

Design of iterative methods with memory for solving nonlinear systems

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In this paper, we design two parametric classes of iterative methods without memory to solve nonlinear systems, whose convergence order is 4 and 7, respectively. From their error equations and to increase the convergence order without performing new functional evaluations, memory is introduced in these families of different forms. That allows us to increase from 4 to 6 the convergence order in the first family and from 7 to 11 in the second one. We perform some numerical experiments with big size systems for confirming the theoretical results and comparing the proposed methods along other known schemes.

KEYWORDS

basins of attraction, dynamical planes, iterative schemes, memory schemes, nonlinear systems

MSC CLASSIFICATION

65H10, 65B99

1 | INTRODUCTION

It is very common that, when solving real problems in Engineering or Science, we need to solve $F(x) = 0$, being $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, where D is a neighborhood of a solution of $F(x) = 0$. In general, it is difficult to solve these systems in exact way, so often an attempt is made to obtain an approximate solution rather than the exact solution. One way to achieve this is by iterative methods. These methods obtain a sequence of approximations, $\{x^{(k)}\}$, which, under certain conditions, converges to the solution of the system. One of the best known schemes is Newton's procedure, which is expressed as

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, \dots, \quad (1)$$

denoting as $F'(x)$ the Jacobian matrix related to F .

Newton's scheme is well-known for its efficiency and simplicity, as well as for its quadratic convergence. When the derivative in (1) is replaced by the divided difference $[x^{(k)} + F(x^{(k)}), x^{(k)}; F]$, we obtain Steffensen's scheme [1], which is a derivative-free and also has quadratic convergence. The calculation of an inverse of F' in the iterative expression of a method can be a drawback when the function to be studied cannot be derived, its derivative is too expensive to calculate, or the Jacobian matrix is singular.

Different techniques have been used to construct Newton-like procedures, as weight functions, direct composition, estimations of the Jacobian matrices by using divided difference operators, and so on. Therefore, some iterative methods for estimating the solutions of $F(x) = 0$ have been analyzed with different order of convergence. The aim of these proposals is to accelerate the convergence or to improve the computational efficiency.

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Recently, new parametric families of iterative methods were proposed in earlier studies [2–4], including a fast procedure for solving this kind of problems. Also, iterative methods avoiding the Jacobian matrix have been proposed in earlier research [5–7], with good orders of convergence. In them, Jacobian matrix $F'(\cdot)$ is replaced by $[\cdot, \cdot; F]$, the divided difference operator. Matrix weight function procedure plays also a key role in the design of iterative schemes, as can be seen in other research [8, 9].

In the works cited in the previous paragraph, and many others in the literature, iterative methods with high order of convergence are presented, but they considerably increase the computational cost. To avoid this increase, we resort to memory methods, that is, schemes in which the iterative expression uses several previous iterations. Iterative schemes for solving systems by means of memory have been very recently constructed in some paper. In them, a new iterate is obtained from using at least two previous iterations. In this way, it is proven that the convergence order increases but no new functional evaluations are added, that means, without performing any functional evaluations other than those already performed by the original method; see previous works [10–13].

In order to increase the quadratic convergence of Newton's method, Traub [14] proposed the following scheme

$$\begin{aligned} y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - F'(x^{(k)})^{-1}F(y^{(k)}), \quad k = 0, 1, \dots, \end{aligned} \quad (2)$$

which has order of convergence three.

In this paper, we design a derivative-free variant of Traub's method by replacing the derivatives by a divided difference with a parameter and a weight function. This yields the following parametric family, which as we shall see below is a class of iterative methods of fourth-order, that we denote by $M_{4,\gamma}$.

$$\begin{aligned} y^{(k)} &= x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)}), \quad k = 0, 1, \dots, \end{aligned} \quad (3)$$

where $w^{(k)} = x^{(k)} + \gamma F(x^{(k)})$, $\gamma \neq 0$, $\gamma \in \mathbb{R}$ and the variable of the weight function is $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[y^{(k)}, w^{(k)}; F]$. The first step of this method corresponds to Steffensen's scheme when $\gamma = 1$.

For increasing the order of convergence, we introduce one more step to the parametric family $M_{4,\gamma}$, obtaining the following class which, as we will see, has order 7, and it is denoted by $M_{7,\gamma}$.

$$\begin{aligned} y^{(k)} &= x^{(k)} - [w^{(k)}, x^{(k)}; F]^{-1}F(x^{(k)}), \\ z^{(k)} &= y^{(k)} - H(\mu^{(k)})[y^{(k)}, x^{(k)}; F]^{-1}F(y^{(k)}), \\ x^{(k+1)} &= z^{(k)} - G(\mu^{(k)}, \nu^{(k)})[z^{(k)}, y^{(k)}; F]^{-1}F(z^{(k)}), \quad k = 0, 1, \dots, \end{aligned} \quad (4)$$

where $\nu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[z^{(k)}, y^{(k)}; F]H(\mu^{(k)})$.

In addition, to design these families of iterative methods, we analyze several ways to introduce memory into their iterative formulas, replacing the parameter with an expression that uses the previous iterates and their functional evaluations. In this way, the order of the schemes is increased without adding new evaluations.

To prove the convergence order of the methods with memory, we use the Ortega–Rheinboldt Theorem, which appears in Ortega and Rheinboldt [15].

Theorem 1. *Let us consider a with memory iterative procedure ϕ , generating the sequence $\{x^{(k)}\}$ that converges to its root α . If there exists a constant η nonzero and positive t_i , $i = 0, \dots, m$ such that*

$$|e_{k+1}| \leq \eta \prod_{i=0}^m |e_{k-i}|^{t_i},$$

is held, then the R-order of convergence of ϕ is at least p , being p the unique positive root of

$$p^{m+1} - \sum_{i=0}^m t_i p^{m-i} = 0.$$

This manuscript is structured as follows. In Section 2, we study the order of convergence of the new classes of iterative methods to solve nonlinear systems. We also study in Section 2 how to introduce memory to these parametric families in order to increase the order of convergence without performing new functional evaluations. In Section 3, we perform some numerical experiments for confirming the theoretical results. The paper finishes with some conclusions and the references used in it.

2 | CONVERGENCE ANALYSIS

The tools we are going to use, in demonstrating the convergence of these schemes, were defined and introduced by the authors in Cordero and Torregrosa [16].

Let us consider $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ as a sufficiently differentiable function in D , a neighborhood of a root α of $F(x) = 0$; we consider the operator

$$[x + h, x; F] = \int_0^1 F'(x + th)dt, \tag{5}$$

defined by Genochi–Hermite on Ortega and Rheinboldt [15]. Using the Taylor development of $F'(x + th)$ at x and by direct integration, we get

$$[x + h, x; F] = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3). \tag{6}$$

Let us also consider $X = \mathbb{R}^{n \times n}$ be the Banach space of size $n \times n$ real square matrixes and $H : X \rightarrow X$ a function defined as follows:

- $H'(u)(v) = H_1uv$, where $H' : X \rightarrow \mathcal{L}(X)$ and $H_1 \in \mathbb{R}$,
- $H''(u, v)(w) = H_2uvw$, being $H'' : X \times X \rightarrow \mathcal{L}(X)$ and $H_2 \in \mathbb{R}$,

where it is denoted by $\mathcal{L}(X)$ the set of linear operators in X . When $k \rightarrow \infty$, $\mu^{(k)} \rightarrow 0$, being 0 the zero matrix. Therefore, there exist real numbers H_0, H_1, H_2 such that H can be expanded around 0 as follows:

$$H(\mu^{(k)}) = H_0I + H_1\mu^{(k)} + \frac{1}{2}H_2(\mu^{(k)})^2 + O((\mu^{(k)})^3),$$

being I is the $n \times n$ identity matrix.

In the same way, we define a multivariable matrix function $G(\mu^{(k)}, v^{(k)})$ so there exist real numbers $G_0, G_{11}, G_{12}, G_{2i}$ for $i = 1, 2, 3$ and G_{3j} for $j = 1, 2, 3, 4$ such that G can be expanded around $(0, 0)$ as follows:

$$\begin{aligned} G(\mu^{(k)}, v^{(k)}) &= G_0I + G_{11}\mu^{(k)} + G_{12}v^{(k)} + \frac{1}{2}(G_{21}(\mu^{(k)})^2 + G_{22}\mu^{(k)}v^{(k)} \\ &+ G_{23}(v^{(k)})^2) + \frac{1}{6}(G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2v^{(k)} \\ &+ G_{33}\mu^{(k)}(v^{(k)})^2 + G_{34}(v^{(k)})^3) + O_4(\mu^{(k)}, v^{(k)}), \end{aligned}$$

where $O_4(\mu^{(k)}, v^{(k)})$ denotes all terms in which the sum of exponents of $\mu^{(k)}$ and $v^{(k)}$ is at least 4.

2.1 | Convergence analysis of $M_{4,\gamma}$

Next, we prove the order of convergence of parametric class $M_{4,\gamma}$.

Theorem 2. *Let us consider $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ a differentiable enough function defined in a neighborhood D of α , such that $F(\alpha) = 0$. Let us also assume that $F'(\alpha)$ is nonsingular. Let $H(\mu)$ be a real matrix function satisfying $H_0 = 1, H_1 = 1$ and $|H_2| < \infty$, being I the $n \times n$ identity matrix. Therefore, being $x^{(0)}$ an initial guess sufficiently near to α , sequence $\{x^{(k)}\}$ defined by $M_{4,\gamma}$ converges to α with order 4, for any nonzero value of parameter γ , and its error equation is as follows:*

$$e_{k+1} = \left(-C_3(I + \gamma F'(\alpha)) + C_2 \left(\left(3I - \frac{H_2}{2}I + \gamma F'(\alpha) \right) C_2 + \gamma C_2 F'(\alpha) \right) \right) C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5), \tag{7}$$

being $C_i = \frac{1}{i!}[F'(\alpha)]^{-1}F^{(i)}(\alpha), i = 2, 3, \dots$, and $e_k = x^{(k)} - \alpha$.

Proof. Let us consider the Taylor expansion of $F(x^{(k)})$ around α

$$F(x^{(k)}) = F'(\alpha) (e_k + C_2 e_k^2 + C_3 e_k^3 + C_4 e_k^4 + O(e_k^5)). \quad (8)$$

We also consider, equivalently to (8), the expansion of $F(w^{(k)})$ around α , where $e_w = w^{(k)} - \alpha$. We calculate the Taylor expansion of $[w^{(k)}, x^{(k)}; F]$ by using (6)

$$[w^{(k)}, x^{(k)}; F] = F'(\alpha) (I + C_2(2I + \gamma F'(\alpha))e_k + (\gamma C_2 F'(\alpha)C_2 + C_3(3I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2)) e_k^2) + O(e_k^3).$$

Next, for the inverse of the divided difference operator $[w^{(k)}, x^{(k)}; F]$, we obtain

$$[w^{(k)}, x^{(k)}; F]^{-1} = (I + X_2 e_k + X_3 e_k^2 + O(e_k^3))F'(\alpha)^{-1}, \quad (9)$$

where

$$\begin{aligned} X_2 &= -C_2(2I + \gamma F'(\alpha)), \\ X_3 &= 4C_2^2 + \gamma C_2 F'(\alpha)C_2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3(3I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2). \end{aligned}$$

Then,

$$\begin{aligned} y^{(k)} - \alpha &= e_k - [w^{(k)}, x^{(k)}; F]^{-1} F(x^{(k)}) \\ &= C_2(I + \gamma F'(\alpha))e_k^2 - (2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2 (C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2)) e_k^3 + O(e_k^4). \end{aligned} \quad (10)$$

In order to obtain the Taylor expansion of $\mu^{(k)} = I - [w^{(k)}, x^{(k)}; F]^{-1}[y^{(k)}, w^{(k)}; F]$, we have

$$[y^{(k)}, w^{(k)}; F] = F'(\alpha)(I + C_2(I + \gamma F'(\alpha))e_k + (\gamma C_2^2 F'(\alpha) + \gamma C_2 F'(\alpha)C_2 + C_2^2 + C_3(I + 2\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2)) e_k^2) + O(e_k^3).$$

So,

$$\mu^{(k)} = C_2 e_k + (-C_2(C_2(3I + \gamma F'(\alpha)) + \gamma F'(\alpha)C_2) + C_3(2I + \gamma F'(\alpha)))e_k^2 + O(e_k^3).$$

Therefore, if we denote $M_3 = -C_2(C_2(3I + \gamma F'(\alpha)) + \gamma F'(\alpha)C_2) + C_3(2I + \gamma F'(\alpha))$, then $\mu^{(k)} = C_2 e_k + M_3 e_k^2 + O(e_k^3)$ and

$$\begin{aligned} H(\mu^{(k)}) &= H_0 + H_1 \mu^{(k)} + \frac{1}{2} H_2 (\mu^{(k)})^2 + O_3(\mu^{(k)}) = I + \mu^{(k)} + \frac{H_2}{2} (\mu^{(k)})^2 + O_3(\mu^{(k)}) \\ &= I + C_2 e_k + \left(M_3 + \frac{H_2}{2} C_2^2 \right) e_k^2 + O(e_k^3). \end{aligned}$$

Applying (6),

$$[y^{(k)}, x^{(k)}; F] = F'(\alpha) (I + C_2 e_k + (C_3 + C_2^2(I + \gamma F'(\alpha)))e_k^2) + O(e_k^3).$$

Then, $[y^{(k)}, x^{(k)}; F]^{-1} = (I + R_2 e_k + R_3 e_k^2 + O(e_k^3))F'(\alpha)^{-1}$, where

$$\begin{aligned} R_2 &= -C_2, \\ R_3 &= -C_3 + C_2(I + \gamma F'(\alpha)C_2 - \gamma C_2 F'(\alpha)). \end{aligned}$$

If we denote by $e_y = y^{(k)} - \alpha$, then

$$\begin{aligned} e_{k+1} &= e_y - \left(I + C_2 e_k + \left(M_3 + \frac{H_2}{2} C_2^2 \right) e_k^2 + O(e_k^3) \right) ((I + R_2 e_k + R_3 e_k^2 + O(e_k^3))(e_y + C_2 e_y^2 + O(e_y^3))) \\ &= \left(-C_3(I + \gamma F'(\alpha)) + C_2 \left((I + \gamma F'(\alpha)) C_2 + C_2 \left(2I - \frac{H_2}{2} I + \gamma F'(\alpha) \right) \right) \right) C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \end{aligned} \quad (11)$$

It is proven that family $M_{4,\gamma}$ has order 4 under these standard conditions. Moreover, if we assume that $H_2 = 2I$, then the expression of the error equation is

$$e_{k+1} = \left(-C_3(I + \gamma F'(\alpha)) + C_2 \left((I + \gamma F'(\alpha)) C_2 + C_2(I + \gamma F'(\alpha)) \right) \right) C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \tag{12}$$

□

2.2 | Introducing memory to $M_{4,\gamma}$

We have shown that parametric family $M_{4,\gamma}$ has order 4. Now, we are going to design other higher order methods from this one by introducing memory.

From the error equation, if parameter γ is $\gamma = -[F'(\alpha)]^{-1}$, then we increase the order, but as we do not know the solution α , we must estimate this expression.

A known way to approximate this expression is by using divided difference operators, whereby we approximate the parameter as follows:

$$\gamma_k = -[x^{(k)}, x^{(k-1)}; F]^{-1}.$$

By replacing this on the family $M_{4,\gamma}$, we get a method with memory, denoted by M_4D .

Another way to approximate this parameter is by using Kurchatov's divided difference operator, that is,

$$\gamma_k = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1}.$$

If we replace the parameter of the family $M_{4,\gamma}$ by this approximation, we obtain a method with memory that we denote by M_4K .

Let us now calculate the order of these two methods with memory.

In the proof of these results, we use Theorem 1 and the following notation: If $\lim_{k \rightarrow \infty} \frac{f(x_k)}{g(x_k)} = C$, being C a nonzero constant, we denote is by $f \sim Cg$.

Theorem 3. *Let us consider $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ a differentiable enough function defined in an neighborhood D of the root of F , α . We assume that $F'(\alpha)$ is nonsingular. Let $H(\mu)$ be a real matrix function that satisfies $H_0 = 1$, $H_1 = 1$, and $H_2 = 2$, being I the $n \times n$ identity matrix. Therefore, being the initial guess $x^{(0)}$ sufficiently close to α , sequence $\{x^{(k)}\}$ obtained by M_4D converges to α with order $p = 2 + \sqrt{6} \approx 4.44949$ and method M_4K converges to the root α with convergence order $p = 2 + 2\sqrt{2} \approx 4.82843$.*

Proof. Let us now consider the Taylor expansion of $F(x^{(k-1)})$, $F'(x^{(k-1)})$ and $F''(x^{(k-1)})$ around α (8), in the same way as in Theorem 2. Let us now calculate $[x^{(k)}, x^{(k-1)}; F]$ using (6) with $h = e_k - e_{k-1}$,

$$[x^{(k)}, x^{(k-1)}; F] = F'(\alpha) (I + C_2(e_k + e_{k-1})) + O_2(e_k, e_{k-1}).$$

Then,

$$[x^{(k)}, x^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{k-1}). \tag{13}$$

Therefore, $\gamma_k = -(I - C_2(e_k + e_{k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{k-1})$ and

$$I + \gamma_k F'(\alpha) = C_2(e_k + e_{k-1}) + O_2(e_{k-1}, e_k). \tag{14}$$

Thus, $I + \gamma_k F'(\alpha) \sim e_{k-1}$. By the error equation (7) and (14), we have

$$e_{k+1} \sim e_{k-1}^2 e_k^4. \tag{15}$$

In addition, suppose that the R-order of the method is at least p . Therefore, it is satisfied that

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p , when $k \rightarrow \infty$. Then,

$$e_{k+1} \sim D_{k,p}(D_{k-1,p}e_{k-1}^p)^p = D_{k,p}D_{k-1,p}^p e_{k-1}^{p^2}. \quad (16)$$

In the same way that relation (15) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^2(D_{k-1,p}e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+2}. \quad (17)$$

Then, by equating the exponents of e_{k-1} in (16) and (17), we get that

$$p^2 = 4p + 2,$$

whose only positive solution defines convergence order of M_4D , being $p = 2 + \sqrt{6} \approx 4.44949$.

On the other hand, applying (6),

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F] = F'(\alpha) (I + 2C_2e_k - 2C_3e_{k-1}e_k + C_3e_{k-1}^2 + 4C_3e_k^2) + O_3(e_k, e_{k-1}).$$

Then,

$$[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1} = (I - 2C_2e_k - C_3e_{k-1}^2 + 2C_3e_{k-1}e_k + 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1} + O_3(e_k, e_{k-1}).$$

Therefore,

$$I + \gamma_k F'(\alpha) = (2C_2e_k + C_3e_{k-1}^2 - 2C_3e_{k-1}e_k - 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1} + O_3(e_k, e_{k-1}).$$

Thus, $I + \gamma_k F'(\alpha)$ can have the behavior of e_k , $e_k e_{k-1}$, e_k^2 , or e_{k-1}^2 . Obviously, the factors $e_k e_{k-1}$ and e_k^2 tend faster to 0 than e_k , so we have to see whether e_k or e_{k-1}^2 converges faster. Let us assume now that the R-order of the scheme is, at least, p . Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p}e_k^p,$$

where $D_{k,p}$ tends to D_p , the asymptotic error constant, when $k \rightarrow \infty$. Then, we have

$$\frac{e_k}{e_{k-1}^2} \sim \frac{D_{k-1,p}e_{k-1}^p}{e_{k-1}^2}.$$

Then, if $p > 2$, we obtain that $\frac{D_{k-1,p}e_{k-1}^p}{e_{k-1}^2}$ converges to 0 when $k \rightarrow \infty$. Thus, if $p > 2$, then $I + \gamma_k F'(\alpha) \sim e_{k-1}^2$.

From error Equation (7) and the above relation, we obtain

$$e_{k+1} \sim e_{k-1}^4 e_k^4. \quad (18)$$

In addition, by assuming that the R-order of the scheme is, at least, p , we have relation (16). In the same way that we obtain relation (18), we have

$$e_{k+1} \sim e_{k-1}^4 (D_{k-1,p}e_{k-1}^p)^4 = D_{k-1,p}^4 e_{k-1}^{4p+4}. \quad (19)$$

Then, equating the exponents of e_{k-1} in (16) and (19), it follows that

$$p^2 = 4p + 4,$$

whose only positive solution is the order of convergence of method M_4K (Theorem 1), being $p = 2 + 2\sqrt{2} \approx 4.82843$. \square

The previous two methods with memory have been obtained using the variable $x^{(k-1)}$. Now, we are going to see what happens when the approximation $y^{(k-1)}$ is used instead, that is, we choose

$$\gamma_k = -[x^{(k)}, y^{(k-1)}; F]^{-1},$$

and M_4D_y is defined by replacing this into the parametric family and we also choose

$$\gamma_k = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F]^{-1},$$

and M_4K_y is defined by replacing this into the parametric family.

Now, we establish the order of these two methods with memory, M_4D_y and M_4K_y , whose demonstration is similar to the previous one.

Theorem 4. *Let us consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined in an neighborhood D of the root α of F . We assume that $F'(\alpha)$ is nonsingular. Let $H(\mu)$ be a real matrix function satisfying $H_0 = 1, H_1 = 1$ and $H_2 = 2$, where I is the $n \times n$ identity matrix. Then, taking an estimate $x^{(0)}$ close enough to α , sequence $\{x^{(k)}\}$ generated by method M_4D_y converges to α with order $p = 5$ and the sequence generated by M_4K_y converges to the root α with order $p = 6$.*

2.3 | Convergence analysis of $M_{7,\gamma}$

In the next result, we establish the convergence of parametric class $M_{7,\gamma}$, which is independent of the value of parameter γ .

Theorem 5. *Let us consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined in a neighborhood D of the roof α of F . We assume that $F'(\alpha)$ is nonsingular. Let $H(\mu)$ be a real matrix function that satisfies $H_0 = 1, H_1 = 1$, and $|H_2| < \infty$, where I is the $n \times n$ identity matrix. Let us also consider a multivariate matrix function $G(\mu, \nu)$ such that $G_0 = 1, G_{11} = G_{12} = 0, G_{21} = 0, G_{22} = 2, G_{23} = 0$, and $|G_{3i}| < \infty$ for $i = 1, \dots, 4$. Then, taking an initial guess $x^{(0)}$ sufficiently close to the root α , sequence $\{x^{(k)}\}$ obtained by $M_{7,\gamma}$ converges to α with order 7.*

Proof. We have already proven that

$$y^{(k)} - \alpha = C_2(I + \gamma F'(\alpha))e_k^2 - (2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2(C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2)) e_k^3 + O(e_k^4) \tag{20}$$

and

$$z^{(k)} - \alpha = \left(-C_3(I + \gamma F'(\alpha)) + C_2 \left(\left(3I - \frac{H_2}{2}I + \gamma F'(\alpha) \right) C_2 + \gamma C_2 F'(\alpha) \right) \right) C_2 (I + \gamma F'(\alpha)) e_k^4 + O(e_k^5). \tag{21}$$

We denote by Z_1 the coefficient of e_k^4 in the error equation.

Applying (6), we obtain

$$[z^{(k)}, y^{(k)}; F] = F'(\alpha) (I + C_2^2(I + \gamma F'(\alpha))e_k^2 + D_3 e_k^3) + O(e_k^4),$$

being $D_3 = -(2C_2^2 + 2\gamma C_2^2 F'(\alpha) + \gamma^2((C_2 F'(\alpha))^2 - C_3(2I + 3\gamma F'(\alpha) + \gamma^2 F'(\alpha)^2) + C_2^4(I + \gamma F'(\alpha)))$.

Calculating the inverse of this divided difference operator as we have done above, we obtain

$$[z^{(k)}, y^{(k)}; F]^{-1} = (I - C_2^2(I + \gamma F'(\alpha))e_k^2 + J_3 e_k^3 + O(e_k^4)) F'(\alpha)^{-1}, \tag{22}$$

being $J_3 = C_2^2(I + \gamma F'(\alpha))^2 - D_3$.

Now, we calculate $v^{(k)}$ and obtain that

$$\begin{aligned} v^{(k)} &= I - [w^{(k)}, x^{(k)}; F]^{-1} [z^{(k)}, y^{(k)}; F] H(\mu^{(k)}) \\ &= I - (I + X_2 e_k + X_3 e_k^2)(I + C_2^2(I + \gamma F'(\alpha)) e_k^2) \left(I + C_2 e_k + \left(M_3 + \frac{H_2}{2} \right) e_k^2 \right) + O(e_k^3) \\ &= -(X_2 + C_2) e_k - \left(X_3 - C_2(I + \gamma F'(\alpha)) + X_2 C_2 + \left(M_3 + \frac{H_2}{2} \right) \right) e_k^2 + O(e_k^3) \\ &= I - (X_2 + C_2) e_k - V_2 e_k^2 + O(e_k^3), \end{aligned}$$

being $V_2 = X_3 - C_2(I + \gamma F'(\alpha)) + X_2 C_2 + \left(M_3 + \frac{H_2}{2} \right)$.

Then,

$$\begin{aligned} G(\mu^{(k)}, v^{(k)}) &= I + \mu^{(k)} v^{(k)} + \frac{1}{6} (G_{31}(\mu^{(k)})^3 + G_{32}(\mu^{(k)})^2 v^{(k)} + G_{33} \mu^{(k)} (v^{(k)})^2 \\ &\quad + G_{34} (v^{(k)})^3) + O_4(\mu^{(k)}, v^{(k)}) \\ &= I + (C_2 e_k + M_3 e_k^2) (-X_2 + C_2) e_k - V_2 e_k^2 \\ &\quad + \frac{1}{6} (G_{31} C_2^3 - G_{32} C_2^2 (X_2 + C_2) + G_{33} C_2 (X_2 + C_2)^2 - G_{34} (X_2 + C_2)^3) e_k^3 \\ &= I - C_2 (X_2 + C_2) e_k^2 + (-M_3 (X_2 + C_2) - C_2 V_2 \\ &\quad + \frac{1}{6} (G_{31} C_2^3 - G_{32} C_2^2 (X_2 + C_2) + G_{33} C_2 (X_2 + C_2)^2 - G_{34} (X_2 + C_2)^3)) e_k^3. \end{aligned}$$

By denoting $R = -M_3(X_2 + C_2) - C_2 V_2 + \frac{1}{6} (G_{31} C_2^3 - G_{32} C_2^2 (X_2 + C_2) + G_{33} C_2 (X_2 + C_2)^2 - G_{34} (X_2 + C_2)^3)$, we have

$$G(\mu^{(k)}, v^{(k)}) = I - C_2 (X_2 + C_2) e_k^2 + R e_k^3 + O(e_k^4).$$

From that

$$\begin{aligned} x^{(k+1)} - \alpha &= e_z - G(\mu^{(k)}, v^{(k)}) [z^{(k)}, y^{(k)}; F]^{-1} F(z^{(k)}) \\ &= e_z - (I - C_2 (X_2 + C_2) e_k^2 + R e_k^3) (I - C_2^2 (I + \gamma F'(\alpha)) e_k^2 + J_3 e_k^3) (e_z + C_2 e_z^2) + O(e_k^8) \\ &= e_z - (I + (-C_2 (X_2 + C_2) - C_2^2 (I + \gamma F'(\alpha))) e_k^2 + (R + J_3) e_k^3) (e_z + C_2 e_z^2) + O(e_k^8). \end{aligned} \quad (23)$$

As $X_2 = -C_2(2I + \gamma F'(\alpha))$, then $X_2 + C_2 = -C_2(I + \gamma F'(\alpha))$, so $-C_2(X_2 + C_2) - C_2^2(I + \gamma F'(\alpha)) = 0$. From this,

$$\begin{aligned} x^{(k+1)} - \alpha &= e_z - G(\mu^{(k)}, v^{(k)}) [z^{(k)}, y^{(k)}; F]^{-1} F(z^{(k)}) \\ &= e_z - (I + (R + J_3) e_k^3) (e_z + C_2 e_z^2) + O(e_k^8) \\ &= -(R + J_3) e_k^3 e_z + O(e_k^8) \\ &= -(R + J_3) Z_1 e_k^7 + O(e_k^8). \end{aligned} \quad (24)$$

Thus, it is proven that parametric family $M_{7,\gamma}$ has order of convergence 7.

In particular, if $G_{31} = G_{32} = G_{34} = 0$ and $G_{33} = 13$, then

$$e_{k+1} \sim (I + \gamma F'(\alpha))^4 e_k^7. \quad (25)$$

□

2.4 | Introducing memory to $M_{7,\gamma}$

As we did with class $M_{4,\gamma}$, in this section, we introduce memory, in different ways, to family $M_{7,\gamma}$.

- If we choose $\gamma_k = -[x^{(k)}, x^{(k-1)}; F]^{-1}$, then replacing the parameter of family $M_{7,\gamma}$ by this value, we obtain a method with memory, denoted by M_7D .

- Choosing $\gamma = -[2x^{(k)} - x^{(k-1)}, x^{(k-1)}; F]^{-1}$ and replacing it in family $M_{7,\gamma}$ yield a method with memory denoted by M_7K .
- If we choose $\gamma_k = -[x^{(k)}, y^{(k-1)}; F]^{-1}$ and replacing it in $M_{7,\gamma}$, a new scheme with memory M_7D_y is obtained.
- Finally, choosing $\gamma = -[2x^{(k)} - y^{(k-1)}, y^{(k-1)}; F]^{-1}$ and replacing it in $M_{7,\gamma}$, a new scheme with memory, M_7K_y , is obtained.

The order of convergence of all these methods with memory is established in the next result, whose proof is similar to that of the previous results.

Theorem 6. *Let us consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined in a neighborhood D of the root α of F . We assume that $F'(\alpha)$ is nonsingular. Let H and G be real matrix functions that satisfy $H_0 = 1, H_1 = 1$, and $H_2 = 2$ and $G_0 = 1, G_{11} = G_{12} = 0, G_{21} = 0, G_{23} = 0, G_{22} = 2, G_{33} = 13$, and $G_{3i} = 0$ for $i = 1, 2, 4$. Therefore, considering an initial guess $x^{(0)}$ sufficiently close to α , we have the following:*

- sequence $\{x^{(k)}\}$ generated by M_7D converges to α with order $p = \frac{7+\sqrt{65}}{2} \approx 7.53113$.
- sequence $\{x^{(k)}\}$ defined by M_7K converges to α with order $p = 8$.
- sequence $\{x^{(k)}\}$ got by M_7D_y converges to α with order $p = 4 + \sqrt{17} \approx 8.12310$.
- sequence $\{x^{(k)}\}$ generated by M_7K_y converges to α with order $p = \frac{9+\sqrt{89}}{2} \approx 9.21699$.

In these methods with memory, we could also use variable $z^{(k-1)}$ in order to obtain a better approximation of the parameter. Thus, if we choose

$$\gamma_k = -[x^{(k)}, z^{(k-1)}; F]^{-1},$$

and replace the parameter of family $M_{7,\gamma}$ by this expression, we obtain a new method with memory denoted by M_7D_z .

In the same way, the approximation by the Kurchatov divided difference

$$\gamma = -[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]^{-1}$$

gives us a scheme with memory, M_7K_z , whose convergence we are going to establish.

Theorem 7. *Let us consider a sufficiently differentiable function $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined in a neighborhood D of the root α of F . We assume that $F'(\alpha)$ is nonsingular. Let H and G be real matrix functions that satisfy $H_0 = 1, H_1 = 1$, and $H_2 = 2$ and that $G_0 = 1, G_{11} = G_{12} = 0, G_{21} = 0, G_{23} = 0, G_{22} = 2, G_{33} = 13$ and $G_{3i} = 0$ for $i = 1, 2, 4$. Therefore, choosing an initial guess $x^{(0)}$ sufficiently close to the root α , sequence $\{x^{(k)}\}$ defined by M_7D_z converges to α with order $\frac{9+\sqrt{89}}{2} \approx 9.21699$, and sequence $\{x^{(k)}\}$ defined by M_7K_z converges to α with order 11.*

Proof. Let us consider the Taylor development of $F(z^{(k-1)})$, $F'(z^{(k-1)})$ and $F''(z^{(k-1)})$ around α as was done in Theorem 3. Applying (6), we obtain

$$[x^{(k)}, z^{(k-1)}; F] = F'(\alpha) (I + C_2(e_k + e_{z,k-1})) + O_2(e_k, e_{z,k-1}).$$

Then, we get

$$[x^{(k)}, z^{(k-1)}; F]^{-1} = (I - C_2(e_k + e_{z,k-1}))F'(\alpha)^{-1} + O_2(e_k, e_{z,k-1}).$$

Therefore,

$$I + \gamma_k F'(\alpha) = C_2(e_k + e_{z,k-1}) + O_2(e_{z,k-1}, e_k). \tag{26}$$

Let us assume that sequence $\{z^{(k)}\}$ has R-order p_1 and the R-order of the scheme is p .

Then, it follows that

$$\frac{e_k}{e_{z,k-1}} = \frac{e_{k-1}^p}{e_{k-1}^{p_1}} = e_{k-1}^{p-p_1}.$$

Thus, $I + \gamma_k F'(\alpha) \sim e_{z,k-1}$ if it is verified that $p > p_1$. By the error Equation (25) and the above relation, we have

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7. \tag{27}$$

Assuming that the R -order of the scheme is p , we have (16). In the same way that relation (27) is obtained and supposing that the sequence $\{z^{(k)}\}$ has R -order at least p_1 , we obtain

$$e_{k+1} \sim e_{z,k-1}^4 e_k^7 \sim (e_{k-1}^{p_1})^4 (e_{k-1}^p)^7 \sim e_{k-1}^{7p+4p_1}. \quad (28)$$

By other way, from the error equation of $e_{z,k}$, we have

$$e_{z,k} \sim (-C_3(I + \gamma F'(\alpha)) + C_2((I + \gamma F'(\alpha)) C_2 + C_2(I + \gamma F'(\alpha)))) C_2(I + \gamma F'(\alpha)) e_k^4 \sim e_{z,k-1}^2 e_k^4 \sim e_{k-1}^{4p+2p_1}. \quad (29)$$

Assuming that $\{z^{(k)}\}$ has R -order p_1 , we assure that

$$e_{z,k} \sim e_k^{p_1} \sim e_{k-1}^{pp_1}. \quad (30)$$

Then, by equaling the exponents of e_{k-1} of (16) and (28) and by equaling the exponents of e_{k-1} of (29) and (30), it follows that

$$\begin{aligned} p^2 &= 7p + 4p_1, \\ pp_1 &= 4p + 2p_1, \end{aligned}$$

whose only positive solution is the order of M_7D_z scheme, where $p = \frac{9+\sqrt{89}}{2} \approx 9.21699$.

Now, we calculate $[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]$ by using (6)

$$[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F] = F'(\alpha) \left(I + 2C_2e_k - 2C_3e_{z,k-1}e_k + C_3e_{z,k-1}^2 + 4C_3e_k^2 \right) + O_3(e_k, e_{z,k-1}).$$

Then, the inverse of this divided difference operator is as follows:

$$[2x^{(k)} - z^{(k-1)}, z^{(k-1)}; F]^{-1} = (I - 2C_2e_k - C_3e_{z,k-1}^2 + 2C_3e_{z,k-1}e_k + 4(C_2^2 - C_3)e_k^2)F'(\alpha)^{-1} + O_3(e_k, e_{z,k-1}).$$

Therefore,

$$I + \gamma_k F'(\alpha) = 2C_2e_k + C_3e_{z,k-1}^2 - 2C_3e_{z,k-1}e_k - 4(C_2^2 - C_3)e_k^2 F'(\alpha)^{-1} + O_3(e_k, e_{z,k-1}).$$

Thus, $I + \gamma_k F'(\alpha)$ can have the behavior of e_k or $e_{z,k-1}^2$, since the factors $e_k e_{z,k-1}$ and e_k^2 tend to have higher speed at 0 than e_k , so we have to see whether e_k or $e_{z,k-1}^2$ converges faster. Assume that the R -order of the scheme is p . On the other hand, as sequence $z^{(k)}$ has R -order p_1 , we have

$$\frac{e_k}{e_{z,k-1}^2} \sim \frac{D_{k-1,p} e_{k-1}^p}{e_{k-1}^{2p_1}}.$$

Then, if we assume that $p > 2p_1$, we have that the behavior will be like that of $e_{z,k-1}^2$, that is, $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$.

From (25) and the above relationship, the following relation is obtained

$$e_{k+1} \sim e_{z,k-1}^8 e_k^7. \quad (31)$$

In addition, relation (16) holds since the R -order of the procedure is p .

In the same way that relation (31) is obtained and taking into account that sequence $z^{(k)}$ has R -order p_1 , we obtain

$$e_{k+1} \sim e_{z,k-1}^8 (e_{k-1}^p)^7 \sim e_{k-1}^{8p_1} e_{k-1}^{7p} \sim e_{k-1}^{7p+8p_1}. \quad (32)$$

On the other hand, by the error equation of $e_{z,k}$, it is obtained that

$$e_{z,k} \sim e_{z,k-1}^4 e_k^4. \tag{33}$$

Then by equaling the exponents of e_{k-1} of (16) and (32) and by equaling the exponents of e_{k-1} of (33) and (30), it is obtained that

$$\begin{aligned} p^2 &= 7p + 8p_1, \\ pp_1 &= 4p + 4p_1, \end{aligned}$$

whose only positive solution is $p \approx 11.3523$ and $p_1 \approx 6.17$; therefore, it does not satisfy the property for which $I + \gamma_k F'(\alpha) \sim e_{z,k-1}^2$; thus, $I + \gamma_k F'(\alpha) \sim e_k$, and therefore,

$$e_{k+1} \sim e_k^4 e_k^7 \sim e_k^{11}. \tag{34}$$

Thus, we conclude that the order of method M_7K_z is $p = 11$. □

As we can see, by introducing memory to families $M_{4,\gamma}$ and $M_{7,\gamma}$, we have managed to increase the order up to 2 and 4 units, thus obtaining methods with memory up to order 6 and 11, respectively. Let us note that $p = 11$ is the maximum order that can be reached by introducing memory in family $M_{7,\gamma}$.

3 | NUMERICAL EXPERIMENTS

Now, we perform several numerical experiments in order to see the behavior of our $M_{4,\gamma}$, $M_{7,\gamma}$ families and the methods derived from them when introducing memory. We present two numerical experiments, one of them applied to the Hammerstein equation and other applied to academical nonlinear system, in which we also make a comparison with two known methods of order 8.

The first thing we would like to point out is that in this case, Matlab 2020b has been used to carry out the numerical experiments, with an arithmetical precision variable of 1000 digits. As stopping criterion, we choose that $\|x^{(k+1)} - x^{(k)}\|_2 + \|F(x^{(k+1)})\|_2$ is less than a chosen tolerance. We use also a maximum of 100 iterations.

For all methods and all numerical experiments, the following matrix functions have been selected as weight functions:

- $H(\mu) = \mu^2 + \mu + I$,
- $G(\mu, \nu) = I + \mu\nu + \frac{13}{6}\mu\nu^2$,

being I the identity matrix.

In the different tables we show,

- the norm $\|F(x^{(k+1)})\|_2$,
- the norm $\|x^{(k+1)} - x^{(k)}\|_2$,
- the number of iterations necessary to satisfy the required tolerance,
- and the ACOC, approximated computational order of convergence, defined by Cordero and Torregrosa [17], which has the following expression:

$$p \approx ACOC = \frac{\ln (\|x^{(k+1)} - x^{(k)}\|_2 / \|x^{(k)} - x^{(k-1)}\|_2)}{\ln (\|x^{(k)} - x^{(k-1)}\|_2 / \|x^{(k-1)} - x^{(k-2)}\|_2)}.$$

3.1 | Hammerstein equation

In this test, we consider the Hammerstein integral equation, appearing, for example, in Ortega and Rheinboldt [15],

$$x(s) = 1 + \frac{1}{5} \int_0^1 F(s, t)x(t)^3 dt, \tag{35}$$

being

$$F(s, t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t \end{cases}$$

and $s, t \in [0, 1]$, $x \in C[0, 1]$.

By means of Gauss–Legendre quadrature, Hammerstein equation is transformed in

$$\int_0^1 f(t)dt \approx \sum_{i=1}^7 \omega_i f(t_i),$$

appearing the abscissas t_i and the weights ω_i for $n = 7$ in the following table.

i	Weight ω_i	Nodes t_i
1	0.0647424831	0.0254460438
2	0.1398526957	0.1292344072
3	0.1909150252	0.2970774243
4	0.2089799185	0.5
5	0.1909150252	0.7029225757
6	0.1398526955	0.8707655928
7	0.0647424831	0.9745539561

Let us also denote by x_i , $i = 1, \dots, 7$ the approximations of $x(t_i)$. Then,

$$5x_i - 5 - \sum_{j=1}^7 a_{ij}x_j^3 = 0,$$

where $i = 1, \dots, 7$ and

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i) & j \leq i, \\ w_j t_i (1 - t_j) & i < j. \end{cases}$$

We initialize the process with $x^{(0)} = (0.5, \dots, 0.5)^T$, we choose as initial approximations for $x^{(-1)}$, $y^{(-1)}$, and $z^{(-1)}$ vector $(0.4, \dots, 0.4)^T$, and we use a tolerance of 10^{-50} . In Table 1, we can see the results obtained by each method for Hammerstein's equation.

We can see that in all cases, the ACOC is close to the theoretical convergence order demonstrated in Section 2 and that the number of iterations required is similar for the methods of the same family, being one unit higher in the case of the methods without memory.

We notice that in all cases, the required tolerance is well below the results obtained. It can be seen that the best results for these numerical experiments are given by the memory methods that use the Kurchatov divided difference operator to

TABLE 1 Numerical results of Hammerstein's equation.

Method	$\ x^{(k+1)} - x^{(k)}\ _2$	$\ F(x^{(k+1)})\ _2$	Iteration	ACOC
$M_{4,-1}$	1.01573e-166	2.41351e-666	5	3.99986
M_4D	2.05575e-154	4.58506e-688	4	4.4952
M_4K	7.31331e-177	2.37182e-853	4	4.9600
M_4D_y	7.12038e-204	1.62192e-1021	4	4.9971
M_4K_y	1.31159e-295	3.12493e-1776	4	5.9975
$M_{7,-1}$	4.53896e-171	9.75609e-1027	4	6.9996
M_7D	2.42252e-79	3.32054e-516	3	7.5291
M_7K	2.45812e-82	3.43156e-571	3	7.8613
M_7D_y	2.35271e-88	4.7518e-622	3	8.1898
M_7K_y	3.16092e-101	7.79194e-813	3	9.1692
M_7D_z	3.17456e-99	2.21039e-796	3	9.2162
M_7K_z	2.85847e-114	1.65181e-1032	3	10.9981

Abbreviation: ACOC, approximated computational order of convergence.

Method	$\ x^{(k+1)} - x^{(k)}\ _2$	$\ F(x^{(k+1)})\ _2$	Iteration	ACOC
$M_{4,-1}$	2.61975e-71	2.275e-102	5	3.92262
M_4D	1.84221e-51	6.63527e-83	4	4.36222
M_4K	5.58564e-57	5.32079e-89	4	4.71234
M_4D_y	3.72943e-54	1.45955e-85	4	5.1739
M_4K_y	1.51937e-63	1.46077e-95	4	5.9701
$M_{7,-1}$	4.32096e-77	1.40986e-123	4	6.93731
M_7D	1.10549e-89	2.70395e-137	4	7.53147
M_7K	8.76112e-51	6.9915e-401	3	7.85679
M_7D_y	1.52107e-94	1.2392e-139	4	8.19609
M_7K_y	8.76112e-51	6.9915e-401	3	9.18679
M_7D_z	5.21322e-97	8.1106e-144	4	9.22566
M_7K_z	8.76112e-51	6.9915e-401	3	10.9754
CCGT1	2.64372e-64	1.59087e-516	4	8.09479
NM8	2.81063e-292	3.0869e-2337	4	8.0

TABLE 2 Numerical results for system F.

Abbreviation: ACOC, approximated computational order of convergence.

approximate the family parameter. These methods give the closest approximations to the solution and the biggest ACOC.

3.2 | Academic problems

We also approximate the solution of the following academic system of nonlinear equations. In this case, we compare the results obtained with the different methods proposed with those provided by two known schemes, both of order 8. These schemes are the CCGT1 method which can be found in Cordero et al. [18] and the NM8 method which can be found in Wang [19].

The system that we use in our experiment, denoted by System F, is as follows:

$$F_i(x) = x_i^2 x_{i+1} - 1,$$

$$F_{200}(x) = x_{200}^2 x_1 - 1,$$

a system with 200 unknown and 200 equations.

For this example, we use a tolerance of 10^{-50} , an initial estimation $x^{(0)} = (0.9, \dots, 0.9)^T$, and as initial approximations for $x^{(-1)}$, $y^{(-1)}$, and $z^{(-1)}$ vector $(0.7, \dots, 0.7)^T$.

The results obtained for System F and for each method are shown in Table 2. We can see from the table that the number of iterations change for the family $M_{7,\gamma}$ and their memory methods. In this case, the iterations range between 3 and 4, making the methods that perform four iterations have the ACOC closer to the theoretical convergence order and those that perform three iterations not so close since the tolerance is small.

It can be seen that the best results for these numerical experiments are given by the memory methods that use Kurchatov's divided difference operator to approximate the family parameter, although these are also the ones that perform the fewest iterations, which means that they are still closer to the solution than the rest, although their ACOC is not adequate for three iterations.

As we can see in the tables, our methods M_7K , M_7K_y , and M_7K_z are quite similar to the results obtained by the CCGT1 method and that the NM8 method performs one more iteration than them to verify the tolerance, so it would be more advisable to use the methods derived from the parametric family.

3.3 | An example of real dynamics

In the previous section, we have introduced memory to two parametric families and studied the order of convergence of the resulting methods. These are important concepts of iterative methods but not the only ones. Another important concept is the behavior of the method according to the initial estimation chosen, since we would like to know a priori if the method converge to any of the solutions according to the estimate taken. This analysis is the study of the stability of the method, also called dynamical study.

In this section, we will only show the dynamical planes associated with each of the methods when they are applied on a simple system of quadratic polynomials. In this case, the mentioned polynomial system is

$$\begin{aligned} x_1^2 - 1 &= 0, \\ x_2^2 - 1 &= 0, \end{aligned}$$

where $(x_1, x_2)^T \in \mathbb{R}^2$.

The appearance of an iterative scheme with memory using only two previous iterations is as follows:

$$x^{(k+1)} = \phi(x^{(k-1)}, x^{(k)}), \quad k \geq 1,$$

being the initial estimations $x^{(0)}$ and $x^{(1)}$. It is clear that no fixed point can be defined in a function from $\mathbb{R}^n \times \mathbb{R}^n$ to \mathbb{R}^n . Therefore, we define an auxiliary function O defined in vectorial form as follows:

$$O(x^{(k-1)}, x^{(k)}) = (x^{(k)}, \phi(x^{(k-1)}, x^{(k)})), \quad k = 1, 2, \dots$$

If $(x^{(k-1)}, x^{(k)})$ is a fixed point of O , then $O(x^{(k-1)}, x^{(k)}) = (x^{(k-1)}, x^{(k)})$, and from the definition of O , we have that $(x^{(k-1)}, x^{(k)}) = (x^{(k)}, x^{(k+1)})$. Thus, the discrete dynamical system $O : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$O(z, x) = (x, \phi(z, x)),$$

being ϕ is the operator of the iterative scheme with memory. Then, a point (z, x) is a fixed pint of O if $z = x$ and $x = \phi(z, x)$. If a fixed point (z, x) of the operator O does not verify that $F(x) = 0$, it is called strange fixed point.

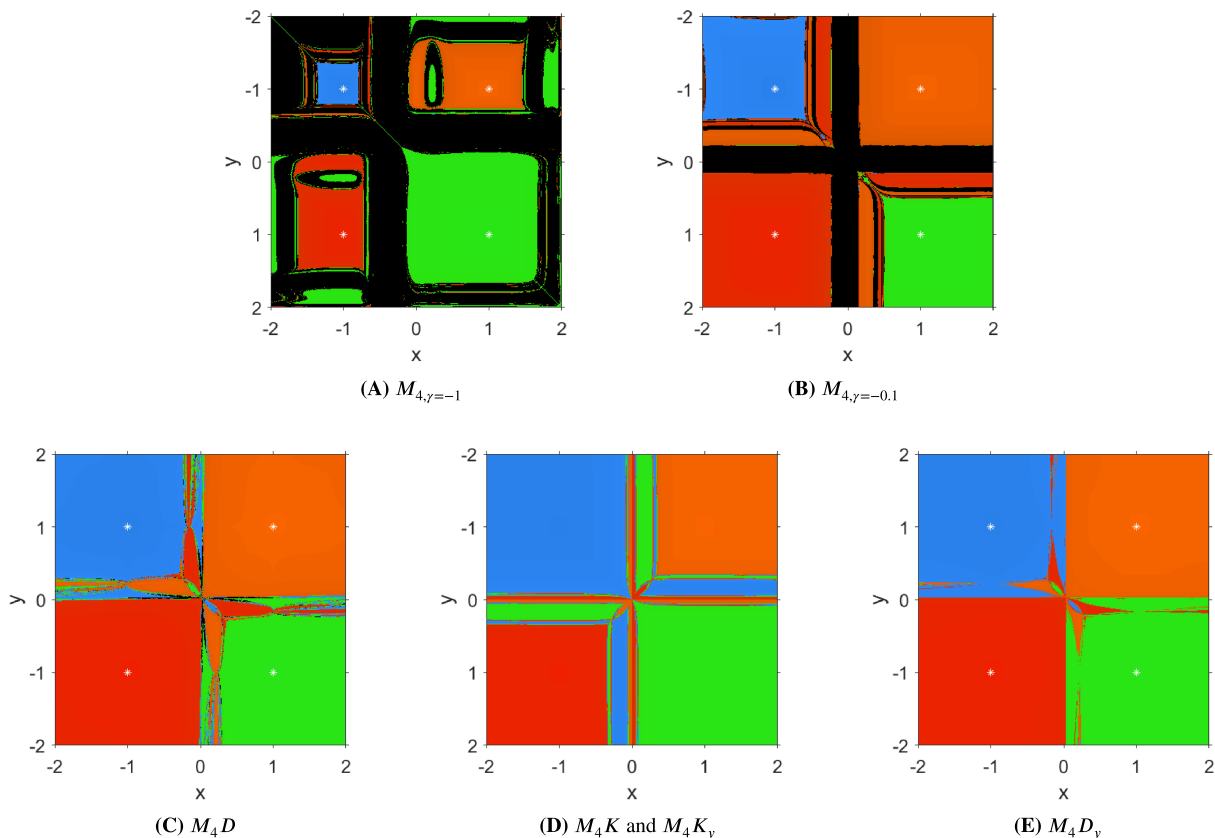


FIGURE 1 Dynamical planes of $M_{4,\gamma}$ and their methods with memory. [Colour figure can be viewed at wileyonlinelibrary.com]

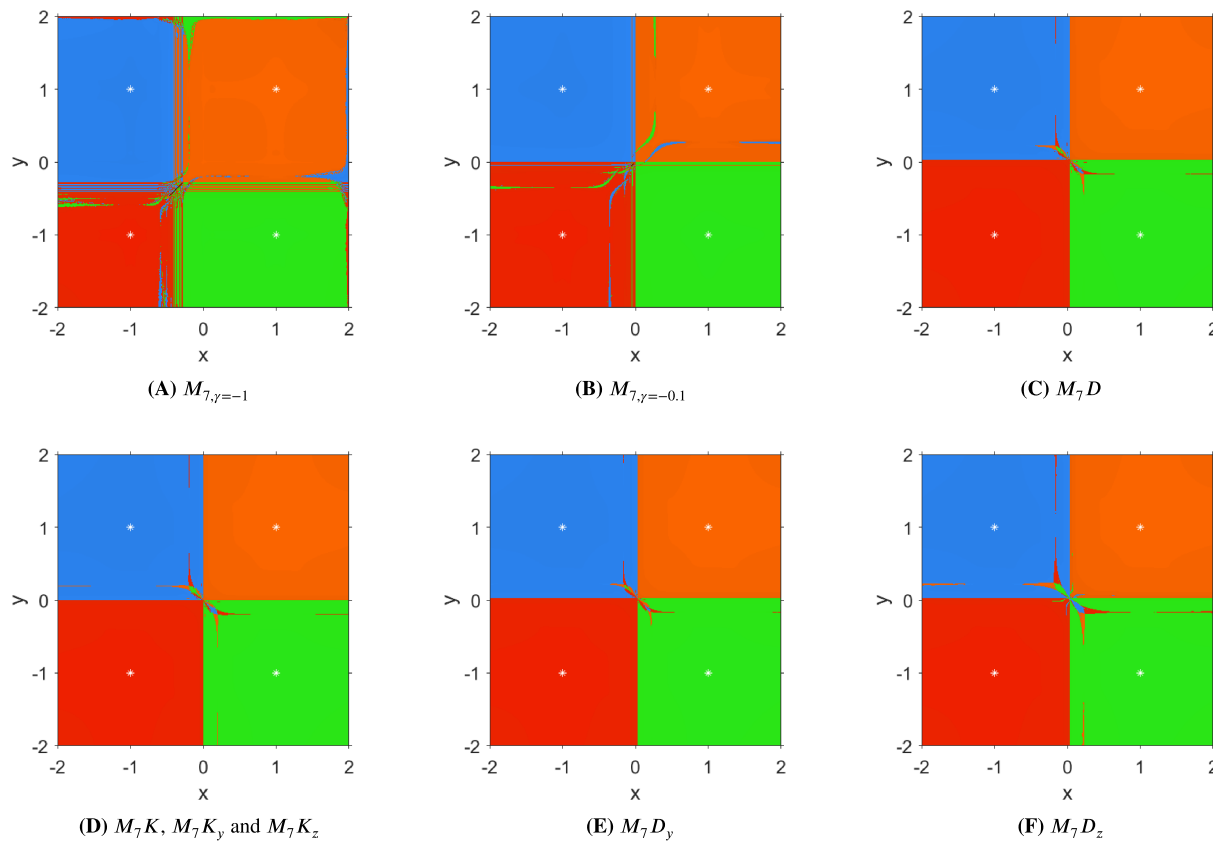


FIGURE 2 Dynamical planes of $M_{7,\gamma}$ and their methods with memory. [Colour figure can be viewed at wileyonlinelibrary.com]

The basin of attraction of a fixed point x^* is defined as follows:

$$\mathcal{A}(x^*) = \{y \in \mathbb{R}^n : O^m(y) \rightarrow x^*, m \rightarrow \infty\}.$$

We know that the roots of the polynomial system are as follows:

- $(1, -1)^T$,
- $(-1, 1)^T$,
- $(-1, -1)^T$,
- $(1, 1)^T$.

For all methods, the following matrix functions have been selected as weight functions:

- $H(\mu) = \mu^2 + \mu + I$,
- $G(\mu, \nu) = I + \mu\nu + \frac{13}{6}\mu\nu^2$,

being I is the 2×2 identity matrix.

To generate the dynamical planes, we have chosen a mesh of 400×400 points, and what we will do is apply our methods to each of these points, taking the point as the initial estimate. We have also defined that the maximum number of iterations that each initial estimate must do is 80 and that we will determine that the initial point converges to one of the solutions if the distance to that solution is less than 10^{-3} . We paint in orange the initial points converging to $(1, 1)^T$, in green the initial points converging to $(1, -1)^T$, in blue the initial points converging to $(-1, 1)^T$, in red the initial points converging to $(-1, -1)^T$, and in black the initial guesses not converging to any of the roots.

In Figure 1, we see the dynamical planes of $M_{4,\gamma}$, for $\gamma = -1$ and $\gamma = -0.1$, and their corresponding methods with memory. Analogously, in Figure 2, we show the same dynamical planes for $M_{7,\gamma}$.

In addition, to having increased the convergence order by introducing memory, it has been seen in the above dynamical planes that for the selected system, the introduction of memory has also helped to obtain a more stable behavior and easier to predict behavior as the convergence zones of the roots are simpler.

4 | CONCLUSIONS

In this work, two parametric classes of iterative methods with orders of convergence 4 and 7, respectively, for solving nonlinear systems, have been designed.

Memory has been introduced, in different ways, to these two families in order to obtain iterative methods with higher convergence order without the need to increase the number of functional evaluations per iteration. These methods with memory have managed to increase the order by up to 2 units for the family of order 4 and up to 4 units for the family of order 7.

But not only does the introduction of memory improve the order of convergence, but as we have seen in the dynamical planes that have been carried out, it has also improved the behavior of the method, since we obtain that more points converge when it comes to the methods with memory or else the attraction zones of the roots are simpler.

In the numerical experiments, the theoretical results are confirmed, and when comparing our methods with other known ones of high order (order of convergence 8), it can be seen that most of the proposed methods obtain a closer approximation to the solution than the known methods.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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