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Three-step iterative weight function scheme with memory for solving nonlinear problems

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In this manuscript, we present a parametric family of derivative-free three-step iterative methods with a weight function for solving nonlinear equations. We study various ways of introducing memory to this parametric family in order to increase the order of convergence without new functional evaluations. We also performed numerical experiments to compare the iterative methods from different points of view.

KEYWORDS

divided differences, iterative method, memory methods, nonlinear equation

MSC CLASSIFICATION

65H05

1 | INTRODUCTION

In most problems appearing in Applied Mathematics, nonlinear equations of the form $f(x) = 0$ must be solved, where $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$. Sometimes it is not possible or is very complicated to solve these equations exactly. This is the reason why methods that obtain an approximation to the zero α of function f have arisen. Iterative schemes aim us to obtain a sequence of iterations, which under certain conditions, converges to this solution [1].

These methods are categorized in different ways, depending on whether they use derivatives to obtain the approximations, if they are methods with memory [2], that is, whether they use only one previous iteration or if they use more than one previous iteration to obtain the next one and whether they have an auxiliary function or weight function in the iterative expression.

In this work, we present an extension of the derivative-free two-step parametric family with a weight function, obtained in Cordero et al. [3], which has the following expression:

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f[w_k, x_k]}, \quad \text{where} \quad w_k = x_k + \beta f(x_k) \ \forall \beta \in \mathbb{R} \backslash \{0\}, \\ x_{k+1} = y_k - H(\mu_k) \dfrac{f(y_k)}{f[y_k, x_k]}, \quad \text{where} \quad \mu_k = \dfrac{f(y_k)}{f(w_k)}, k = 0, 1, \ldots \end{cases} \tag{1}$$

By adding one more step with a similar feature, we define the following parametric family of iterative methods:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[w_k, x_k]}, \quad \text{where} \ w_k = x_k + \beta f(x_k), \\ z_k = y_k - H(\mu_k) \frac{f(y_k)}{f[y_k, x_k]}, \quad \text{where} \ \mu_k = \frac{f(y_k)}{f(w_k)}, \\ x_{k+1} = z_k - G(v_k) \frac{f(z_k)}{f[z_k, y_k]}, \quad \text{where} \ v_k = \frac{f(z_k)}{f(y_k)}, \end{cases} \tag{2}$$

being $f[a,b] = \frac{f(a)-f(b)}{a-b}$ with $a \neq b$, $a, b \in \mathbb{R}$, being $H(t)$ and $G(t)$ real functions. We denote this parametric family by $M_{6,\beta}$, as it is further proven that it has order of convergence 6 for any real value of $\beta \neq 0$.

Usually, methods with higher order of convergence tend to converge in fewer iterations. For this reason, we want to increase the order, but without adding functional evaluations, as that would increase the computational cost. For this reason, we introduce memory to class $M_{6,\beta}$ by replacing the parameter with an expression that depends on the previous iterations and their functional evaluations.

The development of iterative schemes with memory is quite recent, far from classical Secant method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_{k-1}, x_k]}, \quad \text{given} \quad x_0, x_1, k \geq 1.$$

whose order of convergence is $p = \frac{1+\sqrt{5}}{2} \approx 1.618$. In 1964, Traub [4] developed the scheme

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{\Gamma_k}, \\ \Gamma_k &= f[x_k + \gamma_k f(x_k), x_k], \\ \gamma_k &= -\frac{1}{\Gamma_{k-1}}, \quad \text{given } x_0, \gamma_0, \end{aligned}$$

that is a version of known Steffensen's method including memory, by using an accelerating parameter by first time. The order of convergence of this procedure was $p = 1 + \sqrt{2} \approx 2.414$. In 2011, Grau-Sánchez et al. [5] designed a generalized Secant method, with order of convergence $p = \frac{1+\sqrt{1+4n}}{2}$ consisting in n steps, defined by

$$x_k^{(n)} = x_k^{(n-1)} - [x_{k-1}, x_k; F]^{-1} F\left(x_k^{(n-1)}\right), \qquad k > 1,$$

where $x_k^{(0)} = x_k$ and, in the last step, the last computed term is $x_{k+1} = x_k^{(n)}$. Also, more general high-order methods have been generated in the last decades, as those proposed by Neta [6], by using inverse interpolation, with order $p \approx 10.815$ by means of several accelerating parameters. Also, by using Newton's interpolation approach in the replacement of the accelerating parameters, Petković et al. [7, 8] constructed, among others, a general scheme with one accelerating parameter, with high order of convergence. However, these high increasing of the order of convergence usually involves a greater computational cost. This is the reason why a balanced relation between order of convergence and computational efficiency is needed.

To prove the order of the introduced memory methods we use the Ortega–Rheinboldt theorem, which can be found in Ortega and Rheinboldt [9]:

Theorem 1. *Let ϕ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converge to α. If there exist a nonzero constant η and positive numbers t_i, $i = 0, \ldots, m$ such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i}$$

holds, then the R-order of convergence of the iterative method ϕ is at least p, where p is the unique positive root of the equation

$$p^{m+1} - \sum_{i=0}^{m} t_i p^{m-i} = 0.$$

Along this manuscript, the convergence analysis of class $M_{6,\beta}$ is made in Section 2; meanwhile, the memorizing process and the demonstration of its order of convergence are made in Section 3. Finally, in Section 4, we conclude with some numerical experiments where the solutions obtained by the different methods with and without memory are compared. Also, the approximations obtained and the number of iterations necessary to obtain them are analyzed for the sake of comparison, as well as the wideness of the set of converging initial guesses and the total efficiency index of the methods applied to specific problems.

2 | CONVERGENCE ANALYSIS

Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently differentiable function in an open set D that contains a root α of $f(x) = 0$. Let us consider the expression

$$f[x+h,x] = \int_0^1 f'(x+th)dt, \tag{3}$$

obtained by Genochi–Hermite in Ortega and Rheinboldt [9]. Using the Taylor's expansion $f'(x+th)$ around x and integrating, we obtain the following development:

$$f[x+h,x] = f'(x) + \frac{1}{2}f''(x)h + \frac{1}{6}f'''(x)h^2 + O(h^3), \tag{4}$$

which we use to prove that the order of convergence of methods $M_{6,\beta}$, defined in (2), is 6 for any $\beta \in \mathbb{R}\backslash\{0\}$.

Theorem 2. *Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently differentiable function in a neighborhood D of α, such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfy that $H(0) = 1$, $H'(0) = 1$ and $|H''(0)| < \infty$, and let $G(t)$ be a real function that satisfy that $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an initial approximation x_0 close enough to α, the sequence of iterated $\{x_k\}$ generated by the proposed method (2) converge to α with order 6, and its error equation is*

$$\begin{aligned} e_{k+1} = \frac{-C_2}{4}&\left(1+\beta f'(\alpha)\right)\left(-2C_3(1+\beta f'(\alpha)) + C_2^2(6+4\beta f'(\alpha) - H_2)\right) \\ &\cdot \left(-2C_3(1+\beta f'(\alpha))G_1 + C_2^2(-2+6G_1+2\beta f'(\alpha)(-1+2G_1) - G_1H_2)\right) e_k^6 + O\left(e_k^7\right), \end{aligned} \tag{5}$$

where $C_j = \frac{1}{j}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$ for $j = 2, 3, \ldots$, $e_k = x_k - \alpha$, $H_2 = H''(0)$, and $G_1 = G''(0)$.

Proof. As we proved for the parametric family of the two first steps of (1) (see Cordero et al. [3]), error equation e_z is

$$e_z = \frac{1}{2}C_2(1+\beta f'(\alpha))(-2C_3(1+\beta f'(\alpha)) + C_2^2(6+4\beta f'(\alpha) - H_2))e_k^4 + O\left(e_k^5\right), \tag{6}$$

being $e_z = z_k - \alpha$. Let us therefore calculate e_{k+1}.

Let us now consider the Taylor expansion of $f(y_k)$ around α:

$$f(y_k) = f'(\alpha)\left(e_y + C_2e_y^2 + C_3e_y^3 + C_4e_y^4 + C_5e_y^5 + O\left(e_y^6\right)\right), \tag{7}$$

being $e_y = y_k - \alpha$.

Let us consider the Taylor expansion of $f(z_k)$ around α, in the same way that is done for $f(y_k)$, being $e_z = z_k - \alpha$. Now, we calculate $f[z_k, y_k]$ and $v_k = \frac{f(z_k)}{f(y_k)}$ using the above equations:

$$\begin{aligned} f[z_k, y_k] = \frac{f(z_k) - f(y_k)}{z_k - y_k} &= f'(\alpha)(1 + C_2^2(1+\beta f'(\alpha))e_k^2 \\ &+ C_2f'(\alpha)\left((-C_2^2 + C_3)(2+2\beta f'(\alpha) + \beta^2 f'(\alpha)^2) + C_3\beta f'(\alpha)\right)e_k^3 + O\left(e_k^4\right). \end{aligned}$$

On the other hand,

$$\frac{f(z_k)}{f(y_k)} = -C_3(1+\beta f'(\alpha)) + C_2^2\left(3 + 2\beta f'(\alpha) - \frac{H_2}{2}\right)\right)e_k^2 + O\left(e_k^3\right).$$

It follows that

$$\begin{aligned} G(v_k) &= 1 + G_1 v_k + O(v_k^2) \\ &= 1 + G_1 \left(-C_3(1+\beta f'(\alpha)) + C_2^2\left(3 + 2\beta f'(\alpha) - \frac{H_2}{2}\right)\right) e_k^2 + O\left(e_k^3\right). \end{aligned}$$

Using the above expressions, we calculate $e_{k+1} = e_z - G(v_k)\frac{f(z_k)}{f[z_k,y_k]}$. It follows that

$$\begin{aligned} e_{k+1} = \frac{-C_2}{4}&\left(1+\beta f'(\alpha)\right)\left(-2C_3(1+\beta f'(\alpha)) + C_2^2(6+4\beta f'(\alpha) - H_2)\right) \\ &\cdot \left(-2C_3(1+\beta f'(\alpha))G_1 + C_2^2(-2+6G_1+2\beta f'(\alpha)(-1+2G_1) - G_1H_2)\right) e_k^6 + O\left(e_k^7\right). \end{aligned} \tag{8}$$

Thus, it is proved that method (2) has order 6 under these conditions.

In particular, if $H_2 = H''(0) = 2$, then

$$e_{k+1} = C_2(2C_2^2 - C_3)(C_2^2(1-2G_1) + C_3G_1)(1+\beta f'(\alpha))^3 e_k^6 + O\left(e_k^7\right). \tag{9}$$

□

3 | INTRODUCING MEMORY

From the error equation, we note that if $\beta = -\frac{1}{f'(\alpha)}$, then the order increases at least one unit. Since the value of α is unknown, we approximate the value of $f'(\alpha)$ in order to increase the order of the iterative scheme. In this way, we obtain a method with memory. We then approximate the parameter to obtain a new method with memory. If we take

$$\beta_k = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})},$$

and replace it in (2), we obtain a method with memory, denoted by M_6N_1.

Theorem 3. *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in a neighborhood D of α such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfy $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking a initial approximation x_0 close enough to α, the sequence of iterates $\{x_k\}$ generated by the method M_6N_1 converges to α with order $p = 3 + 2\sqrt{3} \approx 6.4641$.*

Proof. From the error equation (9),

$$e_{k+1} \sim (1+\beta_k f'(\alpha))^3 e_k^6.$$

Using Taylor's series developments of $f(x_k)$ and $f(x_{k-1})$ and by the Genochi–Hermite formula, it follows that

$$1 + \beta_k f'(\alpha) \sim C_2 e_{k-1}.$$

The error equation (9) and the above relations give

$$e_{k+1} \sim (C_2 e_{k-1})^3 e_k^6 \sim e_{k-1}^3 e_k^6. \tag{10}$$

Suppose that the R-order of the methods is at least p. Therefore,

$$e_{k+1} \sim D_{k,p} e_k^p,$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, when $k \to \infty$.

Assuming that the R-order of the method is at least p yields the following relation:

$$e_{k+1} \sim D_{k,p} e_k^p \sim D_{k,p} \left(D_{k-1,p} e_{k-1}^p \right)^p \sim e_{k-1}^{p^2}. \tag{11}$$

In the same way as the relation (10) is obtained, it follows that

$$e_{k+1} \sim e_{k-1}^3 \left(D_{k-1,p} e_{k-1}^p \right)^6 = D_{k-1,p}^6 e_{k-1}^{6p+3}. \tag{12}$$

Then, by equating the exponents of e_{k-1} of (11) and (12), it follows that

$$p^2 = 6p + 3,$$

whose only positive solution is the order of convergence of the M_6N_1 method (see Theorem 1), being $p \approx 6.4641$. □

Instead of using the divided differences above, we can use either ones $f[x_k, y_{k-1}]$ or $f[x_k, z_{k-1}]$.

By replacing the parameter for the approximations in (2), we obtain two methods with memory, which we denote by M_6N_{1Y} and M_6N_{1Z}, respectively.

In the same way that it has been proved that M_6N_1 has order $3 + 2\sqrt{3}$, we can prove that, under the same conditions as the previous result, M_6N_{1Y} has order 7 and that M_6N_{1Z} has order 8, that is, we can prove the following Theorem 4 in a similar way as Theorem 3.

Theorem 4. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of α, such that $f(\alpha) = 0$. Suppose that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an initial point x_0, close enough to α, the sequence of iterates generated by M_6N_{1Y} converges to α with order 7, and the sequence of iterates generated by M_6N_{1Z} converges to α with order 8.*

Another way to approximate the derivative of a function is to use the Kurchatov's divided difference instead of the usual divided difference. For this reason, we approximate the parameter using a Kurchatov's divided difference. We replace the parameter in (2) by

$$\beta_k = -\frac{1}{f[2x_k - x_{k-1}, x_{k-1}]},$$

and we obtain a memory method, denoted by M_6K_1.

Theorem 5. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of α, such that $f(\alpha) = 0$. Suppose that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an initial point x_0, close enough to α, sequence of iterates, $\{x_k\}$, generated by M_6K_1 converges to α with order $p = 3 + \sqrt{15} \approx 6.873$.*

Proof. From the error equation (9),

$$e_{k+1} \sim (1 + \beta_k f'(\alpha))^3 e_k^6.$$

Using Taylor's expansion of $f(x_k)$ and $f(x_{k-1})$ and by the Genochi–Hermite formula, we obtain

$$1 + \beta_k f'(\alpha) \sim e_{k-1}^2.$$

From the error equation (9) and the last relation, it follows that

$$e_{k+1} \sim \left(e_{k-1}^2 \right)^3 e_k^6 \sim e_{k-1}^6 e_k^6. \tag{13}$$

By assuming that the R-order of the method is at least p, we obtain relation (11). In the same way that the relationship (13) is obtained, we obtain that

$$e_{k+1} \sim \left(e_{k-1}^2 \right)^3 \left(D_{k-1,p} e_{k-1}^p \right)^6 = D_{k-1,p}^6 e_{k-1}^{6p+6}. \tag{14}$$

Then, by equating exponents of e_{k-1} in (11) and in (14), it follows that

$$p^2 = 6p + 6,$$

whose only positive solution is the order of convergence of the M_6K_1 method, where $p = 3 + \sqrt{15} \approx 6.873$. □

We can also apply Kurchatov's divided differences using y_{k-1} and z_{k-1} instead of x_{k-1}. By replacing the parameter by these approximations in (2), we obtain two methods with memory, which we denote by M_6K_Y and by M_6K_Z, respectively.

In the same way that it has been proved that M_6K has order $3 + \sqrt{15}$, we can prove that, under the same conditions as the previous theorem, M_6K_Y has order 8 and M_6K_Z has order 9, that is, we can prove Theorem 6 in a similar way to that of the Theorem 5.

Theorem 6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of α, denoted by $D \subset \mathbb{R}$, such that $f(\alpha) = 0$. Suppose that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfies $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function that satisfies $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking an initial point x_0, close enough to α, the sequence of iterates by M_6K_Y converges to α with order 8, and the sequence of iterates by M_6K_Z converges to α with order 9.*

In the methods with memory discussed above, we have used divided difference operators to approximate the value of $f'(\alpha)$, but there are more ways to approximate this value. One of them is to use interpolating polynomials. In this case, we use a Newton interpolating polynomial of degree two at the nodes x_k, x_{k-1} and y_{k-1}, which is defined as follows:

$$N_2(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k) + f[x_k, x_{k-1}, y_{k-1}](t - x_k)(t - x_{k-1}).$$

Then, $N_2'(x_k)$ is an approximation of $f'(\alpha)$, so that, taking

$$\beta_k = -\frac{1}{N_2'(x_k)},$$

and replace this in (2), we obtain a method with memory, denoted by M_6N_2.

Theorem 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an neighborhood of α which we denote by $D \subset \mathbb{R}$ such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfy that $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function that satisfy that $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking a initial approximation x_0 close enough to α, the sequence of iterates $\{x_k\}$ generated by the method M_6N_2 converges to α with order $p = \frac{1}{2}\left(7 + \sqrt{61}\right) \approx 7.4051$.*

Proof. From the error equation (9),

$$e_{k+1} \sim (1 + \beta_k f'(\alpha))^3 e_k^6.$$

Using Taylor's series developments of $f(x_k)$, $f(x_{k-1})$, and $f(y_{k-1})$, it follows that

$$\begin{aligned} N_2'(x_k) = {} & f'(\alpha) + 2C_2 f'(\alpha)e_k + C_3 f'(\alpha)e_k e_{y,k-1} + C_3 f'(\alpha)(e_k - e_{y,k-1})e_{k-1} \\ & + O\left(e_{k-1}^2\right) + O\left(e_k^2\right) + O\left(e_{y,k-1}^2\right) + O_3(e_{y,k-1}, e_k, e_{k-1}). \end{aligned}$$

This means that $1 + \beta_k f'(\alpha)$ behaves as e_k, as $e_k e_{y,k-1}$, as $e_{k-1}e_k$, or as $e_{k-1}e_{y,k-1}$.

It is obvious that $e_k e_{y,k-1}$ tends faster to zero than e_k when $k \to \infty$ and that $e_{k-1}e_k$ tends to zero faster than $e_{k-1}e_{y,k-1}$. For this reason, you have to look at if e_k converges faster to zero or does it $e_{k-1}e_{y,k-1}$.

Suppose the R-order of the method is at least p. Consider the sequence $\{y_k\}$ generated by the first step of the method, and suppose that it converges to R-order at least p_1. Therefore, it is satisfied

$$e_{k+1} \sim D_{k,p} e_k^p \qquad \text{and} \qquad e_{y,k} \sim D_{k,p_1} e_k^{p_1},$$

where $D_{k,p}$ tends to the asymptotic error constant, D_p, and where D_{k,p_1} tends to the asymptotic error constant, D_{p_1}, when $k \to \infty$.

Therefore,

$$\frac{e_k}{e_{k-1}e_{y,k-1}} \sim \frac{D_{k-1,p}e_{k-1}^{p}}{D_{k-1,p_1}e_{k-1}e_{k-1}^{p_1}}.$$

Then, if $p > p_1 + 1$, it follows that

$$1 + \beta_k f'(\alpha) \sim e_{k-1}e_{y,k-1}.$$

The error equation (9) and the above relation give

$$e_{k+1} \sim (e_{k-1}e_{y,k-1})^3 e_k^6 \sim e_{k-1}^3 e_{y,k-1}^3 e_k^6. \tag{15}$$

Assuming that the R-order of the method is at least p, we obtain the relation (11), and assuming that the sequence $\{y_k\}$ converges with R-order of at least p_1, we obtain the relation

$$e_{y,k+1} \sim D_{k,p_1}e_k^{p_1} \sim D_{k,p_1}\left(D_{k-1,p}e_{k-1}^{p}\right)^{p_1} \sim e_{k-1}^{pp_1}. \tag{16}$$

In the same way that the relation (15) is obtained, we obtain that

$$e_{k+1} \sim e_{k-1}^3\left(D_{k-1,p_1}e_{k-1}^{p_1}\right)^3\left(D_{k-1,p}e_{k-1}^{p}\right)^6 = D_{k-1,p_1}^3 D_{k-1,p}^6 e_{k-1}^{3p_1}e_{k-1}^{6p+3}. \tag{17}$$

Moreover, we know that

$$e_{k,y} \sim (1+\beta_k f'(\alpha))e_k^2 \sim e_{k-1}e_{y,k-1}e_k^2 \sim e_{k-1}\left(D_{k-1,p_1}e_{k-1}^{p_1}\right)\left(D_{k-1,p}e_{k-1}^{p}\right)^2 \sim e_{k-1}^{2p+1+p_1}. \tag{18}$$

Then, equating the exponents of e_{k-1} of (11) and (17), and equating those of (16) and (18), we get that

$$p^2 = 6p + 3 + 3p_1,$$
$$pp_1 = 2p + 1 + p_1,$$

whose only positive solution is the order of convergence of the method M_6N_2, see Theorem 1, being $p \approx 7.4051$ and $p_1 \approx 2.4684$. □

Other Newton polynomials interpolation can be used of degree 2 or 3:

- $N_{2Z}(t) = f(x_k) + f[x_k, x_{k-1}](t - x_k) + f[x_k, x_{k-1}, z_{k-1}](t - x_k)(t - x_{k-1})$.
- $N_3(t) = N_2(t) + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}](t - x_k)(t - x_{k-1})(t - y_{k-1})$.
- $N_{3Z}(t) = N_{2Z}(t) + f[x_k, x_{k-1}, z_{k-1}, w_{k-1}](t - x_k)(t - x_{k-1})(t - z_{k-1})$.

Then, by choosing β_k as $-\frac{1}{N'_{2Z}(x_k)}$ and substituting this into (2), we get a new method with memory, which we denote by M_6N_{2Z}. In the same way, with polynomials N_3 and N_{3Z}, we obtain methods with memory, denoted by M_6N_3 and M_6N_{3Z}, respectively.

In the same way that the order of convergence of M_6N_2 is demonstrated, we can prove the following result where the orders of convergence obtained for the other methods with memory obtained by using interpolating polynomials are given.

Theorem 8. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in a neighborhood of α such that $f(\alpha) = 0$. We assume that $f'(\alpha) \neq 0$. Let $H(t)$ be a real function that satisfy that $H(0) = 1$, $H'(0) = 1$, $H''(0) = 2$, and $|H'''(0)| < \infty$, and let $G(t)$ be a real function satisfying $G(0) = 1$ and $|G'(0)| < \infty$. Then, taking a initial approximation x_0 close enough to α,*

- *the sequence of iterates generated by M_6N_{2Z} converges to α with order $4 + \sqrt{19} \approx 8.3589$,*

- *the sequence of iterates generated by M_6N_3 converges to α with order 8, and*
- *the sequence of iterates generated by M_6N_{3Z} converges to α with 9.*

4 | NUMERICAL EXPERIMENTS

In this section, we solve some nonlinear equations to compare the results obtained with the parametric family and the results obtained with the methods with memory designed previously. We also compare results with new schemes obtained in Chicharro et al. [10] and in Grau-Sánchez et al. [5], denoted by *CC* and *GS*, respectively. This methods have orders 4 and 6, respectively.

For the computational calculations, we use Matlab R2020b with variable precision arithmetic of 1000 digits and initial estimation x_0 and as stoping criterion the distance between consecutive iterations less than or equal to 10^{-100}. The numerical results we are going to compare the methods in the different examples are:

- the approximation obtained,
- the norm of the equation evaluated in that approximation,
- the norm of the distance between the last two approximations,
- the number of iterations necessary to satisfy the required tolerance, and
- the computational time and the approximate computational convergence order (ACOC), defined by Cordero and Torregrosa [11], which has the following expression:

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}.$$

The test functions used are as follows:

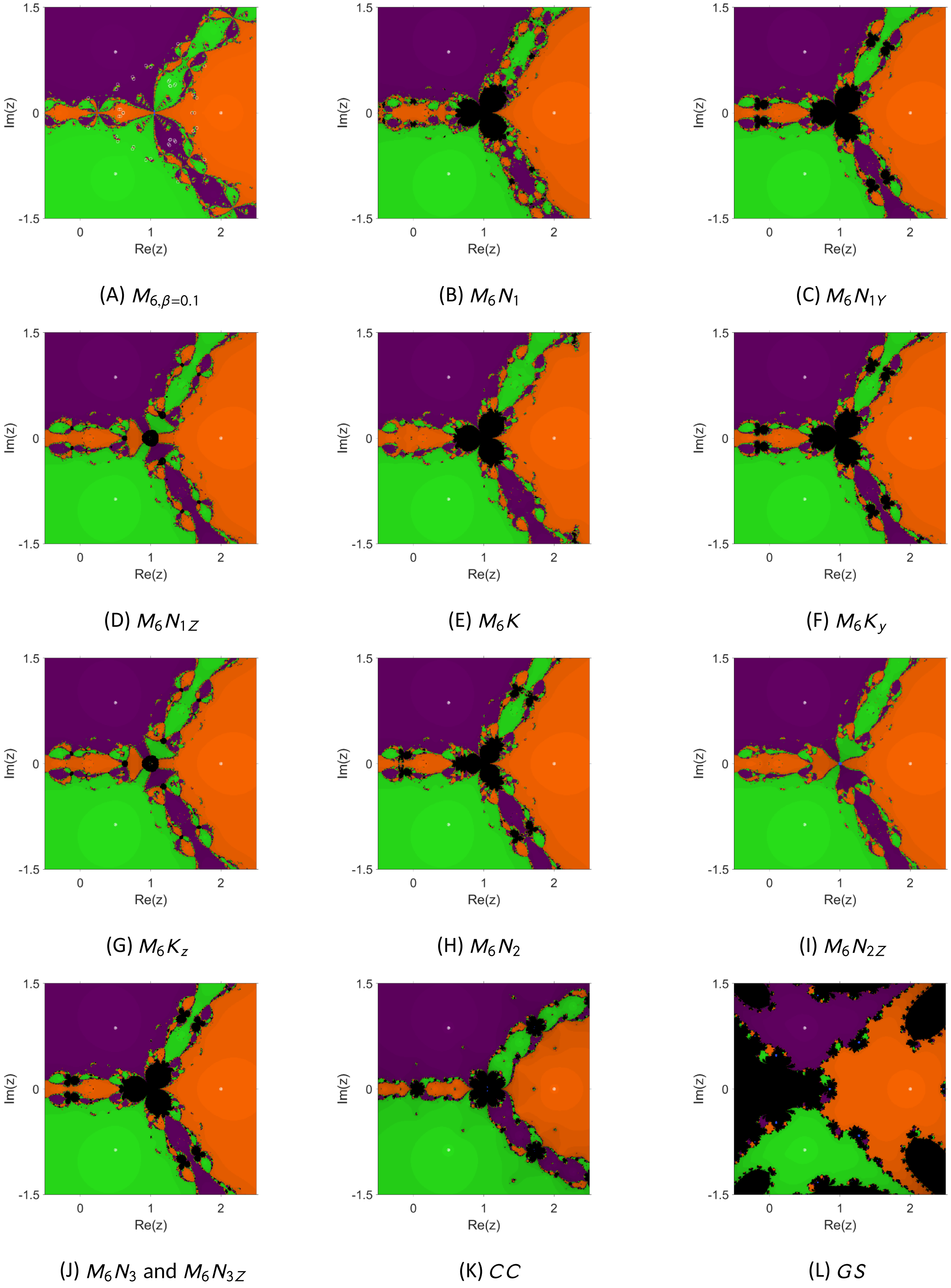
- Equation $(x-1)^3 - 1 = 0$, which has a real root at 2, and we take as initial estimates for all methods $x_1 = 1.5$, $x_0 = 0$, $w_0 = 0.5$, $y_0 = 1.1$, and $z_0 = 1.3$.
- Equation $\arctan(x) = 0$, which has a real root at 0, and we take as initial estimates for all methods $x_1 = 0.75$, $x_0 = 1.5$, $w_0 = 1.25$, $y_0 = 1$, and $z_0 = 0.8$.
- Equation $\cos(x) - x = 0$, which has a real root at 0.73908513, and we take as initial estimates for all methods $x_1 = 2$, $x_0 = 2.4$, $w_0 = 2.3$, $y_0 = 2.2$, and $z_0 = 2.1$.

Let us look at the results obtained for the equation $(x-1)^3 - 1 = 0$ in Table 1. As can be seen in the ACOC column, the theoretical order of convergence coincides with the ACOC in all the methods except in M_6N_3 in which the order increases by one unit.

For the computational time, the M_6 and M_6N_1 methods stand out at least, although they are not the methods that obtain the best approximation, as can be seen in the second and third columns. The conclusion of this numerical experiment is

TABLE 1 Results for the equation $(x-1)^3 - 1 = 0$.

Method	$\lvert x^{(k+1)} - x^{(k)}\rvert$	$\lvert f(x^{(k)})\rvert$	Iteration	ACOC	Time
$M_{6,\beta=-0.1}$	1.28999×10^{-225}	1.2288×10^{-1348}	5	6	0.3016
M_6N_1	1.07719×10^{-218}	4.0404×10^{-1409}	5	6.47927	0.3063
M_6K_1	3.78521×10^{-242}	$1.62794 \times 10^{-1660}$	5	6.85364	0.5469
M_6N_{1y}	8.79934×10^{-361}	$1.22538 \times 10^{-2520}$	6	7.0162	0.6266
M_6K_2	1.38495×10^{-584}	$1.35351 \times 10^{-4672}$	5	7.98531	0.5766
M_6N_{1z}	7.76951×10^{-186}	$1.65981 \times 10^{-1480}$	5	7.98009	0.5500
M_6K_3	8.21326×10^{-537}	$2.26765 \times 10^{-4824}$	5	8.99996	0.5609
M_6N_2	2.44389×10^{-303}	$2.62354 \times 10^{-2242}$	6	7.36596	0.4422
M_6N_{2Z}	2.2108×10^{-192}	$1.54585 \times 10^{-1602}$	5	8.53031	0.3656
M_6N_3	9.73317×10^{-479}	$1.04526 \times 10^{-4301}$	5	9.0	0.4281
M_6N_{3Z}	9.73317×10^{-479}	$1.04526 \times 10^{-4301}$	5	9.0	0.3984
CC	1.8372×10^{-266}	1.7135×10^{-1062}	21	3.999	0.4501
GS	1.5075×10^{-455}	1.0952×10^{-2728}	5	6.0	0.3204



(A) $M_{6,\beta=0.1}$

(B) M_6N_1

(C) M_6N_{1Y}

(D) M_6N_{1Z}

(E) M_6K

(F) M_6K_y

(G) M_6K_z

(H) M_6N_2

(I) M_6N_{2Z}

(J) M_6N_3 and M_6N_{3Z}

(K) CC

(L) GS

FIGURE 1 Dynamical planes generated by all the methods on equation $(x-1)^3-1=0$. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Results for the equation $\arctan(x) = 0$.

Method	$\vert x^{(k+1)} - x^{(k)}\vert$	$\vert f(x^{(k)})\vert$	Iteration	ACOC	Time
M_6	9.63399×10^{-303}	$2.88138 \times 10^{-2122}$	4	7	0.3594
M_6N_1	2.40435×10^{-233}	$6.78309 \times 10^{-1810}$	4	7.81347	0.3203
M_6K_1	7.26769×10^{-255}	$5.77411 \times 10^{-1977}$	4	7.82898	0.3422
M_6N_{1y}	1.10005×10^{-357}	$4.06924 \times 10^{-3367}$	4	9.36642	0.2656
M_6K_2	1.54723×10^{-383}	$1.25996 \times 10^{-3610}$	4	9.39834	0.3141
M_6N_{1z}	3.96266×10^{-481}	$2.54191 \times 10^{-4789}$	4	11.1413	0.2766
M_6K_3	2.60934×10^{-544}	$1.81813 \times 10^{-4890}$	4	11.1632	0.3187
M_6N_2	2.76956×10^{-308}	$2.72641 \times 10^{-2642}$	4	8.5653	0.3531
M_6N_{2Z}	7.34124×10^{-375}	8.011×10^{-2937}	4	9.40829	0.3312
M_6N_3	1.13781×10^{-375}	7.5513×10^{-2939}	4	9.3993	0.4094
M_6N_{3Z}	2.53673×10^{-448}	$7.51537 \times 10^{-4609}$	4	10.2653	0.3438
CC	3.0773×10^{-1394}	7.7912×10^{-694}	5	5.0	0.3107
GS	1.9359×10^{-126}	4.5284×10^{-881}	4	7.0	0.3825

that the most recommended methods are M_6N_3 and M_6N_{3Z}, as they have the highest ACOC, obtain the best approximation, and are among the methods that perform the fewest iterations.

In Figure 1, we draw the dynamical planes obtained for the nonlinear equation $(x - 1)^3 - 1 = 0$ by all the methods we have discussed above.

To generate the dynamical planes, we choose an initial point and what we do is apply our methods taking this point as the initial estimation. To choose the point, we generate a mesh of 400 × 400 points of the set $[-0.5, 2.5] \times [-1.5, 1.5]$, and then, each point of the mesh corresponds to a complex point, $z = \mathrm{Re}(z) + \gimel\, \mathrm{Im}\,(z)$, which we will take as a starting point.

We have also defined that the maximum number of iterations that each initial estimate must do is 80 and that we determine that the initial point converges to one of the solutions if the distance to that solution is less than 10^{-3}. We represent in green the initial points that converge to the root $\frac{1-\sqrt{3}i}{2}$, in orange the initial points that converge to the root 2, in purple the initial points that converge to the root $\frac{1+\sqrt{3}i}{2}$, and in black the initial points that do not converge to any root in less than 80 iterations.

In these dynamical planes, Figure 1, we can observe a priori what will happen to the initial points. In this case, we can see that although M_6 and GS have the same order, the convergence zones of the M_6 method are simpler and larger than those of the GS method, which has a larger black zone.

On the other hand, in the dynamical planes, it can be seen that the methods with memory in which the iteration z_{k-1} intervenes have a smaller black area than the rest of the methods with memory.

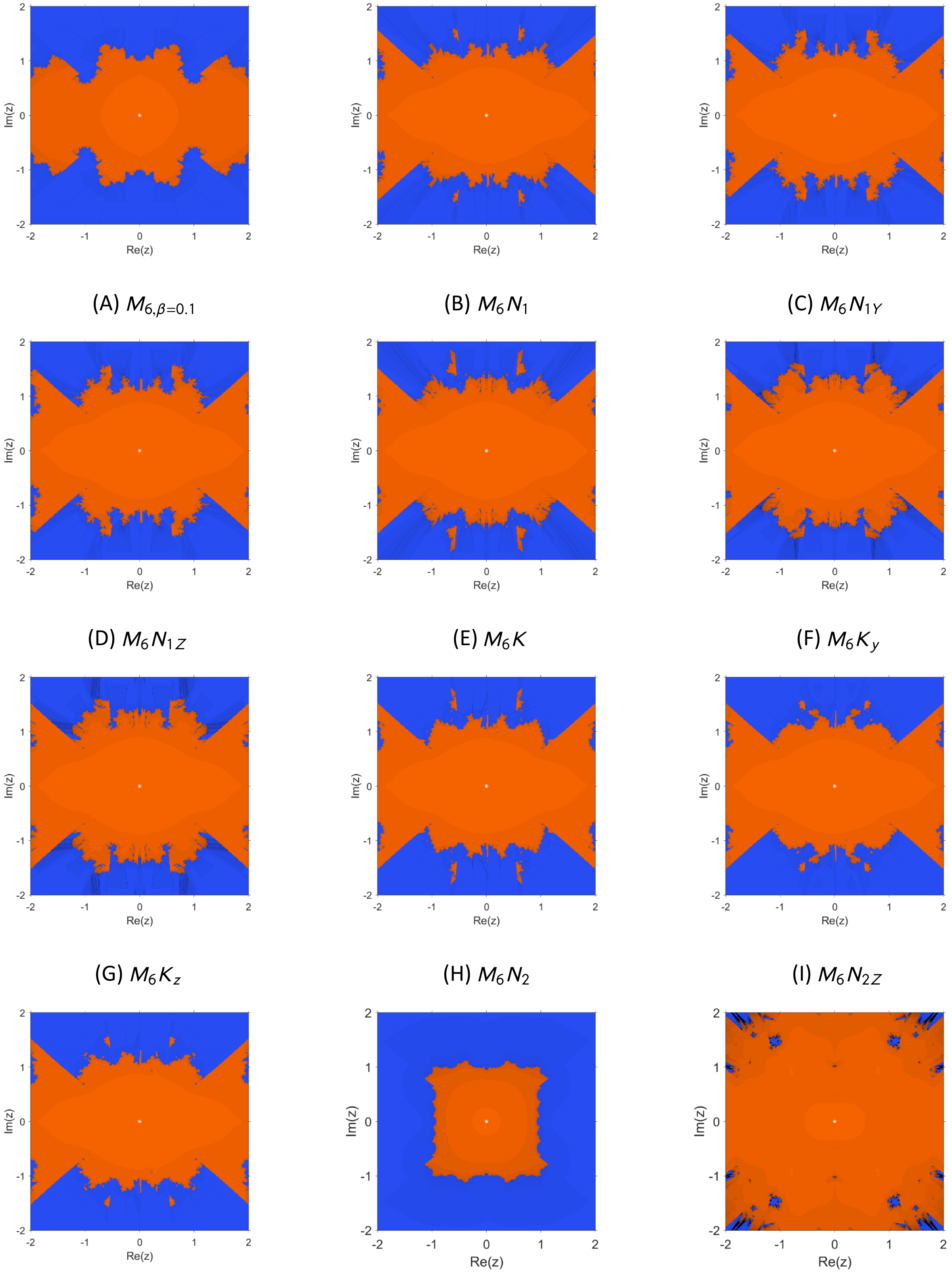
Let us now see the results obtained for the equation $\arctan(x) = 0$ in Table 2. In this case, all methods converge to the solution although there are differences between them. We can see that the method without memory increases its order by one unit according to the ACOC for this numerical example and that the methods with memory increase it by more than one unit, but not more than two, obtaining that the method with the highest ACOC is the M_6N_{3Z} method. If we look at the computational time, we can see that there is practically no difference between the methods. In this case, all the methods perform the same number of iterations, and if we look at Columns 2 and 3, we can see that the method that gives a better approximation is the method M_6N_{3Z}. The conclusion of this numerical experiment is that it is advisable in this case to use the M_6N_{3Z} method.

In Figure 2, we draw the dynamical planes obtained for the studied nonlinear equation $\arctan(x) = 0$ by all the methods we have discussed above.

We generate the dynamical planes in the same way as was done in Figure 2. We represent in orange the initial points that converge to the root 0 and in black the initial points that do not converge to 0 in less than 80 iterations.

In these dynamical planes, Figure 2, we can see a priori what will happen with the initial points. In this case, we can see that the GS method has a larger orange zone, so it has more points that converge to 0 in less than 80 iterations, something that did not happen in the previous example in which it was the method that obtained more points that did not converge than the rest. We can see that there is no great difference between the M_6 family and the methods with memory obtained from it, we can see that there is no great difference, although it is true that the methods with memory have a slightly larger orange area.

As the last part of the numerical experiments, we compare the efficiency of the discussed iterative methods.



(A) $M_{6,\beta=0.1}$

(B) M_6N_1

(C) M_6N_{1Y}

(D) M_6N_{1Z}

(E) M_6K

(F) M_6K_y

(G) M_6K_z

(H) M_6N_2

(I) M_6N_{2Z}

(J) M_6N_3 and M_6N_{3Z}

(K) CC

(L) GS

FIGURE 2 Dynamical planes generated by all the methods on equation $\arctan(x) = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 Table of orders and efficiency indexes.

Method	ACOC	Number evaluations	Efficiency index	Number iterations	Total efficiency index
$M_{6,\beta=-0.1}$	6	4	1.5651	6	1.0775
M_6N_1	6.4968	5	1.4539	4	1.0981
M_6K_1	7.1016	6	1.3864	4	1.0851
M_6N_{1Y}	7.0064	5	1.4760	4	1.1022
M_6K_Y	7.9888	6	1.4139	4	1.0904
M_6N_{1Z}	8.0229	5	1.5166	4	1.1097
M_6K_Z	9.0	6	1.4422	4	1.0959
M_6N_2	7.4167	6	1.3965	4	1.0871
M_6N_{2Z}	8.3741	6	1.4250	4	1.0926
M_6N_3	7.9697	7	1.3452	4	1.0769
M_6N_{3Z}	8.9911	7	1.3685	4	1.0816
CC	4.0	3	1.5874	5	1.0968
GS	6.0	5	1.4309	5	1.0743

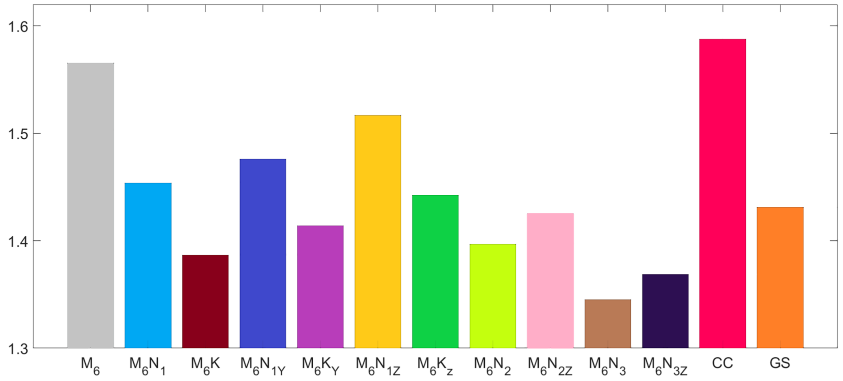


FIGURE 3 Efficiency index comparative. [Colour figure can be viewed at wileyonlinelibrary.com]

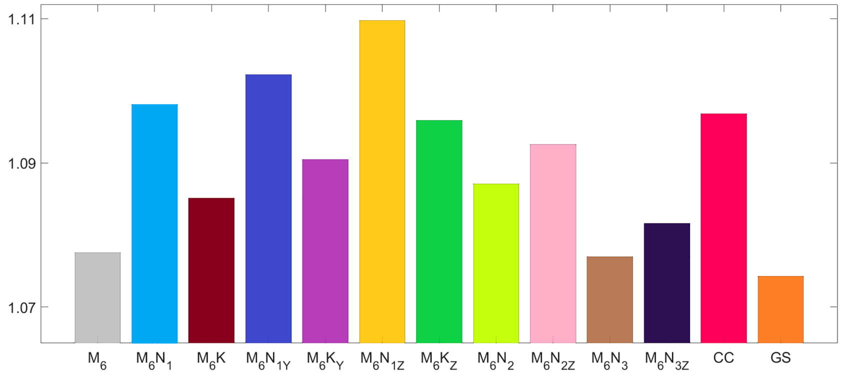


FIGURE 4 Total efficiency index comparative. [Colour figure can be viewed at wileyonlinelibrary.com]

We do this with the efficiency index in one iteration as well as with the total efficiency index, where the number of iterations that our method will perform is also important, given that the more iterations we evaluate, the more evaluations, and therefore the higher the computational cost. We compare the efficiency indexes of the methods with a real example, which is equation $\cos(x) - x = 0$.

Thus, in Table 3, we calculate the efficiency index for this numerical experiment by calculating $p^{\frac{1}{d}}$ where p is the approximate computational order of convergence, or ACOC, and where d is the number of evaluations per iteration.

We also compute the total efficiency index, which is calculated in the same way as the efficiency index but taking into account the number of iterations required to verify the convergence criteria, that is, $p^{\frac{1}{d \cdot k}}$, where k is the number of iterations needed.

In Figures 3 and 4, we show the efficiency indices appearing in Table 3.

As can be seen in Table 3, all methods with memory perform four iterations while the method M_6 performs six, and the *CC* and *GS* methods perform five iterations. As can be seen in Figures 3 and 4, the method with the highest efficiency index is the *CC* method and the M_6 method, followed by the M_6N_{1Z} method, but when we calculate the total efficiency index, things change quite a lot since the method with the highest values is the M_6N_{1Z} method, and one of the methods with the lowest value is M_6, since it needs to perform two more iterations than the rest of the methods, which means that the total efficiency index is lower.

5 | CONCLUSIONS

In this manuscript, we have designed a parametric family of derivative-free three-step iterative methods with convergence order 6.

We have introduced memory to this parametric family, through various divided difference operators and Newton interpolating polynomials, in order to increase the order of convergence without adding new functional evaluations. Thus, we have managed to increase the order by three units; in other words, from a method of order 6, we have managed to obtain a method with memory of order 9.

Furthermore, in the numerical experiments, for different problems, we compare the approximations obtained by each method, the convergence zones to the roots, and the efficiency of the methods according to the number of evaluations they make in each iteration and the number of iterations they make to verify the convergence conditions.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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