $U_{\mathrm{ZY}2}$ representing the principle of minimum deviation in terms of Euclidean distances; (3) $U_{\mathrm{ZY}\infty}$ representing the

Table 3
Best portfolios from Table 2 according to additive and geometric utility functions for different risk preferences

w_1	w_2	Risk profile	U_{ZY1}	U_{ZY2}	$U_{ZY\infty}$	U_{CD}	U_{ZY3}	U_{ZY10}
0.10	0.90	Extreme safety	3	6	7	7	7	7
0.20	0.80	Very strong safety	6	8	10	9	9	10
0.30	0.70	Strong safety	9	10	11	11	11	11
0.40	0.60	Moderate safety	11	12	12	12	12	12
0.50	0.50	Neutral	13	13	13	13	13	13
0.60	0.40	Moderate risk	14	14	14	14	14	14
0.70	0.30	Strong risk	16	15	15	15	15	15
0.80	0.20	Very strong risk	17	16	16	16	16	16
0.90	0.10	Extreme risky	18	17	17	17	17	17

principle of maximum fairness; (4) U_{CD} representing the principle of weighted compensability; and (5) U_{ZY3} and U_{ZY10} representing approximate cases of maximum fairness. Recall that in the context of CP functions U_{ZY1}, U_{ZY2}, $U_{ZY\infty}$, U_{ZY3} and U_{ZY10} are additive utility functions and that U_{CD} is a geometric utility function. This classification of functions allows us to evaluate the differences in the final selection due to the change of perspective introduced by GCP in this paper.

The first conclusion that we extract from the results of Table 3 is that neutral investors are not affected by the selection of any of the different utility functions considered. Note that this conclusion is restricted to the context of the particular form of the empirical efficient frontier depicted in Fig. 3. However, it is also worth noting that we expect small deviations from this behaviour because efficient frontiers derived from the mean-variance optimisation by Markowitz (1952) usually present this quadratic form in the normalised returns–safety space due to the minimisation of the variance of the returns. Indeed, Merton (1972) showed that the analytic derivation of the efficient frontier in portfolio selection is a parabola. As a result, all utility functions considered in the example recommend the choice of portfolio 13, including geometric utility U_{G2} in which weights have no impact. Then, we conclude that the choice of any of these utility functions, either additive or multiplicative, is not a critic step when dealing with neutral investors with no preference for safety or risk.

We also find that the use of utility U_{ZY1} tends to recommend portfolios that are closer to the extremes of the efficient frontier, especially when the risk profile is biased to extreme safety. This behaviour can be partially offset by using utility U_{ZY2}. In the case of additive utility functions, we observe the most balanced portfolios (non-corner solutions) when utility function $U_{ZY\infty}$ is used. These empirical results agree with other theoretical results presented by Ballestero (2007) and Salas-Molina et al. (2019) in the same context of portfolio selection. These results can be summarised as follows: the larger the value of p in Zeleny–Yu utility U_{ZY} from Equation (5), the more balance is achieved by recommended portfolios or, in other words, the further the portfolios from corner solutions.

An additional interesting finding is that increasing p from values greater than 2 as in U_{ZY2} to larger values to approach ∞ as in $U_{ZY\infty}$, has a very low impact in the selection of the best portfolios. For comparative purposes, we include in Table 3 utility U_{ZY3} we find that $p = 3$ results in only one change in the set of recommended portfolios for the whole range of risk profiles. Similarly, $p = 10$ for utility U_{ZY10} results in the same set of recommended portfolios. Then, we conclude that setting p

to small integer values, provided that $p > 2$, results in similar portfolio recommendations to setting $p = \infty$ when using additive utility functions.

We derive a final conclusion from the use of geometric utility U_{CD}: the balance of recommended portfolios is not only achieved by increasing the value p in Equation (5) but also by setting p to zero, hence changing the perspective to geometric utility. The principle of maximum fairness is not only achieved by applying the implicit maximisation of the minimum distance to the ideal when setting $p = \infty$ but also by considering the principle of compensability introduced by geometric utility U_{CD}. Summarising, the main point that motivates us to propose the use of geometric utility functions is that it allows to incorporate the principle of compensability in CP.

4.4. *Practical implications in portfolio selection*

In this section, we discuss the practical implications derived from the use of GCP in relation to portfolio selection publications using CP from a recent review by Aouni et al. (2018). To this end, we first assume that preferences for profitability and safety are already known. Then, portfolio selection using CP usually follows a two-step method: (1) construction of the efficient frontier; and (2) selection of one point of the efficient frontier according to the risk preferences. We here focus on the second step to extend the range of principles that may help investors in the selection of the best portfolio.

It is usually assumed in CP that the optimum portfolio belongs to the compromise set given by bounds L_1 and L_∞ of the efficient frontier. We here extend the concept of compromise set to account for compensability by considering point L_0. Maximum compensability is achieved in point L_1 and limited compensability is achieved in point L_0 as shown in Section 4.1. Indeed, an investor concerned by compensability that wants to select one of the portfolios from Table 2 should select U_{G2}. This comment relates to the results reported by Ballestero and Pla-Santamaria (2003), but the same reasoning is appropriate to similar efficient frontier results in Ballestero and Romero (1998), Ballestero and Pla-Santamaria (2004) and Ballestero et al. (2012).

Within a fuzzy environment Bilbao-Terol et al. (2006), reported points L_1 and L_∞ in a numerical example as alternative solutions for a an investor. This set of alternative solutions could be straightforwardly extended by considering point L_0 as proposed in this paper. As a result, point L_0 can be viewed as an extension of the compromise set reflecting preferences for compensability.

Amiri et al. (2011) proposed nadir CP by changing the usual perspective of minimising distance to the ideal values to maximising distance from the nadir (anti-ideal) values. The authors rely on a parametric distance function similar to Equation (5). Similarly, Xia et al. (2001) and Hasuike and Katagiri (2014) restrict their analysis for simplicity to cases $p = 2$ and $p = 1$, respectively. Salas-Molina et al. (2019) proposed different practical tools to deal with discrete efficient frontiers and uncertain risk preferences for cases $p = 1$ and $p = 2$. Caçador et al. (2021) proposed a new methodology for computing relative-robust portfolios based on the minimax regret, which is equivalent to setting $p = \infty$ in Equation (5). From the review of the previous works, we argue that further insight can be obtained from the analysis of an alternative set of solutions by setting $p = 0$ as proposed by GCP.

We also consider that a prominent field of application of the GCP model is in the field of socially responsible investments. In recent years, some scholars have attempted to develop new approaches for integrating sustainable targets within investment and financial decisions (Hallerbach et al., 2004; Bilbao-Terol et al., 2014; Utz et al., 2014; Ballestero et al., 2015; Garcia-Bernabeu et al., 2019; Liagkouras et al., 2020). In these investigations, the sustainability component is integrated as an objective or constraint to determine the set of solutions of the efficient frontier. The GCP approach is a suitable way to better reflect the preferences of those investors because bias for financial or ESG criteria can be described in extreme positions such as ESG agnostic or ESG motivated investor profiles. Then, an additional added value of GCP in portfolio selection is the possibility of offering investment solutions that integrates information on the preferences about the level of compensability. Thus, investors showing limited or weighted compensability among a given set of criteria could use U_{G2} or U_{CD}, respectively, as shown in Table 1. On the other hand, extreme seekers are better characterised by utility function U_{G1} because both extremes are equally desirable for them. However, recall that using U_{G1} as a measure of utility is equivalent to searching for the maximum achievement of a single-criterion disregarding the achievement of the rest of the criteria under consideration.

Along the lines of the insights derived from Section 4.3, we also argue that GCP adds meaning to the traditional CP approach. First, we show that the final solution not only depends on the risk preferences but also on the utility function used to evaluate the available options. Finally, the solutions obtained by applying the principle of maximum fairness are very similar to those obtained by applying the principle of weighted compensability.

5. Concluding remarks

In this paper, we extend conventional CP by proposing a new class of solutions based on geometric utility functions. We set up our decision-making model by pivoting around a common assumption in CP, namely, the bounds of the compromise set. To this end we consider metric $p = 0$ as an additional way to find compromise solutions that derive from multiplicative utility functions. One of our main findings is the existence of some kind of duality in the optimisation directions derived from the use of multiplicative functions. By minimising geometric distances to the ideal point, we find a new class of compromise solutions that fits well the behaviour of extreme seekers without no preference for any of the extremes.

GCP implies a generalisation of Zeleny's axiom of choice by means of additive and multiplicative utility functions. GCP incorporates the principle of limited compensability as a desired requirement in many multiple criteria decision-making contexts. An additional interesting feature of GCP is that, under reasonable assumptions, it is connected to the principle of minimising the maximum regret (the so-called maximin principle). This connection leads us to extend the $L_1 - L_\infty$ concept of compromise set to propose the notion of geometric compromise set based on point L_1, with minimum additive distance to the ideal point, and point L_0, with maximum geometric distance to the anti-ideal point.

We believe that the GCP approach presented in this paper can be applied to any other field in which CP is used to evaluate solutions in terms of the distance to a reference point. A non-comprehensive list of suitable fields of application include credit risk management (Pla-Santamaria et al., 2021), cash management (Salas-Molina, 2019), water management (Fattahi and Fayyaz, 2010), forestry and agricultural planning (Diaz-Balteiro and Romero, 2008) and public policy design (André et al., 2010). In addition, we consider that the design of additional utility functions to characterise in a different way the behaviour of extreme seekers is a natural extension of this work. Furthermore, we firmly believe that alternative values of p in the range of negative or fractional numbers could lead to an interesting extension of CP.

References

Amiri, M., Ekhtiari, M., Yazdani, M., 2011. Nadir compromise programming: a model for optimization of multi-objective portfolio problem. *Expert Systems with Applications* 38, 6, 7222–7226.

André, F.J., Cardenete, M.A., Romero, C., 2010. *Designing Public Policies: An Approach Based on Multi-Criteria Analysis and Computable General Equilibrium Modeling*, Vol. 642. Springer Science & Business Media, Berlin.

Aouni, B., Doumpos, M., Pérez-Gladish, B., Steuer, R.E., 2018. On the increasing importance of multiple criteria decision aid methods for portfolio selection. *Journal of the Operational Research Society* 69, 10, 1525–1542.

Ballestero, E., 2007. Compromise programming: a utility-based linear-quadratic composite metric from the trade-off between achievement and balanced (non-corner) solutions. *European Journal of Operational Research* 182, 3, 1369–1382.

Ballestero, E., Bravo, M., Pérez-Gladish, B., Arenas-Parra, M., Pla-Santamaria, D., 2012. Socially responsible investment: a multicriteria approach to portfolio selection combining ethical and financial objectives. *European Journal of Operational Research* 216, 2, 487–494.

Ballestero, E., Pérez-Gladish, B., Garcia-Bernabeu, A., 2015. Socially responsible investment. *A Multi-Criteria Decision Making Approach*. *International Series in Operations Research & Management Science*. Springer, Berlin.

Ballestero, E., Pla-Santamaria, D., 2003. Portfolio selection on the Madrid exchange: a compromise programming model. *International Transactions in Operational Research* 10, 1, 33–51.

Ballestero, E., Pla-Santamaria, D., 2004. Selecting portfolios for mutual funds. *Omega* 32, 5, 385–394.

Ballestero, E., Romero, C., 1996. Portfolio selection: a compromise programming solution. *Journal of the Operational Research Society* 47, 11, 1377–1386.

Ballestero, E., Romero, C., 1998. *Multiple Criteria Decision Making and Its Applications to Economic Problems*. Kluwer Academic Publishers, Dordrecht.

Barthold, T.A., Hochman, H.M., 1988. Addiction as extreme-seeking. *Economic Inquiry* 26, 1, 89–106.

Bentham, J., 1789. *An Introduction to the Principles of Morals and Legislation*. Payne and Son, London.

Bilbao-Terol, A., Arenas-Parra, M., Cañal-Fernández, V., Antomil-Ibias, J., 2014. Using TOPSIS for assessing the sustainability of government bond funds. *Omega* 49, 1–17.

Bilbao-Terol, A., Pérez-Gladish, B., Arenas-Parra, M., Rodríguez-Uría, M.V., 2006. Fuzzy compromise programming for portfolio selection. *Applied Mathematics and computation* 173, 1, 251–264.

Boyd, S., Kim, S.J., Vandenberghe, L., Hassibi, A., 2007. A tutorial on geometric programming. *Optimization and Engineering* 8, 1, 67–127.

Caçador, S., Dias, J.M., Godinho, P., 2021. Portfolio selection under uncertainty: a new methodology for computing relative-robust solutions. *International Transactions in Operational Research* 28, 3, 1296–1329.

Ceriani, L., Verme, P., 2012. The origins of the Gini index: extracts from Variabilità e Mutabilità (1912) by Corrado Gini. *The Journal of Economic Inequality* 10, 3, 421–443.

Diaz-Balteiro, L., Romero, C., 2008. Making forestry decisions with multiple criteria: a review and an assessment. *Forest Ecology and Management* 255, 8-9, 3222–3241.

Duffin, R.J., 1970. Linearizing geometric programs. *SIAM Review* 12, 2, 211–227.

Ehrgott, M., 2005. *Multicriteria Optimization*, Vol. 491. Springer Science & Business Media, Berlin.

Fattahi, P., Fayyaz, S., 2010. A compromise programming model to integrated urban water management. *Water Resources Management* 24, 6, 1211–1227.

Garcia-Bernabeu, A., Hilario-Caballero, A., Pla-Santamaria, D., Salas-Molina, F., 2020. A process oriented MCDM approach to construct a circular economy composite index. *Sustainability* 12, 2, 618.

Garcia-Bernabeu, A., Salcedo, J.V., Hilario, A., Pla-Santamaria, D., Herrero, J.M., 2019. Computing the mean-variance-sustainability nondominated surface by ev-MOGA. *Complexity* 2019, 12.

Gharakhani, M., Sadjadi, S., 2013. A fuzzy compromise programming approach for the Black-Litterman portfolio selection model. *Decision Science Letters* 2, 1, 11–22.

Gini, C., 1912. *Variabilità e Mutabilità. Contributo allo Studio delle Distribuzioni e delle Relazioni Statistiche*. C. Cuppini, Bologna.

Gong, X., Yu, C., Min, L., 2021. A cloud theory-based multi-objective portfolio selection model with variable risk appetite. *Expert Systems with Applications* 176, 114911.

González-Pachón, J., Romero, C., 2011. The design of socially optimal decisions in a consensus scenario. *Omega* 39, 2, 179–185.

González-Pachón, J., Romero, C., 2016. Bentham, Marx and Rawls ethical principles: In search for a compromise. *Omega* 62, 47–51.

Gumbel, E.J., 1958. *Statistics of Extremes*. Columbia University Press, New York.

Hallerbach, W., Ning, H., Soppe, A., Spronk, J., 2004. A framework for managing a portfolio of socially responsible investments. *European Journal of Operational Research* 153, 2, 517–529.

Hasuike, T., Katagiri, H., 2014. Risk-controlled multiobjective portfolio selection problem using a principle of compromise. *Mathematical Problems in Engineering* 2014, 7.

Kluyver, T., Ragan-Kelley, B., Pérez, F., Granger, B.E., Bussonnier, M., Frederic, J., Kelley, K., Hamrick, J.B., Grout, J., Corlay, S., Ivanov, P., Avila, D., Abdalla, S., Willing, C., Jupyter Development Team, 2016. Jupyter notebooks—a publishing format for reproducible computational workflows. 20th International Conference on Electronic Publishing, IOS Press, Amsterdam, pp. 87–90.

Kraft, D., 1988. A software package for sequential quadratic programming. *Forschungsbericht- Deutsche Forschungs- und Versuchsanstalt fur Luft- und Raumfahrt* 28.

Li, J., Xu, J., 2013. Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm. *Information Sciences* 220, 507–521.

Liagkouras, K., Metaxiotis, K., Tsihrintzis, G., 2020. Incorporating environmental and social considerations into the portfolio optimization process. *Annals of Operations Research* 1–26. https://doi.org/10.1007/s10479-020-03554-3.

Markowitz, H., 1952. Portfolio selection. *The Journal of Finance* 7, 1, 77–91.

Merigó, J.M., Gil-Lafuente, A.M., 2008. Using the OWA operator in the Minkowski distance. *International Journal of Computer Science* 3, 3, 149–157.

Merton, R.C., 1972. An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis* 7, 4, 1851–1872.

Munda, G., 2005. Measuring sustainability: a multi-criterion framework. *Environment, Development and Sustainability* 7, 1, 117–134.

Parra, M.A., Terol, A.B., Gladish, B.P., Urıa, M.R., 2005. Solving a multiobjective possibilistic problem through compromise programming. *European Journal of Operational Research* 164, 3, 748–759.

Pla-Santamaria, D., Bravo, M., Reig-Mullor, J., Salas-Molina, F., 2021. A multicriteria approach to manage credit risk under strict uncertainty. *TOP* 29, 2, 494–523.

Qi, Y., Steuer, R.E., Wimmer, M., 2017. An analytical derivation of the efficient surface in portfolio selection with three criteria. *Annals of Operations Research* 251, 1, 161–177.

Rawls, J., 1973. *A Theory of Justice*. Oxford University Press, Oxford.

Romero, C., 2001. A note on distributive equity and social efficiency. *Journal of Agricultural Economics* 52, 2, 110–112.

Salas-Molina, F., 2019. Selecting the best risk measure in multiobjective cash management. *International Transactions in Operational Research* 26, 3, 929–945.

Salas-Molina, F., Rodriguez-Aguilar, J.A., Pla-Santamaria, D., 2019. Characterizing compromise solutions for investors with uncertain risk preferences. *Operational Research* 19, 3, 661–677.

Schelling, T.C., 1978. Egonomics, or the art of self-management. *The American Economic Review* 68, 2, 290–294.

Steuer, R.E., 1986. *Multiple Criteria Optimization*. John Wiley & Sons, New York.

Steuer, R.E., Qi, Y., Hirschberger, M., 2005. Multiple objectives in portfolio selection. *Journal of Financial Decision Making* 1, 1, 5–20.

Stigler, G.J., Becker, G.S., 1977. De gustibus non est disputandum. *The American Economic Review* 67, 2, 76–90.

Tirole, J., 1988. *The Theory of Industrial Organization*. MIT Press, Cambridge, MA.
Utz, S., Wimmer, M., Hirschberger, M., Steuer, R.E., 2014. Tri-criterion inverse portfolio optimization with application to socially responsible mutual funds. *European Journal of Operational Research* 234, 2, 491–498.
Xia, Y., Wang, S., Deng, X., 2001. A compromise solution to mutual funds portfolio selection with transaction costs. *European Journal of Operational Research* 134, 3, 564–581.
Xian, S., Sun, W., Xu, S., Gao, Y., 2016. Fuzzy linguistic induced OWA Minkowski distance operator and its application in group decision making. *Pattern Analysis and Applications* 19, 2, 325–335.
Yu, P.L., 1973. A class of solutions for group decision problems. *Management Science* 19, 8, 936–946.
Yu, P.L., 1985. *Multiple-Criteria Decision Making: Concepts, Techniques, and Extensions*. Plenum Press, New York.
Zeleny, M., 1973. Compromise programming. In *Multiple Criteria Decision Making*. University of South Carolina Press, Columbia, SC, pp. 262–301.
Zopounidis, C., Doumpos, M., 2002. Multi-criteria decision aid in financial decision making: methodologies and literature review. *Journal of Multi-Criteria Decision Analysis* 11, 4–5, 167–186.

Appendix

In this Appendix, we show how setting $p = 0$ leads to two different geometric utility functions in Equations (7) and (8). In both cases, we consider the natural logarithm of the limit and apply L'Hopital's rule.

First case: weights raised to p.

$$\ln(\mathcal{L}_0) = \lim_{p\to 0} \frac{\ln\left(\sum_{j=1}^{q} w_j^p z_j^p\right)}{p} = \lim_{p\to 0} \frac{\sum_{j=1}^{q} w_j^p z_j^p \ln(w_j z_j)}{\sum_{j=1}^{q} w_j^p z_j^p} \tag{A1}$$

$$\ln(\mathcal{L}_0) = \frac{\sum_{j=1}^{q} \ln(w_j z_j)}{q} = \ln\left(\left[\prod_{j=1}^{q} w_j z_j\right]^{1/q}\right) \tag{A2}$$

$$\mathcal{L}_0 = \left[\prod_{j=1}^{q} w_j z_j\right]^{1/q}. \tag{A3}$$

Second case: weights not raised to p and $\sum_{j=1}^{q} w_j = 1$.

$$\ln(\mathcal{L}_0) = \lim_{p\to 0} \frac{\ln\left(\sum_{j=1}^{q} w_j z_j^p\right)}{p} = \lim_{p\to 0} \frac{\sum_{j=1}^{q} w_j z_j^p \ln(z_j)}{\sum_{j=1}^{q} w_j z_j^p} \tag{A4}$$

$$\ln(\mathcal{L}_0) = \frac{\sum_{j=1}^{q} w_j \ln(z_j)}{\sum_{j=1}^{q} w_j} = \sum_{j=1}^{q} \ln(z_j^{w_j}) = \ln(\prod_{j=1}^{q} z_j^{w_j}) \tag{A5}$$

$$\mathcal{L}_0 = \prod_{j=1}^{q} z_j^{w_j}. \tag{A6}$$