Nikishin's theorem, Orlicz type and factorization through Marcinkiewicz spaces

Mieczysław Mastyło and Enrique A. Sánchez Pérez

Abstract

Consider L^0, the F-space of all equivalence classes of measurable functions on a finite measure space equipped with the topology of convergence in measure. Inspired by Nikishin's classical result on the factorization of sublinear continuous operators from a Banach space to L^0, we prove a theorem that characterizes those maps from any quasi-metric space into L^0 that factor strongly through Marcinkiewicz weighted spaces. We show applications to sublinear operators on a certain class of quasi-Banach spaces with generalized Rademacher type generated by Orlicz sequence spaces.

1 Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite complete measure space. By $L^0(\Omega, \mathcal{A}, \mu)$ ($L^0(\Omega, \mu)$ or $L^0(\mu)$ for short) we denote the space of all (classes) of scalar valued $\mathcal{A}$-measurable almost everywhere finite functions on Ω with the topology of convergence on sets of finite measure. This paper is motivated by Nikishin's classical theorem on the factorisation of operators into $L^0(\mu)$. It states that, under certain mild requirements, every sublinear operator of a Banach function space on $L^0(\mu)$ can be factored through a weak L_p-space. This result has found many applications in the theory of Banach spaces, operator theory and harmonic analysis (see, for example, [6, 7, 13, 15, 16]). Following the current trend of leaving results from Functional

2010 *Mathematics Subject Classification*: Primary 46B28, 46E30.

Key words and phrases: Lipschitz operator, Marcinkiewicz space, Orlicz space, Nikishin theorem, Rademacher type and cotype.

The first named author was supported by the National Science Centre, Poland, Project no. 2019/33/B/ST1/00165. This work is part of the project/grant R+D&I PID2022-138342NB-I00 funded by MCIN/ AEI/10.13039/501100011033/ (Spain), which financially supports the second author.

Analysis on linear operators to the general context of metric spaces, the natural question is whether one can extend this result to general maps, or at least to those that satisfy the Lipschitz condition. Thus, the main aim of this paper is to prove a general variant of Nikishin's theorem that characterize those operators from a quasi-metric spaces to $L^0(\mu)$ that satisfy a factorization through a Marcinkiewicz space. Factorizations through weak L_p-spaces are, of course, a particular case of our results. That is, to establish clearly what requirements are necessary to impose on the operator and on the metric space involved in order to obtain a factorization through that space, thus focusing on the fundamental question of what aspects of the linear structure must be preserved for the "Nikishin machinery" to continue working. Before discussing more precisely the problem and contents of the paper, we introduce some notation. Let X be a nonempty set. A function $d\colon X \times X \to [0, \infty)$ is called a quasi-metric on X if $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and

$$d(x, y) \leq k(d(x, z) + d(z, x)), \quad \text{for some } k \geq 1.$$

If we can choose $k = 1$ in the inequality above, the quasi-metric d is called a metric. A pair (X, d) is called a quasi-metric space. If X is a linear space and the quasi-metric d on X satisfies $d(\lambda x, 0) = |\lambda| d(x, 0)$ for all scalars λ and all $x \in X$, then $\|\cdot\| := d(\cdot, 0)$ is called a quasi-norm on X. If the resulting space (X, d) is complete, then it is called a quasi-Banach space.

It was proved in [4, Theorem 2.1] that if (X, d) is a quasi-metric space, then for a certain $0 < p < \frac{1}{2}\log_{2k} 2$ there is a metric $\tilde{d}_p$ on X such that for all $x, y \in X$, one has

$$\tilde{d}_p(x, y) \leq d^p(x, y) \leq 2\,\tilde{d}_p(x, y), \quad x, y \in X\,.$$

Since the topology of a locally bounded, topological vector space can be given by a quasi-norm, the above-mentioned result is a generalization the Aoki-Rolewicz theorem (see [18, p. 95]). Recall that an F-norm is a subadditive function on a linear space $|\!|\!|\cdot|\!|\!|\colon X \to \mathbb{R}_+$ such that $|\!|\!|\lambda x|\!|\!| \leq |\!|\!|x|\!|\!|$ for $|\lambda| \leq 1$ and $x \in X$, and $|\!|\!|\lambda x|\!|\!| \to 0$ as $\lambda \to 0$. If the function d in the inequalities above defines a quasi-norm, then the classical Aoki-Rolewicz theorem establishes that $\tilde{d}_p$ can be chosen to be an F-norm. If μ is a finite measure, then $L^0(\mu)$ becomes an F-space when endowed with the F-norm given by $|\!|\!|f|\!|\!| = \int_\Omega \frac{|f|}{1+|f|}\,d\mu$. A quasi-Banach space $X \subset L^0(\mu)$ is said to be a quasi-Banach function lattice over Ω if for all $f, g \in L^0(\mu)$ with $|g| \leq |f|$ μ-a.e. and $f \in X$, one has $g \in X$ and $\|g\|_X \leq \|f\|_X$.

If $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ are metric spaces, then a map $T\colon \mathcal{M} \to \mathcal{N}$ is said to be Lipschitz if there exists a constant $\gamma > 0$ such that

$$d_{\mathcal{N}}(Tx, Ty) \leq \gamma\, d_{\mathcal{M}}(x, y), \quad x, y \in \mathcal{M}\,.$$

Local versions of such inequality will appear also in this paper.

We consider the set Φ of all functions $\varphi\colon [0,\infty) \to [0,\infty)$ that are increasing, continuous, unbounded and such that $\varphi(0) = 0$. A convex function $\varphi \in \Phi$ is called an *Orlicz function*. For a map φ in Φ, we denote by $L_{\varphi,\infty}(\mu)$ the *weak Marcinkiewicz space* of all $f \in L^0(\mu)$ such that

$$\|f\|_{\varphi,\infty} := \sup_{\lambda>0} \lambda\, \varphi(\mu_f(\lambda)) < \infty\,,$$

where (and in what follows) $\mu_f(\lambda) := \mu(\{\omega \in \Omega;\, |f(\omega)| > \lambda\})$ for all $\lambda > 0$.

It is well-known that if $\varphi \in \Phi$ satisfies that there exists $C \geq 1$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t \in (0, \mu(\Omega)$, then $(L_{\varphi,\infty}, \|\cdot\|_{\varphi,\infty})$ is a quasi-Banach lattice over $(\Omega, \mathcal{A}, \mu)$. If $0 < p < \infty$ and $\phi(t) = t^{1/p}$ for all $t \geq 0$, then we recover the weak L^p-Marcinkiewcz space denoted by $L_{p,\infty}$.

We recall that Nikishin [15] proved that if a Banach space embeds isomorphically into $L^0(\mu)$ (where μ is nonatomic probability measure), then it is isomorphic to a subspace of $L^p(\mu)$ for every $p \in (0,1)$. This result was the first progress on the unsolved problem posed by Kwapień [9] whether every Banach space which embeds into $L^0((0,1), m)$ is isomorphic to a subspace of $L^1(m)$, where m is Lebesgue measure on the interval $(0,1)$. In [16] Nikishin proved the following factorization result, which improved his first one. It states that if X is a Banach space of type p $(1 \leq p < 2)$ and $T\colon X \to L^0(\mu)$ is any continuous linear operator, then for every $\varepsilon > 0$ there is a set E with $\mu(E) \geq 1 - \varepsilon$ and such that the operator S given by

$$Sx := \mathbf{1}_E Tx, \quad x \in X$$

is bounded from X into $L_{p,\infty}(\mu)$, where $\mathbf{1}_E$ denotes the indicator function of E.

Following Kalton [7] a continuous linear operator T from a Banach space X into a quasi-Banach function lattice E over a measure space $(\Omega, \mathcal{A}, \mu)$ is said to be a *strong embedding* if T is an isomorphism onto its range, and the topology of E on its range coincides with the $L^0(\mu)$-topology. Using the above terminology, it follows in particular from Nikishin's theorem that every Banach subspace of $L^0(\mu)$ can be strongly embedded in $L_{1,\infty}(\mu)$. The mentioned paper [7] gives a valuable contribution to Kwapień's problem. In particular it is shown that, for $1 \leq p < 2$, a Banach space X embeds into $L_p(0,1)$ if and only if $\ell_p(X)$ embeds into $L^0(m)$.

2 Nikishin's factorization theorem

The aim of this section is to prove a new version of Nikishin's theorem for functions from quasi-metric spaces into $L^0(\mu)$ involving Marcinkiewicz spaces instead of the usual weak L_p-spaces. Given a quasi-metric space (X, d), a measure space $(\Omega, \mathcal{A}, \mu)$ and a function $\varphi \in \Phi$, we say that a mapping $T\colon X \to L^0(\mu)$ factors strongly through $L_{\varphi,\infty}(\mu)$ if there exists a function $g \in L^0(\mu)$ such that for all $x, y \in X$ one has

$$\varphi\Big(\mu\Big(\Big\{\omega \in \Omega : \Big|\frac{Tx(\omega) - Ty(\omega)}{g(\omega)}\Big| > \lambda\Big\}\Big)\Big) \leq \frac{d(x,y)}{\lambda}, \quad \lambda > 0\,,$$

or equivalently, for short, $\|(Tx - Ty)/g\|_{\varphi,\infty} \leq d(x,y)$. This is equivalent to the following factorization diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "T_0"'] & & L^0(\mu) \\ & L_{\varphi,\infty}(\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

where M_g is the multiplication mapping given by $M_g(f) := g \cdot f$, for all $f \in L_{\varphi,\infty}(\mu)$. In particular, if X is a Banach space, $T\colon X \to L^0(\mu)$ a sublinear operator and $\varphi(t) := t^{1/p}$ for all $t \geq 0$, then following [19, Definition 4, p. 259] we say that T factors strongly through $L_{p,\infty}(\mu)$.

To state and prove the main result, we recall that if $(\Omega, \mathcal{A}, \mu)$ is a measure space and $\varphi \in \Phi$, then we denote by $L_\varphi(\mu)$ (L_φ for short) the *Orlicz space* of all $f \in L^0(\mu)$ such that for some $\lambda > 0$,

$$\rho_\varphi(\lambda f) := \int_\Omega \varphi(\lambda |f|)\, d\mu < \infty.$$

It is an F-space equipped with the F-norm $|\!|\!| f |\!|\!|_\varphi := \inf\{\varepsilon > 0 :\ \rho_\varphi(f/\varepsilon) \leq \varepsilon\}$ (see (see [14]).

Throughout the paper we consider only such $\varphi \in \Phi$ for which the functional

$$\|f\|_\varphi := \inf\{\varepsilon > 0 :\ \rho(f/\varepsilon) \leq 1\}, \quad f \in L_\varphi$$

is a quasi-norm on L_φ. In what follows the set of all such functions $\varphi \in \Phi$ is denoted by $\widehat{\Phi}$. It is easy to verify that $\varphi \in \widehat{\Phi}$ if there exists $C \geq 1$ such that $\varphi(t/C) \leq \varphi(t)/2$, for all $t > 0$, and furthermore we have

$$\|f + g\|_\varphi \leq C(\|f\|_\varphi + \|g|_\varphi), \quad f, g \in L_\varphi\,.$$

The function $\|\cdot\|_\varphi$ is a norm on L_φ if φ is an Orlicz function.

In the case when μ is the counting measure on $\mathbb{N}$, then the Orlicz space L_φ is called an *Orlicz sequence space* and is denoted by ℓ_φ, so we only consider functions $\varphi \in \Phi$ for which the functional

$$\|\xi\|_{\ell_\varphi} := \inf\Big\{\varepsilon > 0 :\ \sum_{j=1}^{\infty} \varphi(|\xi_j|/\varepsilon) \leq 1\Big\}, \quad \xi = (\xi_j)_{j=1}^{\infty} \in \ell_\varphi$$

is a quasi-norm. In what follows the class of all such φ-functions is denoted by $\widehat{\Phi}$. For any $\varphi \in \widehat{\Phi}$ we set ℓ_φ^n to be $\mathbb{R}^n$ equipped with the (quasi)-norm defined for all $(\xi_j)_{j=1}^n \in \mathbb{R}^n$ by

$$\|(\xi_j)_{j=1}^n\|_{\ell_\varphi^n} := \|\xi\|_\varphi, \quad \xi := (\xi_1, \dots, \xi_n, 0, 0, \dots).$$

Recall that a function $\varphi\colon [0,\infty) \to [0,\infty)$ satisfies the Δ_2-condition ($\varphi \in \Delta_2$ for short), if $\sup_{t>0} \varphi(2t)/\varphi(t) < \infty$. For $\varphi \in \Delta_2$, we define the function $s_\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$s_\varphi(t) := \sup_{u>0} \frac{\varphi(tu)}{\varphi(u)}, \quad t > 0.$$

In what follows, if $\varphi \in \widehat{\Phi} \cap \Delta_2$ satisfies $s_\varphi(t) \to 0$ as $t \to 0$, then we write $\varphi \in \Phi_0$. Observe that if $\varphi \in \Phi \cap \Delta_2$ is an Orlicz function, then for all $t \in (0,1]$ one has $s_\varphi(t) \le t$ and so $\varphi \in \Phi_0$.

The following result is an extension of Nikishin's theorem.

Theorem 2.1. *Let (X,d) be a quasi-metric space and let $T\colon X \to L^0(\mu)$ be a mapping. Let us assume that $(\Omega, \mathcal{A}, \mu)$ is a probability measure space and $\varphi \in \Phi_0$. Then the following statements are equivalent*:

(i) *T factors strongly through $L_{\varphi^{-1},\infty}(\mu)$.*

(ii) *There is a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $C(\lambda) \to 0$ as $\lambda \to \infty$ such that, for each $n \in \mathbb{N}$ and for all real positive scalars $(\alpha_j)_{j=1}^n$ and all finite sequences $(x_j)_{j=1}^n$, $(y_j)_{j=1}^n$ in X, one has*

$$\mu\Big(\Big\{\omega \in \Omega : \sup_{1\le j\le n} \alpha_j\,|Tx_j(\omega) - Ty_j(\omega)| \ge \lambda \|(\alpha_j d(x_j,y_j))_{j=1}^n\|_{\ell_\varphi^n}\Big\}\Big) \le C(\lambda).$$

(iii) *For every $\varepsilon > 0$ there exist a constant $C_\varepsilon > 0$ and a set $E_\varepsilon \in \mathcal{A}$ with $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ such that*

$$\big\|\mathbf{1}_{E_\varepsilon}(Tx - Ty)\big\|_{\varphi^{-1},\infty} \le C_\varepsilon d(x,y), \quad x, y \in X.$$

Proof. (i) $\Rightarrow$ (ii). Suppose that T admits the following factorization

$$T\colon X \xrightarrow{T_0} L_{\varphi^{-1},\infty}(\mu) \xrightarrow{M_g} L^0(\mu).$$

For each n fix any positive sequence $(\alpha_j)_{j=1}^n$ and any sequences $(x_j)_{j=1}^n$ and $(y_j)_{j=1}^n$ in X. Then consider the set

$$\begin{aligned} A_\varphi &:= \Big\{\omega \in \Omega : \sup_{1\le j\le n} \alpha_j\,|Tx_j(\omega) - Ty_j(\omega)| \ge \lambda\,\|(\alpha_j\, d(x_j,y_j))\|_{\ell_\varphi^n}\Big\} \\ &= \Big\{\omega \in \Omega : |g(\omega)| \sup_{1\le j\le n} \alpha_j\,|T_0x_j(\omega) - T_0y_j(\omega)| \ge \lambda\|(\alpha_j\, d(x_j,y_j))\|_{\ell_\varphi^n}\Big\}. \end{aligned}$$

We also have

$$
\begin{aligned}
A_\varphi \subset \big\{\omega\in\Omega:\ |g(\omega)|>\sqrt{\lambda}\big\}\\
\cup \bigcup_{j=1}^{n} \Big\{\omega\in\Omega:\ |T_0x_j(\omega)-T_0y_j(\omega)| \geq \frac{\sqrt{\lambda}}{\alpha_j}\,\big\|(\alpha_j d(x_j,y_j))\big\|_{\ell_\varphi^n}\Big\}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mu(A_\varphi) &\leq \mu_g(\sqrt{\lambda}) + \sum_{j=1}^{n} \mu\Big(\Big\{\omega\in\Omega;\ |T_0x_j(\omega)-T_0y_j(\omega)| \geq \frac{\sqrt{\lambda}}{\alpha_j}\,\|(\alpha_j d(x_j,y_j)\|_{\ell_\varphi^n}\Big\}\Big)\\
&\leq \mu_g(\sqrt{\lambda}) + \sum_{j=1}^{n} \varphi\Big(\frac{\alpha_j d(x_j,y_j)}{\sqrt{\lambda}\,\|(\alpha_j\, d(x_j,y_j)\|_{\ell_\varphi^n}}\Big)\\
&\leq \mu_g\big(\sqrt{\lambda}\big) + s_\varphi\big(1/\sqrt{\lambda}\big)\cdot \sum_{j=1}^{n} \varphi\Big(\frac{\alpha_j d(x_j,y_j)}{\|(\alpha_j\, d(x_j,y_j))_j\|_{\ell_\varphi^n}}\Big)\,.
\end{aligned}
$$

Now note that for any scalar sequence $\xi=(\xi_j)_{j=1}^n \neq 0$, we have

$$
\sum_{j=1}^{n} \varphi\big(|\xi_k|/\|\xi\|_\varphi\big) \leq 1\,,
$$

and so combining with the above estimate of $\mu(A_\varphi)$ yields

$$
\mu(A_\varphi) \leq C(\lambda) := \mu_g(\sqrt{\lambda}) +\ s_\varphi\big(1/\sqrt{\lambda}\big)\,.
$$

Since $C(\lambda)\to 0$ as $\lambda\to\infty$, the statement (ii) is proved.

(ii) $\Rightarrow$ (iii). Let C be a function that satisfies condition (ii). For a given $\varepsilon>0$ choose $R>0$ such that $C(R)<\varepsilon$. Consider a family $\mathcal{F}$ of all measurable subsets $F\subset\Omega$ with the following property (P): there exist x, $y\in X$ such that

$$
\varphi^{-1}(\mu(F))\,|Tx(\omega)-Ty(\omega)| > R\,d(x,y), \quad \omega\in F\,.
$$

If $\mathcal{F}$ is empty, then we can take $E_\varepsilon=\Omega$ and the uniform constant $C_\varepsilon=R$ for all $\varepsilon>0$. Indeed, for every fixed $x,y\in X$ and $\lambda>0$ define $F_{x,y,\lambda}$ by

$$
F_{x,y,\lambda} := \big\{\omega\in\Omega:\ |Tx(\omega)-Ty(\omega)|>\lambda\big\}\,.
$$

Since $F_{x,y,\lambda}$ does not satisfy property (P), there exists $\omega\in F_{x,y,\lambda}$ such that

$$
\lambda\varphi^{-1}(\mu(F_{x,y,\lambda})) \leq \varphi^{-1}(\mu(F_{x,y,\lambda}))\,|Tx(\omega)-Ty(\omega)| \leq R\,d(x,y).
$$

Thus $\lambda\varphi^{-1}(\mu(F_{x,y,\lambda})) \leq R\,d(x,y)$, that is,

$$\|Tx - Ty\|_{\varphi^{-1},\infty} \leq R\,d(x,y)\,.$$

If the family $\mathcal{F}$ is nonempty, then by a standard argument there is a maximal collection (F_j) of pairwise disjoint measurable sets satisfying property (P) with the corresponding pairs $(x_j, y_j) \in X \times X$, that is, for each j

$$\varphi^{-1}(\mu(F_j))\,|Tx_j(\omega) - Ty_j(\omega)| > R\,d(x_j, y_j). \tag{$*$}$$

Clearly, for $(\alpha_j)_j$ given by $\alpha_j := \varphi^{-1}(\mu(F_j))/d(x_j, y_j)$ one has

$$\|(\alpha_j d(x_j, y_j))_j\|_{\varphi} \leq 1\,.$$

Thus if we set $F := \bigcup_j F_j$, then using inequality $(*)$ yields

$$\sup_j \alpha_j |Tx_j(\omega) - Ty_j(\omega)| \geq R, \quad \omega \in F\,.$$

Combining these remarks with condition (ii), we obtain

$$\begin{aligned}
\mu(F) &= \mu\big(\{\omega \in F :\, \sup_j \alpha_j |Tx_j(\omega) - Ty_j(\omega)| \geq R\}\big) \\
&\leq \mu\big(\{\omega \in \Omega :\, \sup_j \alpha_j |Tx_j(\omega) - Ty_j(\omega)| \geq R\,\|((\alpha_j\, d(x_j, y_j))_j\|_{\ell_\varphi}\}\big) \\
&\leq C(R) < \varepsilon\,.
\end{aligned}$$

We claim that $E := \Omega \setminus F$ satisfies condition (iii) for $C_\varepsilon = R$. Otherwise there are $x,\, y \in X$ and $\lambda > 0$ such that the set

$$A := \big\{\omega \in E :\, |Tx(\omega) - Ty(\omega)| \geq \lambda\big\}$$

satisfies $\lambda\,\varphi^{-1}(\mu(A)) > R\,d(x,y)$. Then

$$\varphi^{-1}(\mu(A))|Tx(\omega) - Ty(\omega)| > \frac{\lambda R\,d(x,y)}{\lambda} = R\,d(x,y), \quad \omega \in A\,,$$

and so A satisfies property (P). Since $A \subset E$ is disjoint to F, this is a contradiction by the maximality of the family that constitutes F. This proves (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). For each $n \in \mathbb{N}$ set $\varepsilon_n := 1/n$, and write E_n for the measurable set and $C_n > 0$ for the constant which satisfy the requirements in (iii) for ε_n. Fix a sequence of positive real numbers $(\alpha_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty s_\varphi(C_n/\alpha_n) \leq 1$. Consider the disjoint partition $(F_n)_{n=1}^\infty$ of Ω given by $F_1 = E_1$ and $F_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$ for $n > 1$. Define $g \in L^0(\mu)$ by

$$g = \sum_{n=1}^\infty \alpha_n \mathbf{1}_{F_n}\,.$$

Observe that for all x, $y \in X$ and all $\lambda > 0$ one has

$$\begin{aligned}
&\mu\Big(\Big\{\omega \in \Omega : \Big|\frac{Tx(\omega) - Ty(\omega)}{g(\omega)}\Big| > \lambda\Big\}\Big) \\
&\qquad = \sum_{n=1}^{\infty} \mu\big(\{\omega \in F_n : |Tx(\omega) - Ty(\omega)| > \alpha_n \lambda\}\big) \\
&\qquad \leq \sum_{n=1}^{\infty} \mu\big(\{\omega \in E_n : |Tx(\omega) - Ty(\omega)| > \alpha_n \lambda\}\big) \\
&\qquad \leq \varphi\Big(\frac{d(x,y)}{\lambda}\Big) \sum_{n=1}^{\infty} s_{\varphi}\Big(\frac{C_n}{\alpha_n}\Big) \leq \varphi\Big(\frac{d(x,y)}{\lambda}\Big).
\end{aligned}$$

This yields the required estimate

$$\|(Tx - Ty)/g\|_{\varphi^{-1},\infty} \leq d(x,y), \quad x, y \in X,$$

which completes the proof. □

3 Factorization of sublinear continuous operators and quasi-Banach spaces of type φ

In this section we show how Theorem 2.1 provides a characterization of the strong factorization through weak Marcinkiewicz spaces for a special class of non-linear maps. Together with a well-known extension of the central notion of (Rademacher) type p involving functions $\varphi \in \Phi_0$ for quasi-Banach spaces, our result provides a broad theoretical context in which the circle of ideas of Nikishin's theorem works optimally. Specifically, we prove that under the assumption of being of type φ a continuous sublinear operator from X into $L^0(\mu)$ factors strongly through a weighted Marcinkiewicz space.

Recall that if X is a quasi-Banach space, then an operator $T\colon X \to L^0(\mu)$ is called sublinear if for all x, $y \in X$ and scalars λ,

(a) $|T(x+y)| \leq |Tx| + |Ty|$,

(b) $|T(\lambda x)| = |\lambda|\,|Tx|$.

Note that if $T\colon X \to L^0(\mu)$ is sublinear, then $|T|$ given by $|T|(x) := |Tx|$ for all $x \in X$ satisfies $||T|(x) - |T|(y)| \leq |T|(x-y)$ for all x, $y \in X$. Since $|T|$ is positively homogeneous, the following corollary is an immediate consequence of Theorem 2.1.

Corollary 3.1. *Let X be a quasi-Banach space and let $T\colon X \to L^0(\mu)$ be a sublinear operator. Let us assume that μ is a probability measure and $\varphi \in \Phi_0$. Then the following statements are equivalent:*

(a) *$|T|$ factors strongly through $L_{\varphi^{-1},\infty}(\mu)$.*

(b) *There is a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $C(\lambda) \to 0$ as $\lambda \to \infty$ such that, for any finite sequence $(x_j)_j$ in X, one has*

$$\mu\Big(\Big\{\omega \in \Omega:\ \sup_j |Tx_j(\omega)| \geq \lambda \big\|\big(\|x_j\|_X\big)\big\|_{\ell_\varphi}\Big\}\Big) \leq C(\lambda)\,.$$

The proof of the following lemma is straightforward (see [19, Proposition 2] for the case of Banach spaces).

Lemma 3.2. *Let T be a sublinear operator from a quasi-Banach space into $L^0(\mu)$, with μ a finite measure. Then T is continuous if and only if there exists a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mu\big(\big\{\omega \in \Omega:\ |Tx(\omega)| \geq \lambda\, \|x\|_X\big\}\big) \leq C(\lambda)\,,$$

where $C(\lambda) \to 0$ as $\lambda \to \infty$.

Let us introduce now the second component for the factorization through Marcinkiewicz spaces. Let $\varphi \in \Phi$. The following definition extends the well-known notion of Rademacher type p (type p for short) with $0 < p \leq 2$.

It is said that a quasi-Banach space X is of type φ for if there is a constant C such that for any finite sequence $(x_j)_j$ in X, we have

$$\bigg(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|_X^2\, dt\bigg)^{1/2} \leq C\, \big\|\big(\|x_j\|_X\big)_j\big\|_{\ell_\varphi}\,,$$

where $(r_j)_j$ is the sequence of the Rademacher functions. The least constant C is denoted by $T_\varphi(X)$.

The classical Khintchine inequalities immediately imply that if X has type φ, then the Orlicz space ℓ_φ is continuously included in ℓ^2, that is, for some $\gamma > 0$ and all $t \in (0,1)$ one has $\gamma\, t^2 \leq \varphi(t)$. Note that we can change the exponent 2 in the definition of type φ without altering the definition. This follows from a result due to Kalton [8] who extended a well-known result of Kahane [5] for Banach spaces, and proved that if X is a quasi-Banach space, then for any $0 < p < q < \infty$ there is a constant $K = K(p,q)$ such that for any finite sequence $(x_j)_j$ in X it holds

$$\begin{aligned}\Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|_X^p\,dt\Big)^{1/p} &\le \Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|_X^q\,dt\Big)^{1/q}\\ &\le K\Big(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|_X^p\,dt\Big)^{1/p}.\end{aligned}$$

In this context, let us apply the notion of type φ to get an estimate of the distribution $\mu(\{\sup_j |Tx_j| > \lambda\}$, where $T\colon X \to L^0(\mu)$ is a sublinear operator continuous at zero and (x_j) is any sequence in X. The following result shows our distributional estimate. Its proof is based on the proof of Theorem 6 in [19, p. 261]. We include it with some minor modifications of the arguments for the aim of completeness. From now on we assume that $(\Omega, \mathcal{A}, \mu)$ is a probability measure space.

Proposition 3.3. *Let X is a quasi-Banach space of type φ with $\varphi \in \Phi_0$ and let $T\colon X \to L^0(\mu)$ be a sublinear operator that is continuous at zero. Then there exists a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(\lambda) \to 0$ as $\lambda \to \infty$ and*

$$\mu\Big(\Big\{\sup_j |Tx_j| > \lambda\Big\}\Big) \le 2\,T_\varphi(X)^2\lambda^{-1} + 2\,C(\sqrt{\lambda}), \quad \lambda > 0,$$

for all finite sequences $(x_j)_j$ in X satisfying $\sum_j \varphi(\|x_j\|_X) \le 1$.

Proof. Given a finite sequence $(x_j)_j$ in X with $\sum_j \varphi(\|x_j\|_X) \le 1$. As in the proof of [19, Theorem 6, p. 231], we define for every $t \in [0,1]$ and each $k \in \{1,\dots,n\}$ the functions g_t and g_t^k by $g_t := \sum_j r_j(t)x_j$ and $g_t^k := \sum_j \varepsilon_j r_j(t)x_j$, where $\varepsilon_k = 1$ and $\varepsilon_j = -1$ for each $j \ne k$. Then by sublinearity of T we have

$$2\,|Tx_k| = |T(g_t + g_t^k)| \le |Tg_t| + |Tg_t^k|, \quad t \in [0,1].$$

It is easy to see that for μ-almost all $\omega \in \Omega$ the functions $t \mapsto |Tg_t(\omega)|$ and $t \mapsto |Tg_t^k(\omega)|$ have the same distribution. Since

$$\begin{aligned}&\big\{t \in [0,1] : |Tg_t(\omega)| + |Tg_t^k(\omega)| \ge 2|Tx_k(\omega)|\big\}\\ &\subset \quad \big\{t \in [0,1] : |Tg_t(\omega)| \ge |Tx_k(\omega)|\big\}\\ &\quad \cup \big\{t \in [0,1] : |Tg_t^k(\omega)| \ge |Tx_k(\omega)|\big\},\end{aligned}$$

it follows that $|\{t \in [0,1] : |Tg_t(\omega)| \ge |Tg_t^k(\omega)|\}| \ge 1/2$, where $|A|$ denotes the Lebesgue measure of a set A. In consequence, it follows by the fact that $Tx_k(\omega)$ does not depend on $t \in [0,1]$, that for μ-almost all $\omega \in \Omega$,

$$\Big|\Big\{t \in [0,1] : |Tg_t(\omega)| \ge \sup_k |Tx_k(\omega)|\Big\}\Big| \ge 1/2.$$

Hence, for every $\lambda > 0$ one has for the set $A_\lambda := \{\omega \in \Omega : \sup_k |Tx_k(\omega)| > \lambda\}$,

$$\begin{aligned}
\frac{1}{2}\,\mu(A_\lambda) &\le \int_{A_\lambda} \big|\{t \in [0,1] : |Tg_t(\omega)| \ge \max_{1\le k\le n} |Tx_k(\omega)|\}\big|\, d\mu \\
&\le \int_\Omega |\{t \in [0,1] : |Tg_t| \ge \lambda\}|\, d\mu \le \int_\Omega \big|\{t \in [0,1] : \|g_t\|_X \ge \sqrt{\lambda}\}\big|\, d\mu \\
&+ \int_\Omega \big|\{t \in [0,1] : |Tg_t(\omega)| \ge \sqrt{\lambda}\,\|g_t\|_X\}\big|\, d\mu = \big|\{t \in [0,1] : \|g_t\|_X \ge \sqrt{\lambda}\}\big| \\
&+ \int_0^1 \mu\big(\{\omega \in \Omega : |Tg_t(\omega)| \ge \sqrt{\lambda}\,\|g_t\|_X\}\big)\, dt\,,
\end{aligned}$$

where we have used Fubini's Theorem for the last equality.

Since X has type φ and $\sum_j \varphi(\|x_j\|_X) \le 1$ implies $\big\|\big(\|x_j\|_X\big)\big\|_{\ell_\varphi} \le 1$, we get

$$\big|\{t \in [0,1] : \|g_t\|_X \ge \sqrt{\lambda}\}\big| \le T_\varphi(X)^2\,\lambda^{-1}\,.$$

Now observe that by Lemma 3.2, it follows that there exists a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(\lambda) \to 0$ as $\lambda \to \infty$ and satisfies the estimate given there. Combining all together the above estimates, we obtain

$$\mu(A_\lambda) \le 2\,T_\varphi(X)^2\lambda^{-1} + 2C(\sqrt{\lambda}), \qquad \lambda > 0\,.$$

This completes the proof. □

An immediate consequence of Proposition 3.3 and Corollary 3.1 is the following.

Corollary 3.4. *Let X be a quasi-Banach space of type φ with $\varphi \in \Phi_0$. Then every sublinear continuous operator $T\colon X \to L^0(\mu)$ factors strongly through $L_{\varphi^{-1},\infty}(\mu)$.*

When X is a quasi-Banach space of type $0 < p \le 2$ and $T\colon X \to L^0$ is a continuous linear operator, this variant of the Nikishin's theorem was proved by Kalton in [6]. In this remarkable article, he studied operators from the spaces $L_p = L_p(0,1)$ for $0 < p < 1$ on other spaces of measurable functions. It is worth noting here that one of the main results of this paper is Theorem 7.2 in [6], a consequence of Nikishin's theorem which states that if $0 < p < 1$ and $T\colon L_p \to L^0$ is a continuous linear operator, then if $p < r \le 2$, there exists a subspace V of L_p which is isomorphic to L_r and such that T is an isomorphism on V.

The requirement on the quasi-Banach space X that is needed to obtain our factorization of a sublinear operator that it be of type φ. Since this notion is not so usual, we end this section by showing some examples of spaces (which are even Banach) that satisfy this condition. Let us introduce some new concepts and notations. We can define the notion of cotype φ in the same way that we do for type. For an Orlicz function φ, a Banach space X is said to be of cotype φ if there exists a constant $C>0$ such that for every finite sequence $(x_j)_j$ in X,

$$\big\|\big(\|x_j\|_X\big)_j\big\|_{\ell_\varphi} \le C\left(\int_0^1 \Big\|\sum_j r_j(t)x_j\Big\|_X^2\,dt\right)^{1/2}.$$

Recall that if φ is an Orlicz function, then the map $\varphi_*\colon [0,\infty)\to[0,\infty]$ defined by

$$\varphi_*(t)=\sup\{st-\varphi(s):\, s\ge 0\},\qquad t\ge 0,$$

is called the Young's conjugate function of φ. It takes finite values if $\lim_{t\to\infty}\varphi(t)/t=\infty$, that is, if φ_* is an Orlicz function. In this case, φ is called Young's function.

We refer to [2, p. 259] for the definition of the important concept of K-convexity of Banach spaces introduced by Pisier. He shown in [17] that in fact K-convexity and B-convexity are equivalent properties for a Banach space, and they are also equivalent of being of non-trivial Rademacher type.

Using the well-known Köthe duality formula $(\ell_\varphi)'=\ell_{\varphi_*}$ up to equivalence of norms, we can prove the following statement in a similar way to how it is proved for the Rademacher cotype q $(2\le q<\infty)$ in [2, Theorem 13.17].

Proposition 3.5. *Let X be a K-convex Banach space and φ a convex Young function such that X is of cotype φ. Then the Banach dual space X^* has type φ_*,*

To show how to apply this statement let us recall some definitions from the geometry of Banach spaces. The modulus of convexity $\delta_X\colon[0,2]\to[0,1]$ of a Banach space X is defined by

$$\delta_X(\varepsilon):=\inf\big\{1-\|(x+y)/2\|:\ \|x\|=\|y\|=1,\ \|x-y\|\ge\varepsilon\big\},\qquad \varepsilon\in[0,2]\,;$$

its modulus of smoothness δ_X is given by

$$\rho_X(t)=\sup\Big\{\frac{1}{2}(\|x+ty\|+\|x-ty\|-1:\ \|x\|=\|y\|=1\Big\},\quad t>0.$$

X is said to be uniformly convex if $\delta_X(\varepsilon)>0$ for all $\varepsilon\in(0,2]$, and uniformly smooth if $\lim_{t\to 0}\rho_X(t)/t=0$.

The moduli of convexity and smoothness of a Banach space X and its dual X^* are related by the following duality formulae due to Lindenstrauss [10],

$$\rho_X(t)=\sup\Big\{\frac{t\varepsilon}{2}-\delta_{X^*}(\varepsilon):\ \varepsilon\in(0,2]\Big\},\qquad t>0\,,$$

and

$$\rho_{X^*}(t)\ge\sup\Big\{\frac{t\varepsilon}{2}-\delta_X(\varepsilon):\ \varepsilon\in(0,2]\Big\},\qquad t>0\,.$$

Note that the modulus of smoothness ρ_X of a Banach space is a continuous non-decreasing convex function satisfying the Δ_2-condition at zero, and $\rho_X(t)/t^2$ is equivalent to a decreasing function, that is, there exists $C > 0$ such that $\rho_X(t)/t^2 \le C \le \rho_X(s)/s^2$ for all $0 < s < t$ (see [10]). In particular this implies that if a Banach space X is uniformly smooth and $\varphi := \rho_X$, then the Orlicz sequence space ℓ_{ρ_X} is a Banach space and ℓ_{ρ_X} is continuously included in ℓ_2. Note also that if a Banach space X with $\dim X \ge 2$ is uniformly convex, then its modulus of convexity δ_X is equivalent (on $(0, 2]$) to a canonic Orlicz function $\widetilde{\delta}_X$ (see [11, pp. 59]).

Using the introduced notation we may state the following result essentially due to Figiel and Pisier [3]. It provides an extensive source of examples of spaces with type φ and cotype φ.

Lemma 3.6. *If X is uniformly smooth (resp., uniformly convex) Banach space with modulus of uniform smoothness (resp., modulus of uniform convexity), then X is of ρ_X-type (resp., X is cotype $\widetilde{\delta}_X$).*

This result can be applied to all classical Banach function lattices. For example, for estimates of the moduli of convexity of Lorentz sequence spaces and Orlicz sequence spaces, we refer to [1] and [12], respectively.

4 Maximal estimates of sublinear operators into Orlicz spaces

Let us finish the paper by showing a maximal estimate for operators on Orlicz spaces, which is obtained by adapting the arguments of the circle of ideas of our general version of Nikishin's theorem.

Theorem 4.1. *Let X be a quasi-Banach space of type ψ with $\psi \in \Phi$, and let $\varphi \in \Phi$ be such that $s_\varphi(t) \le \gamma\, t^r$ for some $\gamma > 0$ and $0 < r < \infty$, and for all $t > 0$. Suppose $T\colon X \to L_\varphi(\mu)$ is a continuous sublinear operator. Then there exists a constant C such that*

$$\big\| \sup_k |Tx_k| \big\|_{L_\varphi(\mu)} \le C \, \big\| \big(\|x_j\|_X\big) \big\|_{\ell_\psi}$$

for all finite sequences $(x_k)_k$ in X.

Proof. Fix a finite sequence $(x_k)_k$ in X. By the homogeneity of T, we may assume without loss of generality that $\|(\|x_k\|_X)_k\|_{\ell_\psi} \le 1$. We apply the following well known formula which holds for all $f \in L^0(\mu)$

$$\int_\Omega |f| \, d\mu = \int_0^\infty \mu(\{|f| \ge \lambda\}) \, d\lambda \, .$$

Using the estimate obtained in the proof of Proposition 3.3, which is based on the auxiliary functions g_t and g_t^k, we obtain by means of Fubini's theorem

$$\begin{aligned}
\int_\Omega \varphi\Big(\sup_k |Tx_k|\Big)\, d\mu &= \int_0^\infty \mu\Big(\Big\{\sup_k |Tx_k| \geq \varphi^{-1}(\lambda)\Big\}\Big)\, d\lambda \\
&\leq 2\int_0^\infty \bigg(\int_\Omega \big|\big\{t \in [0,1] :\, |Tg_t(\omega)| \geq \varphi^{-1}(\lambda)\big\}\big|\, d\mu\bigg)\, d\lambda \\
&= 2\int_0^\infty \bigg(\int_0^1 \mu\big(\big\{\omega \in \Omega :\, \varphi(|Tg_t(\omega)|) \geq \lambda\big\}\big)\, dt\bigg)\, d\lambda \\
&= 2\int_0^1 \bigg(\int_0^\infty \mu\big(\big\{\omega \in \Omega :\, \varphi(|Tg_t(\omega)|) \geq \lambda\big\}\big)\, d\lambda\bigg)\, dt \\
&= 2\int_0^1 \bigg(\int_\Omega \varphi\big(|Tg_t(\omega)|\big)\, d\mu\bigg)\, dt\,.
\end{aligned}$$

Now it follows from the Kahane-Khintchine inequality that there is a constant $C_r > 0$ such that

$$\begin{aligned}
\bigg(\int_0^1 \Big\|\sum_k r_k(t)x_k\Big\|_X^r\, dt\bigg)^{\frac{1}{r}} &\leq C_r\bigg(\int_0^1 \Big\|\sum_k r_k(t)x_k\Big\|_X^2\, dt\bigg)^{\frac{1}{2}} \\
&\leq C_r\, T_\psi(X)\, \big\|\big(\|x_k\|_X\big)\big\|_{\ell_\psi} \leq C_r\, T_\psi(X)\,.
\end{aligned}$$

Since $T\colon X \to L_\varphi(\mu)$ is continuous, $\|Tg_t\|_{L_\varphi(\mu)} \leq \|T\|\, \|g_t\|_X$ for all $t \in [0,1]$, and thus $\int_\Omega \varphi\big(|Tg_t(\omega)|/\|T\|\, \|g_t\|_X\big)\, d\mu \leq 1$. Combining all together these estimates with our hypothesis $\varphi(ut) \leq \gamma\, \varphi(u)t^r$ for all $u, t > 0$ gives

$$\int_\Omega \varphi\Big(\sup_k |Tx_k|\Big)\, d\mu \leq \gamma\, \|T\|^r \int_0^1 \|g_t\|_X^r\, dt \leq \gamma\, \|T\|^r\, T_\psi(X)^r\,,$$

and so $\|\sup_k |Tx_k|\|_{L_\varphi(\mu)} \leq C$ with $C = \gamma^{1/r}\, \|T\|\, T_\psi(X)$. This completes proof. $\square$

References

[1] Z. Altshuler, *Uniform convexity in Lorentz sequence spaces*, Israel J. Math. **20** (1975), 260–275.

[2] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge University Press, Cambridge 1995.

[3] T. Figiel and G. Pisier, *Séries alétoires dans les espaces uniformément convexe ou uniformément lisses*, C. R. Acad. Sci., Paris **279**, Serie A (1974), 611–614.

[4] J. Gustavsson, *Metrization of quasi-metric spaces*, Math. Scand. **35** (1974), 56–60.

[5] J. P. Kahane, *Some random series of functions*, Heath Mathematical Monographs, 1968.

[6] N. J. Kalton, *Linear operators on L_p for $0 < p < 1$*, Trans. Amer. Math. Soc. **259** (1980), no. 2, 319–355.

[7] N. J. Kalton, *Banach spaces embedding into L_0*, Israel J. Math. **52** (1985), no. 4, 305–319.

[8] N. J. Kalton, *Convexity, type and the three space problem*, Studia Math. **69** (1980/81), no. 3, 247–287.

[9] S. Kwapień, *Problem* 3, Studia Math. **38** (1969), 469.

[10] J. Lindenstrauss, *On the modulus of smoothness and divergent series in Banach spaces*, Mich. Math. J. **10** (1963), 241–252.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function spaces*, Springer-Verlag, 1979.

[12] R. P. Maleev and S. L. Troyanski, *On the moduli of convexity and smoothness in Orlicz spaces*, Studia Math. **54** (1975), 131–141.

[13] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L_p*, Astérisque, No. 11. Société Mathématique de France, Paris, 1974.

[14] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.

[15] E. M. Nikishin, *Resonance theorems and superlinear operators*, Uspehi Mat. Nauk **25** (1970), no. 6, 129–191 (in Russian).

[16] E. M. Nikishin, *A resonance theorem and series in eigenfunctions of the Laplace operator*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 795–813.

[17] G. Pisier, *Holomorphic semigroups and the geometry of Banach spaces*, Ann. of Math. **115** (1982), no. 2, 375–392.

[18] S. Rolewicz, *Metric linear spaces*, PWN, Warsaw, 1972.

[19] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge University Press, Cambridge, 1996.

Mieczysław Mastyło
Faculty of Mathematics and Computer Science
Adam Mickiewicz University, Poznań
Uniwersytetu Poznańskiego 4
61-614 Poznań, Poland
E-mail: mieczyslaw.mastylo@amu.edu.pl

Enrique A. Sánchez Pérez
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
Camino de Vera s/n, 46022 Valencia, Spain
E-mail: easancpe@mat.upv.es