Distributional chaos of C_0-semigroups of operators

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Summary

Distributional chaos was introduced by Schweizer and Smítal in [SS94] from the notion of Li-Yorke chaos in order to imply positive topological entropy for the mappings from the compact interval into itself. Distributional chaos for linear operators was considered for the first time in [Opr06] and firstly studied in the infinite-dimensional linear setting in [MGOP09].

The concept of distributional chaos for an operator (semigroup) consists on the existence of an uncountable subset and a positive real number δ such that for every pair of distinct elements of the uncountable set, both the upper density of the set of iterations (times) in which the difference of the images by the corresponding operator is greater than δ, and the upper density of the set of iterations (times) in which that difference is as small as we want, are equal to one.

This thesis is divided into six chapters. In the first one, we do a summary of the state of the art about chaotic dynamics for C_0-semigroups of linear operators.

In the second chapter, we show the equivalence between the distributional chaos of a C_0-semigroup and the distributional chaos of each one of its non-trivial operators. We also characterize the distributional chaos of a C_0-semigroup in terms of the existence of a distributionally irregular vector.

The notion of hypercyclicity for an operator (semigroup) consists on the existence of an element with dense orbit by the operator (semigroup). If, in addition, the set of periodic points is dense, we say that the operator (semigroup) is Devaney chaotic. One of the most useful tools to check whether an operator is hypercyclic is the Hypercyclicity Criterion, first stated by Kitai in 1982. In [BBMGP11], Bermúdez, Bonilla, Martínez-Giménez and Peris introduce the Criterion for Distributional Chaos (CDC) for operators. We state and prove a version of the CDC for semigroups.

In addition, in the semigroup setting, Desch, Schappacher and Webb studied in [DSW97] hypercyclicity and Devaney chaos for C_0-semigroups, giving a criterion for Devaney chaos based on the spectrum of the infinitesimal generator of the C_0-semigroup. In the third chapter, we establish a criterion for the existence of a dense distributionally irregular manifold (DDIM) in terms of the spectrum of the infinitesimal generator of the C_0-semigroup.

In Chapter 4, some sufficient conditions for distributional chaos for the translation C_0-semigroup on weighted L^p-spaces are given in terms of the admissible weight function. Moreover, we establish a complete analogy between the study of distributional chaos for the translation C_0-semigroup and for backward shift operators on weighted sequence spaces.

The fifth chapter is devoted to the study of the existence of C_0-semigroups for which every non-zero vector is a distributionally irregular vector. We also give an example of such C_0-semigroups that is not hypercyclic.

In Chapter 6, the DDIM criterion is applied to several examples of C_0-semigroups. Some of them are the solution semigroup of a partial differential equation, like the hyperbolic heat transfer equation or the von Foerster-Lasota equation, and others are the solution of an infinite system of ordinary differential equations used to modelize the dynamics of a population of cells under simultaneous proliferation and maturation.

Resum

El *caos distribucional* va ser introduït per Schweizer i Smítal en [SS94] a partir de la noció de caos de Li-Yorke amb la finalitat d'implicar l'entropia topològica positiva per a aplicacions de l'interval compacte en ell mateix. El caos distribucional per a operadors va ser considerat per primera vegada en [Opr06] i va ser analitzat en el context lineal de dimensió infinita en [MGOP09].

El concepte de caos distribucional per a un operador (semigrup) consisteix en l'existència d'un conjunt no numerable i un nombre real positiu δ tal que per a dos elements distints qualssevol del conjunt no numerable, tant la densitat superior del conjunt d'iteracions (temps) en les quals la diferència entre les òrbites dels elements esmentats és major que δ, com la densitat superior del conjunt d'iteracions (temps) en les quals dita diferència és tan menuda com es vulga, és igual a u.

Aquesta tesi està dividida en sis capítols. Al primer, fem un resum de l'estat actual de la teoria sobre la dinàmica caòtica per a C_0-semigrups d'operadors lineals.

Al segon capítol, mostrem l'equivalència entre el caos distribucional d'un C_0-semigrup i el caos distribucional de cadascun dels seus operadors no trivials. També caracteritzem el caos distribucional d'un C_0-semigrup en termes de l'existència d'un vector distribucionalment irregular.

La noció d'hiperciclicitat d'un operador (semigrup) consisteix en l'existència d'un element l'òrbita per l'operador (semigrup) del qual siga densa. Si, a més, el conjunt de punts periòdics és dens, direm que l'operador (semigrup) es caòtic

en el sentit de Devaney. Una de les eines més útils per comprovar si un operador és hipercíclic és el Criteri d'Hiperciclicitat, enunciat per primera vegada per Kitai en 1982. En [BBMGP11], Bermúdez, Bonilla, Martínez-Giménez i Peris presenten el Criteri per a Caos Distribucional (CDC en anglès) per a operadors. Enunciem i provem una versió del CDC per a C_0-semigrups.

En el context de C_0-semigrups, Desch, Schappacher i Webb també estudien en [DSW97] la hiperciclicitat i el caos de Devaney per a C_0-semigrups, donant un criteri per a caos de Devaney basat en l'espectre del generador infinitesimal del C_0-semigrup. Al tercer capítol, establim un criteri d'existència d'una varietat distribucionalment irregular densa (DDIM en les seues sigles en anglès) en termes de l'espectre del generador infinitesimal del C_0-semigrup.

Al Capítol 4, es donen algunes condicions suficients per a que el C_0-semigrup de translació en espais L^p ponderats siga distribucionalment caòtic en funció de la funció pes admissible. A més a més, establim una analogia completa entre l'estudi del caos distribucional per al C_0-semigrup de translació i per als operadors de desplaçament enrere o "backward shifts" en espais ponderats de successions.

El capítol cinquè està dedicat a l'estudi de l'existència de C_0-semigrups per als quals tot vector no nul és un vector distribucionalment irregular. També donem un exemple dels esmentats C_0-semigrups que a més no és hipercíclic.

Al Capítol 6, el criteri DDIM s'aplica a diversos exemples de C_0-semigrups. Alguns d'aquests són els semigrups de solució d'equacions en derivades parciales, com ara l'equació hiperbòlica de transferència de calor o l'equació de von Foerster-Lasota i altres són la solució d'un sistema infinit d'equacions diferencials ordinàries utilitzat per a modelitzar la dinàmica d'una població de cèl·lules baix proliferació i maduració simultànies.

Resumen

El *caos distribucional* fue introducido por Schweizer y Smítal en [SS94] a partir de la noción de caos de Li-Yorke con el fin de implicar la entropía topológica positiva para aplicaciones del intervalo compacto en sí mismo. El caos distribucional para operadores fue estudiado por primera vez en [Opr06] y fue analizado en el contexto lineal de dimensión infinita en [MGOP09].

El concepto de caos distribucional para un operador (semigrupo) consiste en la existencia de un conjunto no numerable y un numero real positivo δ tal que para dos elementos distintos cualesquiera del conjunto no numerable, tanto la densidad superior del conjunto de iteraciones (tiempos) en las cuales la diferencia entre las órbitas de dichos elementos es mayor que δ, como la densidad superior del conjunto de iteraciones (tiempos) en las cuales dicha diferencia es tan pequeña como se quiera, es igual a uno.

Esta tesis está dividida en seis capítulos. En el primero, hacemos un resumen del estado actual de la teoría de la dinámica caótica para C_0-semigrupos de operadores lineales.

En el segundo capítulo, mostramos la equivalencia entre el caos distribucional de un C_0-semigrupo y el caos distribucional de cada uno de sus operadores no triviales. También caracterizamos el caos distribucional de un C_0-semigrupo en términos de la existencia de un vector distribucionalmente irregular.

La noción de hiperciclicidad de un operador (semigrupo) consiste en la existencia de un elemento cuya órbita por el operador (semigrupo) sea densa. Si además el conjunto de puntos periódicos es denso, diremos que el operador

(semigrupo) es caótico en el sentido de Devaney. Una de las herramientas más útiles para comprobar si un operador es hipercíclico es el Criterio de Hiperciclicidad, enunciado por primerza vez por Kitai en 1982. En [BBMGP11], Bermúdez, Bonilla, Martínez-Giménez y Peris presentan el Criterio para Caos Distribucional (CDC en inglés) para operadores. Enunciamos y probamos una versión del CDC para C_0-semigrupos.

En el contexto de C_0-semigrupos, Desch, Schappacher y Webb también estudiaron en [DSW97] la hiperciclicidad y el caos de Devaney para C_0-semigrupos, dando un criterio para caos de Devaney basado en el espectro del generador infinitesimal del C_0-semigrupo. En el tercer capítulo, establecemos un criterio de existencia de una variedad distribucionalmente irregular densa (DDIM en sus siglas en inglés) en términos del espectro del generador infinitesimal del C_0-semigrupo.

En el Capítulo 4, se dan algunas condiciones suficientes para que el C_0-semigrupo de traslación en espacios L^p ponderados sea distribucionalmente caótico en función de la función peso admisible. Además, establecemos una analogía completa entre el estudio del caos distribucional para el C_0-semigrupo de traslación y para los operadores de desplazamiento hacia atrás o "backward shifts" en espacios ponderados de sucesiones.

El capítulo quinto está dedicado al estudio de la existencia de C_0-semigrupos para los cuales todo vector no nulo es un vector distribucionalmente irregular. También damos un ejemplo de uno de dichos C_0-semigrupos que además no es hipercíclico.

En el Capítulo 6, el criterio DDIM se aplica a varios ejemplos de C_0-semigrupos. Algunos de ellos siendo los semigrupos de solución de ecuaciones en derivadas parciales, como la ecuación hiperbólica de transferencia de calor o la ecuación de von Foerster-Lasota y otros son la solución de un sistema infinito de ecuaciones diferenciales ordinarias usado para modelizar la dinámica de una población de células bajo proliferación y maduración simultáneas.

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I love deadlines.
I like the whooshing sound they make as they fly by.

—Douglas Adams
(1952 - 2001)

In mathematics you don't understand things.
You just get used to them.

—Johann von Neumann
(1903 - 1957)

Today's scientists have substituted mathematics for experiments, and they wander off through equation after equation, and eventually build a structure which has no relation to reality.

—Nikola Tesla
(1857 - 1943)

The world always seems brighter when you've just made something that wasn't there before.

—Neil Gaiman
(1960 -)

海賊王におれはなる! *[Jo seré el rei dels pirates!]*

—Oda Eiichirō
(1975 -)

Contents

Chapter 1

Introduction

This thesis is devoted to the study of distributional chaos of C_0-semigroups defined on infinite-dimensional Banach spaces.

In this section we will present the notions of hypercyclicity, Devaney chaos, the weakly mixing property, Li-Yorke chaos, and distributional chaos for operators and C_0-semigroups as well as some sufficient criteria that yield some of these properties.
First of all we recall the definition of a C_0-semigroup and some of its elementary properties.

1.1 C_0-semigroups

During last years, several notions have been used in order to describe the dynamical behavior of linear operators on infinite-dimensional spaces, such as hypercyclicity, chaos in the sense of Devaney, chaos in the sense of Li-Yorke, subchaos, mixing and weakly mixing properties, and frequent hypercyclicity, among others. These notions have been extended to the frame of C_0-semigroups of linear and continuous operators as far as possible.

Our study will be focused on dynamical systems defined on a separable infinite-dimensional Banach space X. Within this frame, $L(X)$ denotes the set of linear and continuous operators from X to X, in the sequel we refer to them just as operators.

We recall that a one–parameter family $\mathcal{T} = \{T_t \colon X \to X \ ; \ t \in \Delta\}$, where $\Delta = \mathbb{R}_0^+$ or $\mathbb{R}$, is said to be a *strongly continuous semigroup of operators in* $L(X)$ (briefly, C_0*-semigroup of operators*) if the following conditions are satisfied.

(1) $T_0 = I$.

(2) $T_t T_s = T_{t+s}$, for all $s, t \in \Delta$.

(3) $\lim_{t\to s} T_t x = T_s x$, for all $x \in X$ and $s, t \in \Delta$.

The third condition can be expressed saying that the map

$$\begin{aligned} \Delta &\longrightarrow L_S(X) \\ t &\longrightarrow T_t \end{aligned}$$

is continuous for every $s \in \Delta$, where $L_S(X)$ denotes the space $L(X)$ endowed with the strong operator topology, that is, the topology of the pointwise convergence on the elements of X.

If the space $L(X)$ is endowed with the uniform convergence on the bounded sets of X, we will say that the semigroup is *uniformly continuous.* Whenever X is a normed space, this topology coincides with the topology of the convergence of operators in the operator norm.

If $\mathcal{T}$ is a C_0-semigroup in $L(X)$, then there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T_t x\| \leq M e^{\omega t}\|x\|$, for all $t \in \Delta$ and $x \in X$, and hence it is locally equicontinuous, that is,

$$\forall N > 0 \ \ \exists C > 0 \text{ such that } \|T_t x\| \leq C\|x\|, \quad \forall t \in [0, N], \quad \forall x \in X.$$

We refer to [EN00] and [EN06] as basic references on C_0-semigroups.

The notion of a C_0-semigroup can be viewed as the continuous time analog of the discrete time case of iterations of a single operator. One of the main interests on C_0-semigroups is that they describe the asymptotic behaviour of solutions to abstract linear Cauchy problems, which include linear partial differential equations and infinite systems of linear ordinary differential equations.

1.1.1 Linear dynamics for operators and C_0-semigroups

Two main research lines have been followed in the study of the linear dynamics of C_0-semigroups: On the one hand, to determine if the corresponding dynamical behavior of a C_0-semigroup was inherited by their non-trivial operators, since this will let us import known results from the operator case. On the other hand, to discover examples of C_0-semigroups that present a chaotic behavior. The first survey papers in linear dynamics are due to Grosse-Erdmann [GE99; GE03] and Bonet, Martínez-Giménez and Peris [BMGP03]. The recent monograph by Grosse-Erdmann and Peris [GEPM11] is a good reference for researchers interested in the study of linear dynamics. Moreover, it contains a chapter dedicated to analyse the dynamics of C_0-semigroups. See also [BM09] for further topics in the area.

In the sequel, let X be an infinite-dimensional separable Banach space, and let $\mathcal{T} = \{T_t \; ; \; t \in A \subseteq \Delta\}$ be either a C_0-semigroup or a sequence of operators (A countable and unbounded) in a semigroup. We will pay special attention when this sequence is given by the iterates of a fixed operator. In this case we simply say that this operator verifies the corresponding properties.

Godefroy and Shapiro introduced in [GS91] the notion of chaos in linear dynamics from the definition of Devaney chaos [Dev89]. We analyze the three ingredients in the definition of chaos in the sense of Devaney, transitivity, density of periodic points, and sensitive dependence on initial conditions, in this frame.

$\mathcal{T}$ is *transitive* if for any pair of non-empty sets $U, V \subset X$ there is some $t \in A$ such that $T_t(U) \cap V \neq \emptyset$. Transitivity is equivalent to the existence of some element $x \in X$ with dense orbit, i.e. $\overline{\{T_t x \ ; \ t \in A\}} = X$, see for instance [GS91; DSW97]. This phenomenon is usually known in Operator Theory as *hypercyclicity*, and such a vector x is said to be a *hypercyclic vector* for $\mathcal{T}$. We denote by $HC(\mathcal{T})$ the set of these vectors. There are plenty of hypercyclic vectors for a hypercyclic operator: by the Birkhoff transitivity theorem for operators [Bir22] or its analogous for C_0-semigroups, it can be seen that the set of these vectors is a dense G_δ-set.

We recall that a vector $x \in X$ is said to be a *periodic point* for $\mathcal{T}$ if there exists some $0 \neq t \in A$ such that $T_t x = x$. The set of periodic points for $\mathcal{T}$ is denoted by $Per(\mathcal{T})$. We point out that the structure of the set of periods for a C_0-semigroup has been partially analyzed in [BB09; MFSSW12].

Sensitive dependence on initial conditions can be directly obtained from hypercyclicity, see for instance [BBCDS92]. Therefore, hypercyclicity coincides here with the notion of chaos in the sense of Auslander and Yorke (existence of an element with dense orbit and sensitive dependence on initial conditions) [AY80]. Hence $\mathcal{T}$ is said to be *chaotic* if it is hypercyclic and the set of periodic points $Per(\mathcal{T})$ is dense in X.

Furthermore, hypercyclicity is a stronger notion than Li-Yorke chaos. We recall that $\mathcal{T}$ is said to be *Li-Yorke chaotic* if there exists an uncountable subset $\Gamma \subset X$, called the *scrambled* set, such that for every pair $x, y \in \Gamma$ of distinct points we have that

$$\liminf_{\substack{t\to\infty \\ t\in A}} \|T_t x - T_t y\| = 0 \quad \text{and} \quad \limsup_{\substack{t\to\infty \\ t\in A}} \|T_t x - T_t y\| > 0.$$

Every hypercyclic operator or C_0-semigroup is Li-Yorke chaotic, since we have that for a hypercyclic vector x we have that $\{p(T)x \ ; \ p \in \mathbb{K}[x], \ p \neq 0\}$ is a dense linear manifold of hypercyclic vectors, which is also a dense scrambled set.

$\mathcal{T}$ is called *topologically mixing* if, for any pair U, V of non-empty open subsets of X, there exists some $t_0 \in A$ such that $T_t(U) \cap V \neq \emptyset$, for all $t \geq t_0$. $\mathcal{T}$ is *topologically weakly mixing* if $\{T_t \oplus T_t : X \oplus X \to X \oplus X \ ; \ t \geq 0\}$ is transitive.

1.2 Criteria for hypercyclicity and Devaney Chaos

The following criterion for hypercyclicity of C_0-semigroups is inspired in the Hypercyclicity Criterion given by Kitai [Kit82], in particular in its generalization given by Bès and Peris [BP99].

Theorem 1.1. ***Hypercyclicity Criterion for C_0-semigroups [CP09].*** *Let $\mathcal{T}$ be a C_0-semigroup in $L(X)$. If there exist a sequence $(t_n)_n \subset \mathbb{R}^+$ with $\lim_{n\to\infty} t_n = \infty$, dense subsets $Y, Z \subset X$ and maps $S_{t_n} : Z \to X$, $n \in \mathbb{N}$ such that*

i. $\lim_{n\to\infty} T_{t_n} y = 0$ *for all* $y \in Y$,

ii. $\lim_{n\to\infty} S_{t_n} z = 0$ *for all* $z \in Z$,

iii. $\lim_{n\to\infty} T_{t_n} S_{t_n} z = z$ *for all* $z \in Z$,

then $\mathcal{T}$ is hypercyclic.

In fact, $\mathcal{T}$ verifies this criterion if and only if $\mathcal{T}$ is weakly mixing [BP99].

1.2.1 Dynamics of autonomous discretizations

In linear dynamics there exist some results which relate the dynamical properties of a C_0-semigroup with the corresponding properties of certain sequences of its operators. For this purpose we introduce the following notion.

Definition 1.2. A *discretization* of a C_0-semigroup $\{T_t\}_{t\geq 0}$ is a sequence of operators $(T_{t_n})_n$ in the semigroup, where $\lim_{n\to\infty} t_n = \infty$ If there is $t_0 \neq 0$ such that $t_n = nt_0$ for each $n \in \mathbb{N}$, then $(T_{t_n})_n = (T_{t_0}^n)_n$ is called an *autonomous discretization* of $\{T_t\}_{t\geq 0}$.

An easy observation yields that a C_0-semigroup is hypercyclic if and only if it admits a hypercyclic discretization. In fact all the autonomous discretizations are hypercyclic. $(T_{t_n})_n$.

Proposition 1.3 ([CP09; GEPM11])**.** *Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. The following are equivalent:*

i. $\{T_t\}_{t\geq 0}$ *is mixing.*

ii. Every discretization of $\{T_t\}_{t\geq 0}$ *is mixing.*

iii. Every discretization of $\{T_t\}_{t\geq 0}$ *is weakly mixing.*

iv. Every discretization of $\{T_t\}_{t\geq 0}$ *is transitive.*

v. There exists a mixing autonomous discretization of $\{T_t\}_{t\geq 0}$.

Theorem 1.4 ([CP09; GEPM11]). *Let* $\{T_t\}_{t\geq 0}$ *be a* C_0*-semigroup in* $L(X)$. *The following are equivalent:*

i. $\{T_t\}_{t\geq 0}$ *is weakly mixing.*

ii. All autonomous discretizations are weakly mixing.

In the flavour of the last results, the following ones also refer to the dynamics of the autonomous discretizations in a C_0-semigroup.

Theorem 1.5 (Conejero-Müller-Peris [CMP07]). *Let* $\mathcal{T} = \{T_t\}_{t\geq 0}$ *be a hypercyclic* C_0*-semigroup in* $L(X)$*, and let* $x \in HC(\mathcal{T})$*. Then* $x \in HC(T_{t_0})$ *for every* $t_0 > 0$.

This is equivalent to say that the restriction of the orbit of a hypercyclic vector x for $\mathcal{T}$ to the set $\{T_{kt_0}x \ ; \ k \in \mathbb{N}\}$ is still dense in X for any $t_0 > 0$. As a direct consequence, if a C_0-semigroup is weakly mixing, then all its non-trivial operators are weakly mixing. We recall that every Devaney chaotic C_0-semigroup is also weakly mixing, c.f. [BB09, Rem. 5].

However, we do not have an analogous result in the chaotic case.

Theorem 1.6 (Bayart-Bermúdez [BB09]). *There exists a* C_0*-semigroup* $\{T_t\}_{t\geq 0}$ *on a separable Hilbert space* H *and* $t_0 \neq t_1$ *such that* T_{t_0} *is chaotic and* T_{t_1} *is not chaotic.*

One of the most striking results in linear dynamics was the example of De la Rosa and Read of a hypercyclic operator which is not weakly mixing [RR09], see also [BM07]. It is still unknown whether each hypercyclic C_0-semigroup satisfies the hypercyclicity criterion, that is:

Question 1.7. *Are all the non-trivial operators in a hypercyclic C_0-semigroup weakly mixing?*

In fact, it will be enough with finding a single weakly mixing operator in the C_0-semigroup [CP09, Th. 2.4].

1.2.2 Spectral conditions

Let T be an operator on a complex Banach space X. The *spectrum* $\sigma(T)$ of T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} \ ; \ \lambda I - T \text{ is not invertible}\}.$$

The *point spectrum* $\sigma_p(T)$ is the set of eigenvalues of T.

The number

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

is called the *spectral radius* of T.

For the spectral radius we have that

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

The *infinitesimal generator* of a semigroup $\{T_t\}_{t \in \Delta}$ is the operator A given by the following limit:

$$Ax := \lim_{\substack{t \to 0 \\ t \in \Delta}} \frac{T_t x - x}{t},$$

defined wherever that limit exists. It is known that the infinitesimal generator of a C_0-semigroup $\{T_t\}_{t \geq 0}$ is a closed and densely defined operator in the general setting of the Fréchet spaces, see for instance [Yos80, Thm. IX.3.1]. The infinitesimal generator of a C_0-semigroup is important since it allows us to "reconstruct" the semigroup in several cases.

In a Banach space X, it can be seen that every uniformly continuous semigroup can be expressed as $\{T_t\}_{t \geq 0} = \left\{e^{tA}\right\}_{t \geq 0}$, where

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

for a certain operator $A \in L(X)$, see for instance [EN00, Thm. I.3.7].

Sometimes the Hypercyclicity Criterion is hard to be applied. In many situations we can have the infinitesimal generator of a C_0-semigroup but we do not have the explicit representation of its operators. This situation is quite common when we are dealing with the solution C_0-semigroups associated to certain partial differential equations. Desch, Schappacher, and Webb gave a criterion which permits us to state Devaney chaos (and hypercyclicity) of a C_0-semigroup in terms of the abundance of eigenvectors of the infinitesimal generator.

Theorem 1.8. ***Desch-Schappacher-Webb Criterion [DSW97].*** *Let X be a complex separable Banach space, and $\mathcal{T}$ be a C_0-semigroup on X with infinitesimal generator $(A, D(A))$. Assume that there exists an open connected subset $U \subset \mathbb{C}$ and a weakly holomorphic function $f : U \to X$, such that:*

i. $U \cap i\mathbb{R} \neq \emptyset$,

ii. $f(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U$

iii. for any $x^* \in X^*$, *if* $\langle f(\lambda), x^* \rangle = 0$ *for all* $\lambda \in U$, *then* $x^* = 0$.

Then the semigroup $\mathcal{T}$ is chaotic.

Remark 1.9. Improvements and comments regarding the Desch-Schappacher-Webb Criterion can be found in [CM08; CM10] and in [BM05], where the analyticity of f is dropped out if we replace the third condition by the following one: *for some* $\lambda_0 \in U$ $\dim(\ker(A-\lambda_0 I)) = 1$, $f(\lambda_0) \neq 0$, *and* $\bigcup_{n\geq 1} \ker(A-\lambda_0 I)^n$ *is dense in X.*

Kalmes proved that if a C_0-semigroup satisfies the Desch-Schappacher-Webb Criterion, then all their non-trivial operators are Devaney chaotic [Kal06]. As an example, this holds for Devaney chaotic translation C_0-semigroups on weighted spaces of continuous and integrable functions [dE01]. However, this chaotic behavior of a C_0-semigroup is not always inherited by its operators. Furthermore, there are examples of Devaney chaotic C_0-semigroups whose operators are never Devaney chaotic as we have noted in Theorem 1.6.

1.3 Operators

In this section we introduce the notions of *distributional chaos* and *distributionally irregular vector* for operators. In the paper [SS94], Schweizer and Smítal introduced the notion of distributional chaos (it was called strong chaos there). The main point was that, in the case of a self-map f on a compact interval, the existence of a distributionally chaotic pair implies that f has positive topological entropy. Actually, in [Li93] it is shown that the existence of distributionally chaotic pairs implies, within this framework, a very strong behaviour: distributional chaos, positive topological entropy, ω-chaos, and the existence of an infinite invariant subset on which f exhibits chaos in the sense of Devaney are equivalent properties for interval maps. This concept was generalized in [BSv05; Sv04]. We also refer to [Ov08; Opr09a; GKLOP09; Opr09b] for some recent papers dealing with distributional chaos, and to [MGOP09; BBMGP11] for distributional chaos in the linear infinite-dimensional setting, which is the object of this dissertation.

We suppose here that the metric space X has a finite diameter. For any pair (x, y) of points in X and any positive integer n, a distribution function $\Phi_{xy}^{(n)} :]0, +\infty[\to [0, 1]$ is defined by

$$\Phi_{xy}^{(n)}(t) = \frac{|\{0 \le i \le n-1 \ ; \ d(f^i(x), f^i(y)) < t\}|}{n}, \quad t > 0.$$

Then $\Phi_{xy}^{(n)}(t)$ is a non-decreasing function and $\Phi_{xy}^{(n)}(t) = 1$ for t greater than the diameter of X. Let

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t), \qquad \text{and} \qquad \Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t).$$

We say that Φ_{xy} is the *lower distribution function*, and Φ_{xy}^* the *upper distribution function* of x and y. Clearly, $\Phi_{xy}(t) \le \Phi_{xy}^*(t)$ for any $t > 0$. If $\Phi_{xy}(t) < \Phi_{xy}^*(t)$ for all t in an interval, we simply write $\Phi_{xy} < \Phi_{xy}^*$. We are interested in the case when there are an uncountable scrambled set $S \subset X$ and $\delta > 0$ such that, for any pair of different points $x, y \in S$, we have

(D_1C) $\Phi^*_{xy} \equiv 1$ and $\Phi_{xy}(\delta) = 0$.

(D_2C) $\Phi^*_{xy}(t) \equiv 1$ and $\Phi_{xy} < \Phi^*_{xy}$.

(D_3C) $\Phi^*_{xy} > \Phi_{xy}$.

Then f is said to exhibit *distributional chaos of type 1-3*, respectively and will be denoted by (D_1C), (D_2C) or (D_3C). Obviously, (D_1C) implies (D_2C) and (D_2C) implies (D_3C). However, neither of the converses is true (see for instance [BSv05; Sv04]).

From now on, to simplify, when we say distributional chaos we mean (D_1C). This notion can be expressed in terms of the abundance of pairs (x, y) whose orbits are as "close" as we want in certain times and far enough in other occasions. This concept of distributional chaos can be rephrased in terms of densities of sets of integers. The *upper density* $\overline{\text{dens}}(\mathcal{K})$ of a set $\mathcal{K} \subset \mathbb{N}$ is defined by

$$\overline{\text{dens}}(\mathcal{K}) := \limsup_{n\to\infty} \frac{|\mathcal{K} \cap \{1, \dots, n\}|}{n},$$

and the corresponding *lower density* $\underline{\text{dens}}(\mathcal{K})$ by

$$\underline{\text{dens}}(\mathcal{K}) := \liminf_{n\to\infty} \frac{|\mathcal{K} \cap \{1, \dots, n\}|}{n},$$

where $|.|$ denotes the cardinality of a set. Thus, taking into account that $\overline{\text{dens}}(\mathcal{K}) + \underline{\text{dens}}(\mathbb{N} \setminus \mathcal{K}) = 1$, $f : X \to X$ is distributionally chaotic if there is an uncountable scrambled set $S \subset X$ and $\delta > 0$ such that, for any pair of different points $x, y \in S$ and for every $\varepsilon > 0$, we have

$$\overline{\text{dens}}\{n \in \mathbb{N}\ ;\ d(f^n(x), f^n(y)) > \delta\} = 1, \text{ and}$$
$$\overline{\text{dens}}\{n \in \mathbb{N}\ ;\ d(f^n(x), f^n(y)) < \varepsilon\} = 1.$$

Such a pair (x, y) is called a distributionally chaotic pair for f.

Inspired by the concept of irregular vectors of Beauzamy [Bea88], the authors of [BBMGP11] defined a vector $x \in X$ to be *distributionally irregular* for T if there are increasing sequences of integers $J = (n_k)_k$ and $K = (m_k)_k$

such that $\overline{\text{dens}}(J) = \overline{\text{dens}}(K) = 1$, $\lim_k \|T^{n_k}x\| = 0$ and $\lim_k \|T^{m_k}x\| = \infty$, or, equivalently, by Lemma 2.6, for every $\delta > 0$

$$\overline{\text{dens}}\,(\{n \in \mathbb{N}\ ;\ \|T^n x\| > \delta\}) = 1 \qquad \text{and} \qquad \overline{\text{dens}}\,(\{n \in \mathbb{N}\ ;\ \|T^n x\| < \delta\}) = 1.$$

The set of all distributionally irregular vectors for T is denoted by $\mathcal{DI}(T)$.

A linear manifold $Y \subset$ X is a *distributionally irregular manifold* for T if every non-zero vector $y \in Y$ is a distributionally irregular vector for T, i.e., $Y \setminus \{0\} \subset \mathcal{DI}(T)$.

If x is a distributionally irregular vector, then $S = \text{span}\{x\}$ is a distributionally scrambled set for T (see e.g. [BBMGP11]), hence T is distributionally chaotic.

In [BBMP12] the authors introduced the following criterion:

Theorem 1.10 (Criterion for Distributional Chaos (CDC))**.** *Suppose that there exist sequences $(x_m)_m$, $(y_m)_m \subset X$ such that:*

a) there exists a subset $K \subset \mathbb{N}$ with $\overline{\text{dens}}(K) = 1$ such that $\lim\limits_{\substack{n\to\infty\\ n\in K}} T^n x_m = 0$ for all m;

b) $y_m \in \overline{\text{span}\{x_k \colon k \in \mathbb{N}\}}$, $\lim y_m = 0$ and there exist $\delta > 0$ and a sequence of positive integers $(N_m)_m$ increasing to ∞ with

$$|\{n \le N_m\ ;\ d(T^n y_m, 0) > \delta\}| \ge N_m \left(1 - \frac{1}{m}\right) \text{ for all } m \in \mathbb{N}.$$

The authors have proved, as a matter of fact, that for a $T \in L(X)$, to have a distributionally irregular vector, to satisfy the (CDC) and to be distributionally irregular are actually equivalent:

Theorem 1.11. ([BBMP12]) *Let $T : X \to X$ be an operator on a Fréchet space X. The following statements are equivalent.*

(i) *T admits a distributionally chaotic pair.*

(ii) *T has a distributionally irregular vector.*

(iii) *T is distributionally chaotic.*

(iv) *T satisfies the (CDC).*

Chapter 2

Dynamic behaviour of the operators of a C_0-semigroup

The aim of this chapter is to introduce the notions of distributional chaos and distributionally irregular vectors in the C_0-semigroup setting, in order to give a *Criterion for Distributional Chaos* for C_0-semigroups and to study the relation between these properties and the corresponding ones for the non-trivial operators of the C_0-semigroup.

2.1 Definitions

First we will extend the definition of upper and lower density to Lebesgue measurable subsets.

Definition 2.1. If A is a Lebesgue measurable subset of $\mathbb{R}_0^+$, then the *upper density* of A is defined as

$$\overline{\mathrm{Dens}}(A) := \limsup_{t\to\infty} \frac{\mu(A \cap [0,t])}{t},$$

and its *lower density* by

$$\underline{\mathrm{Dens}}(A) := \liminf_{t\to\infty} \frac{\mu(A \cap [0,t])}{t},$$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

Inspired in the notion of distributional chaos for operators we provide the definition for C_0-semigroups.

Definition 2.2. A C_0-semigroup of operators $\mathcal{T} = \{T_t\}_{t\geq 0}$ on X is said to be *distributionally chaotic* if there exist an uncountable subset $S \subset X$ and $\delta > 0$ such that, for each pair of distinct points $x, y \in S$ and for every $\varepsilon > 0$, we have

$$\begin{aligned} &\overline{\mathrm{Dens}}(\{s \geq 0\ ;\ \|T_s x - T_s y\| > \delta\}) = 1, \quad \text{and} \\ &\overline{\mathrm{Dens}}(\{s \geq 0\ ;\ \|T_s x - T_s y\| < \varepsilon\}) = 1. \end{aligned} \tag{2.1}$$

The set S is said to be a *distributionally δ–scrambled set* for $\mathcal{T}$ and the pair $\{x, y\}$ a *distributionally chaotic pair* for $\mathcal{T}$. If the scrambled set S is dense on X, then we say that $\mathcal{T}$ is *densely distributionally chaotic*; and if $S = X$, then we say that $\mathcal{T}$ is *completely distributionally chaotic.*

Inspired by the Criterion for Distributional Chaos *(CDC)* for operators introduced in [BBMP12], we give the corresponding concept for C_0-semigroups.

Definition 2.3. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$. $\mathcal{T}$ satisfies the *(CDC)* for C_0-semigroups if there exist sequences $(x_m)_m$, $(y_m)_m \subset X$ such that:

a) There exists a Lebesgue measurable set $A \subseteq \mathbb{R}^+$ with $\overline{\mathrm{Dens}}(A) = 1$ such that $\lim\limits_{\substack{s\to\infty \\ s\in A}} T_s x_m = 0$ for every m;

b) $(y_m)_m \subset \overline{\mathrm{span}\{x_k \colon k \in \mathbb{N}\}}$, $\lim\limits_{m\to\infty} y_m = 0$ and there exist $\delta > 0$ and an increasing sequence of positive real numbers $(\rho_m)_m$ tending to ∞ with

$$\mu(\{s \in [0, \rho_m] \colon \|T_s y_m\| > \delta\}) \geq \rho_m \left(1 - \frac{1}{m}\right) \text{ for every } m \in \mathbb{N}.$$

We can assume without loss of generality that $(\rho_m)_m$ is a sequence of positive integers.

As we mentioned in the Introduction, the notion of a distributionally irregular vector plays an essential role for distributional chaos in the iterations of a single operator. We will see that it is also the case for C_0-semigroups.

Definition 2.4. A vector $x \in X$ is said to be *distributionally irregular* for the C_0−semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ if the following holds: for every $\delta > 0$

$$\overline{\text{Dens}}\{s \geq 0 \ ; \ \|T_s x\| < \delta\} = 1, \text{ and} \tag{2.2}$$

$$\overline{\text{Dens}}\{s \geq 0 \ ; \ \|T_s x\| \geq \delta\} = 1. \tag{2.3}$$

The set of all distributionally irregular vectors for $\mathcal{T}$ is denoted by $\mathcal{DI}(\mathcal{T})$.

Remark 2.5. We will prove in Section 2.2 that our definition is equivalent to the natural extension of the definition of distributionally irregular vector for an operator.

A linear manifold $Y \subset$ X is said to be a *distributionally irregular manifold* for $\mathcal{T}$ if every non-zero vector $y \in Y$ is a distributionally irregular vector for $\mathcal{T}$, i.e., $Y \setminus \{0\} \subset \mathcal{DI}(\mathcal{T})$.

A C_0-semigroup $\mathcal{T}$ is *completely distributionally irregular* if every vector $x \in X \setminus \{0\}$ is distributionally irregular.

2.2 Distributionally chaotic dynamics of autonomous discretizations

Motivated by the results in the Introduction, we are interested in finding whether the distributional chaos of a C_0-semigroup implies distributional chaos for its non-trivial operators or not, and whether there exists a relation between satisfying the (CDC) and the existence of a distributionally irregular vector.

First we include a Lemma about upper densities for the sake of completeness.

Lemma 2.6. *Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. Then the following conditions are equivalent:*

(i) *there exists $A \subseteq \mathbb{N}$ with $\overline{\text{dens}}(A) = 1$ such that*

$$\lim_{n\to\infty, n\in A} x_n = x \quad \textit{(respectively,} \quad \lim_{n\to\infty, n\in A} d(x_n, x) = \infty),$$

(ii) $\overline{\text{dens}}\,(\{n \in \mathbb{N}\ ;\ d(x_n, x) < \delta\}) = 1$ *for every* $\delta > 0$
(respectively, $\overline{\text{dens}}\,(\{n \in \mathbb{N}\ ;\ d(x_n, x) > \delta\}) = 1$ *for every* $\delta > 0$*).*

Let $f : \mathbb{R}^+ \to X$, $x \in X$. *Then the following conditions are equivalent:*

(a) *there exists* $A \subseteq \mathbb{R}^+$ *with* $\overline{\text{Dens}}(A) = 1$ *such that*

$$\lim_{s\to\infty, s\in A} f(s) = x \text{ (respectively, } \lim_{s\to\infty, s\in A} d(f(s), x) = \infty\text{)},$$

(b) $\overline{\text{Dens}}\,(\{s \in \mathbb{R}^+\ ;\ d(f(s), x) < \delta\}) = 1$ *for every* $\delta > 0$
(respectively, $\overline{\text{Dens}}\,(\{s \in \mathbb{R}^+\ ;\ d(f(s), x) > \delta\}) = 1$ *for every* $\delta > 0$*).*

Proof. We will only show the discrete case. Suppose that (i) holds and let $\delta > 0$. By assumption, there exists $\nu \in \mathbb{N}$ such that for every $n \in A$ with $n > \nu$ we have $d(x_n, x) < \delta$. Then $A \cap [\nu, \infty[\subseteq \{n \in \mathbb{N}\ ;\ d(x_n, x) < \delta\}$ and, since $1 = \overline{\text{dens}}(A) = \overline{\text{dens}}(A \cap [M, \infty[)$, we get the assertion.

Conversely, for every $k \in \mathbb{N}$ let

$$A_k = \left\{ n \in \mathbb{N}\ ;\ d(x_n, x) < \frac{1}{k} \right\}.$$

By the assumption, $\overline{\text{dens}}(A_k) = 1$. We can construct inductively a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of natural numbers such that

$$|A_k \cap [m_{k-1}, m_k[| > m_k \left(1 - \frac{1}{k}\right), \quad \forall k \geq 2. \tag{2.4}$$

Let

$$A := \bigcup_{k\geq 2} (A_k \cap [m_{k-1}, m_k[)\,.$$

Observe that $\overline{\text{dens}}(A) = 1$, since, by (2.4),

$$|A \cap [1, m_k]| \geq |A_k \cap [m_{k-1}, m_k[| > m_k \left(1 - \frac{1}{k}\right), \quad \forall k \geq 2. \tag{2.5}$$

We can write A as a suitable strictly increasing sequence of positive integers $(n_j)_{j\in\mathbb{N}}$. We claim that $d(x_{n_j}, x)$ tends to 0 as $j \to \infty$. If we fix any $j \in \mathbb{N}$, there exists k_j such that $n_j \in A_{k_j} \cap [m_{k_j - 1}, m_{k_j}[$, consequently,

$$d(x_{n_j}, x) \leq \frac{1}{k_j}$$

Therefore, letting $j \to \infty$, we obtain the assertion. ⌟

We want to establish a relation between distributional chaos for a C_0-semigroup $\{T_t\}_{t\geq 0}$ and for their operators T_t. To this aim we also need a lemma that connects some upper densities of subsets of $\mathbb{R}$ with upper densities of suitable subsets of $\mathbb{N}$.

Lemma 2.7. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup of operators on a Banach space X, $t_0 > 0$, $x \in X$. Let $C_{t_0} := \sup_{0\leq t\leq t_0} \|T_t\|$. Then for every $\varepsilon, \delta > 0$ and for all $N > 0$:*

1. $\mu\left(\{t \in [0,N] \ ; \ \|T_t x\| > \delta\}\right) \leq t_0 \left|\left\{k \in \mathbb{N} \ ; \ k \leq \frac{N}{t_0} + 1, \ \|T_{t_0}^{k-1} x\| > \frac{\delta}{C_{t_0}}\right\}\right|$.
2. $t_0 |\{k \in \mathbb{N} \ ; \ k \leq N, \ \|T_{t_0}^k x\| > \delta\}| \leq \mu\left(\left\{t \in [0, N t_0] \ ; \ \|T_t x\| > \frac{\delta}{C_{t_0}}\right\}\right)$.
3. $\mu(\{t \in [0,N] \ ; \ \|T_t x\| < \varepsilon\}) \leq t_0 \left|\left\{k \in \mathbb{N} \ ; \ k \leq \frac{N}{t_0} + 1, \ \|T_{t_0}^k x\| < \varepsilon C_{t_0}\right\}\right|$.
4. $t_0 |\{k \in \mathbb{N} \ ; \ k \leq N \colon \|T_{t_0}^k x\| < \varepsilon\}| \leq \mu\left(\{t \in [0, (N+1)t_0] \ ; \ \|T_t x\| < \varepsilon C_{t_0}\}\right)$.

In consequence,

1'. $\overline{\mathrm{Dens}}(\{t \geq 0 \ ; \ \|T_t x\| > \delta\}) \leq \overline{\mathrm{dens}}\left(\left\{k \in \mathbb{N} \ ; \ \|T_{t_0}^k x\| > \frac{\delta}{C_{t_0}}\right\}\right)$.

2'. $\overline{\mathrm{dens}}(\{k \in \mathbb{N} \ ; \ \|T_{t_0}^k x\| > \delta\}) \leq \overline{\mathrm{Dens}}\left(\left\{t \geq 0 \ ; \ \|T_t x\| > \frac{\delta}{C_{t_0}}\right\}\right)$.

3'. $\overline{\mathrm{Dens}}(\{t \geq 0 \ ; \ \|T_t x\| < \varepsilon\}) \leq \overline{\mathrm{dens}}(\{k \in \mathbb{N} \ ; \ \|T_{t_0}^k x\| < \varepsilon C_{t_0}\})$.

4'. $\overline{\mathrm{dens}}(\{k \in \mathbb{N} \ ; \ \|T_{t_0}^k x\| < \varepsilon\}) \leq \overline{\mathrm{Dens}}(\{t \geq 0 \ ; \ \|T_t x\| < \varepsilon C_{t_0}\})$.

Proof. 1. Let $A = \{t \leq N \ ; \ \|T_t x\| > \delta\}$ and

$$K = \{k \in \mathbb{N} \ ; \ \exists \bar{t} \in A \cap [(k-1)t_0, k t_0[\}.$$

Then

$$K \subseteq \left\{k \in \mathbb{N} \ ; \ 1 \leq k \leq \frac{N}{t_0} + 1, \ \|T_{(k-1)t_0} x\| > \frac{\delta}{C_{t_0}}\right\}.$$

Indeed, if there exists $\bar{t} \in [(k-1)t_0, k t_0[$ such that $\bar{t} \leq N$ and $\|T_{\bar{t}} x\| > \delta$, then $1 \leq k \leq \frac{N}{t_0} + 1$ and

$$\delta < \|T_{\bar{t}} x\| = \|T_{\bar{t}-(k-1)t_0} T_{(k-1)t_0} x\| \leq C_{t_0} \|T_{t_0}^{k-1} x\|.$$

Therefore

$$\begin{aligned}\mu(A) &\leq \sum_{k\in K} \mu\left([(k-1)t_0, kt_0[\right) \leq t_0|K| \\ &\leq t_0 \left| \left\{ k \in \mathbb{N} \; ; \; k \leq \frac{N}{t_0} + 1, \; \|T_{t_0}^{k-1}x\| > \frac{\delta}{C_{t_0}} \right\} \right| .\end{aligned}$$

2. Let $K' = \{k \in \mathbb{N} \; ; \; k \leq N, \; \|T_{t_0}^k x\| > \delta\}$. Then, for every $t \in [(k-1)t_0, kt_0[$, we have that

$$\delta < \|T_{t_0}^k x\| = \|T_{kt_0 - t} T_t x\| \leq C_{t_0} \|T_t x\|.$$

Hence

$$\bigcup_{k\in K'} [(k-1)t_0, kt_0[\subseteq \left\{ t \in [0, Nt_0] \; ; \; \|T_t x\| > \frac{\delta}{C_{t_0}} \right\},$$

thus

$$t_0 |K'| \leq \mu\left(\left\{ t \in [0, Nt_0] \; ; \; \|T_t x\| > \frac{\delta}{C_{t_0}} \right\} \right).$$

3. Let $A = \{t \leq N \; ; \; \|T_t x\| < \varepsilon\}$ and $K = \{k \in \mathbb{N} \, ; \, \exists \bar{t} \in A \cap [(k-1)t_0, kt_0[\}$. Then

$$K \subseteq \left\{ k \in \mathbb{N} \; ; \; 1 \leq k \leq \frac{N}{t_0} + 1, \; \|T_{kt_0} x\| < \varepsilon C_{t_0} \right\}.$$

Certainly, if $k \in K$

$$\|T_{kt_0} x\| = \|T_{kt_0 - \bar{t}} - T_{\bar{t}x}\| \leq C_{t_0} \|T_{\bar{t}} x\| < \varepsilon C_{t_0}.$$

It follows that

$$\begin{aligned}\mu(A) &\leq \sum_{k\in K} \mu\left([(k-1)t_0, kt_0[\right) \leq t_0|K| \\ &\leq t_0 \left| \left\{ k \in \mathbb{N} \; ; \; k \leq \frac{N}{t_0} + 1, \; \|T_{t_0}^k x\| < \varepsilon C_{t_0} \right\} \right| .\end{aligned}$$

4. Let $K' = \{k \in \mathbb{N} \; ; \; k \leq N, \; \|T_{t_0}^k x\| < \varepsilon\}$. Then, for every $s \in [kt_0, (k+1)t_0[$ with $k \in K$, we obtain that

$$\|T_s x\| = \|T_{s-kt_0} T_{kt_0} x\| \leq C_{t_0} \|T_{kt_0} x\| < \varepsilon C_{t_0}.$$

Hence

$$\bigcup_{k\in K'}[kt_0,(k+1)t_0[\subseteq \{t\in[0,(N+1)t_0]\ ;\ \|T_tx\|<\varepsilon C_{t_0}\},$$

therefore

$$t_0|K'|\le \mu\left(\{t\in[0,(N+1)t_0]\ ;\ \|T_tx\|<\varepsilon C_{t_0}\}\right).$$

1'. By 1., we have that

$$\begin{aligned}\overline{\mathrm{Dens}}(\{t\ge 0\ ;\ \|T_tx\|>\delta\}) &= \limsup_{N\to\infty}\frac{\mu\left(\{t\in[0,N]\ ;\ \|T_tx\|>\delta\}\right)}{N}\le\\ &\le \limsup_{N\to\infty}\frac{t_0}{N}\left|\left\{k\in\mathbb{N}\ ;\ k\le\frac{N}{t_0}+1,\ \|T_{t_0}^{k-1}x\|>\frac{\delta}{C_{t_0}}\right\}\right|=\\ &= \limsup_{b\to\infty}\frac{\left|\left\{k\in\mathbb{N}\ ;\ k-1\le b,\ \|T_{t_0}^{k-1}x\|>\frac{\delta}{C_{t_0}}\right\}\right|}{b}\\ &= \overline{\mathrm{dens}}\left(\left\{k\in\mathbb{N}\ ;\ \|T_{t_0}^{k}x\|>\frac{\delta}{C_{t_0}}\right\}\right).\end{aligned}$$

2'. By 2 it follows that

$$\begin{aligned}\overline{\mathrm{dens}}(\{k\in\mathbb{N}\ ;\ \|T_{t_0}^kx\|>\delta\}) &= \limsup_{N\to\infty}\frac{|k\in\mathbb{N}\ ;\ k\le N,\ \|T_{t_0}^kx\|>\delta|}{N}\\ &= \limsup_{N\to\infty}\frac{t_0|k\in\mathbb{N}\ ;\ k\le N,\ \|T_{kt_0}x\|>\delta|}{t_0N}\\ &\le \limsup_{N\to\infty}\frac{\mu\left(\left\{t\in[0,Nt_0]\ ;\ \|T_tx\|>\frac{\delta}{C_{t_0}}\right\}\right)}{Nt_0}\\ &\le \limsup_{b\to\infty}\frac{\mu\left(\left\{t\in[0,b]\ ;\ \|T_tx\|>\frac{\delta}{C_{t_0}}\right\}\right)}{b}\\ &= \overline{\mathrm{Dens}}\left(\left\{t\ge 0\ ;\ \|T_tx\|>\frac{\delta}{C_{t_0}}\right\}\right).\end{aligned}$$

The other inequalities follow analogously. ⌟

The following lemma will be used several times.

Lemma 2.8. *Let $A\subseteq\mathbb{R}^+$ be a measurable subset with $\overline{\mathrm{Dens}}(A)=1$, $\alpha\in]0,1[$, and let $\mathcal{K}_\alpha=\{k\in\mathbb{N}\ ;\ \mu(A\cap[k,k+1])>\alpha\}$. Then $\overline{\mathrm{dens}}(\mathcal{K}_\alpha)=1$.*

Proof. If we suppose that $\overline{\mathrm{dens}}(\mathcal{K}) < c < 1$, then $\underline{\mathrm{dens}}(\mathbb{N}\smallsetminus\mathcal{K}) \geq 1-c$ and hence there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|(\mathbb{N}\smallsetminus\mathcal{K})\cap\{1,\dots,n\}| \geq n(1-c)$. For each $j \in \mathbb{N}\smallsetminus\mathcal{K}$ there exists a measurable set $B_j \subseteq [j, j+1]$ such that $\mu(B_j) \geq 1-\alpha$ and $A \cap B_j = \varnothing$. Define $\mathcal{B} = \bigcup_{j\in\mathbb{N}\smallsetminus\mathcal{K}} B_j \subset \mathbb{R}^+ \smallsetminus A$. Then for each $n \geq n_0$

$$\frac{\mu(\mathcal{B}\cap[1,n+1])}{n} \geq (1-\alpha)\frac{|(\mathbb{N}\smallsetminus\mathcal{K})\cap\{1,\dots,n\}|}{n} > (1-\alpha)(1-c) > 0.$$

Thus $\underline{\mathrm{Dens}}(\mathcal{B}) > 0$, that gives

$$\overline{\mathrm{Dens}}(A) = 1 - \underline{\mathrm{Dens}}(\mathbb{R}^+ \smallsetminus A) \leq 1 - \underline{\mathrm{Dens}}(\mathcal{B}) < 1,$$

which is a contradiction. Consequently, $\overline{\mathrm{dens}}(\mathcal{K}) = 1$. ⌟

Now we are ready to offer the interplay between the continuous and the discrete cases concerning distributional chaos.

Theorem 2.9. *Let $\mathcal{T} := \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$. Then the following properties are equivalent.*

(i) *$\mathcal{T}$ is distributionally chaotic.*

(ii) *T_t is distributionally chaotic for all $t > 0$.*

(iii) *There exists $t_0 > 0$ such that T_{t_0} is distributionally chaotic.*

Proof. (i) implies (ii). Let $S \subset X$ be a distributionally δ–scrambled set for $\mathcal{T}$. By definition we know that for every two distinct points $f, g \in S$ we have that

$$\overline{\mathrm{Dens}}(\{s \geq 0 \ ; \ \|T_s(f-g)\| > \delta\}) = 1.$$

Then for every $t_0 > 0$, by Lemma 2.7

$$\overline{\mathrm{dens}}\left(\left\{k \in \mathbb{N} \ ; \ \|T_{t_0}^k(f-g)\| > \frac{\delta}{C_{t_0}}\right\}\right) = 1.$$

Let $\varepsilon > 0$. By assumption

$$\overline{\mathrm{Dens}}\left(\left\{s \geq 0 \ ; \ \|T_s(f-g)\| < \frac{\varepsilon}{C_{t_0}}\right\}\right) = 1,$$

hence

$$\overline{\text{dens}}\left(\left\{k \in \mathbb{N}\ ;\ \|T_{t_0}^k(f-g)\| < \varepsilon\right\}\right) = 1,$$

by Lemma 2.7. Thus S is a δ'-scrambled set for T_{t_0}, where $\delta' = \delta/C_{t_0}$.

(ii) implies (iii) is trivial.

(iii) implies (i) is analogous to the first implication by Lemma 2.7. ⌟

Remark 2.10. Observe that we have indeed proved that the C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ and each operator T_t share the scrambled set, and hence the distributionally chaotic pairs. In particular, $\mathcal{T}$ is dense distributionally chaotic if and only if every (some) operator T_t is dense distributionally chaotic.
This result can be compared with Theorems 1.5 and 1.6. In this matter, distributional chaos behaves more similarly to hypercyclicity than to Devaney chaos.

The next corollary will be useful, specially in Chapter 6, as it will simplify some proofs.

Corollary 2.11. *Two C_0-semigroups $\mathcal{T} := \{T_t\}_{t\geq 0} \subset L(X)$, $\mathcal{S} := \{S_t\}_{t\geq 0} \subset L(Y)$ are conjugate if there exists a homeomorphism $\phi : X \longrightarrow Y$ such that $\phi \circ T_t = S_t \circ \phi$ for every $t \geq 0$. If ϕ is uniformly continuous, $\mathcal{T}$ is distributional chaotic if and only if $\mathcal{S}$ is distributionally chaotic.*

Proof. If $\mathcal{T}$ is distributionally chaotic, by Theorem 2.9, every non-trivial operator T_t is distributionally chaotic. Since distributional chaos for operators is preserved under uniform conjugacy, as it is shown in [MGOP09], then every operator S_t is distributionally chaotic and by Theorem 2.9 $\mathcal{S}$ is distributionally chaotic. ⌟

By a quasi-conjugacy of a linear operator $S : Y \longrightarrow Y$ to a linear operator $T : X \longrightarrow X$ we mean the existence of a continuous map $\phi : X \to Y$ with dense range such that $S \circ \phi = \phi \circ T$.

Proposition 2.12. *[BMGP03; GEPM11] Hypercyclicity, mixing, and weakly mixing property are preserved under quasi-conjugacy.*

The following result is also a connection between continuous and discrete cases with respect to the Criterion for Distributional Chaos (CDC).

Proposition 2.13. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$. Then the following properties are equivalent.*

(i) *$\mathcal{T}$ satisfies the (CDC) for semigroups.*

(ii) *T_t satisfies the (CDC) for operators for all $t > 0$.*

(iii) *There exists $t_0 > 0$ such that T_{t_0} satisfies the (CDC) for operators.*

Proof. (i) implies (ii): Let $A \subseteq \mathbb{R}^+$ be according to the (CDC) property for semigroups. For the sake of simplicity, let $t = 1$ and $C = \sup_{0\leq t\leq 1} \|T_t\|$. Let

$$K = \{n \in \mathbb{N}\ ;\ \exists s \in A \cap [n-1, n[\}.$$

Then $\overline{\text{dens}}(K) = 1$. Then, since for every $n \in K$ there exist $s \in A$ with

$$\|T_1^n x_m\| \leq \|T_{n-s+s} x_m\| = \|T_{n-s} T_s x_m\| \leq C\|T_s x_m\|,$$

it is clear that $\lim\limits_{\substack{n\to\infty \\ n\in K}} T_1^n x_m = 0$ for every $m \in \mathbb{N}$. Also, Lemma 2.7 easily implies that, for $N_m := \rho_m$ with $m \in \mathbb{N}$,

$$\left|\left\{1 < n \leq N_m\ ;\ \|T_1^n y_m\| > \frac{\delta}{C}\right\}\right| \geq N_m(1 - \frac{1}{m}),$$

for each $m \in \mathbb{N}$, and T_1 satisfies the (CDC).

(ii) implies (iii): Trivial.

(iii) implies (i): Assume that T_1 satisfies condition (CDC) for operators and let $(x_m)_m$ and $(y_m)_m$ be sequences in X according to the definition of (CDC). Let $K \subseteq \mathbb{N}$, with $\overline{\text{dens}}(K) = 1$ and $\lim_{n\to\infty, n\in K} T_n x_m = 0$. Let us define

$$A := \bigcup_{n\in K} [n, n+1[\subseteq \mathbb{R}^+.$$

Clearly $\overline{\text{Dens}}(A) = 1$.

Again, since for every $t \in A$ there exists $n \in K$ with $t - n < 1$, then

$$\|T_t x_m\| = \|T_{t-n} T_n x_m\| \leq C\|T_1^n x_m\|;$$

which gives $\lim\limits_{\substack{t\to\infty \\ t\in A}} \|T_t x_m\| = 0$, for all $m \in \mathbb{N}$, and by Lemma 2.7

$$\mu\left(\left\{t \leq \rho_m\ ;\ \|T_t y_m\| > \frac{\delta}{C}\right\}\right) \geq \rho_m\left(1 - \frac{1}{m}\right)$$

for $\rho_m := N_m$, $m \in \mathbb{N}$, which concludes the result. ⌟

Proposition 2.14. *If the C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ admits a distributionally irregular vector, then $\mathcal{T}$ is distributionally chaotic.*

Proof. It follows immediately from the definition of irregular vector that $S := \text{span}\{x\}$ is a distributionally $\delta-$scrambled set for every δ. Indeed, let $x \in \mathcal{DI}(\mathcal{T})$. Then properties (2.2) and (2.3) hold for every $\delta > 0$. Set $S := \text{span}\{x\}$. If $y, z \in S$ with $y \neq z$, then $y - z = \alpha x$ with $\alpha \neq 0$. So, $\|T_s y - T_s z\| = |\alpha| \|T_s x\|$ for all $s \geq 0$. By (2.2) and (2.3) we obtain, for each $\delta > 0$ and $t > 0$, that

$$\frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s y - T_s z\| < \delta\}\right)}{t} = \frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s x\| < \delta|\alpha|^{-1}\}\right)}{t} \quad \text{and}$$
$$\frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s y - T_s z\| \geq \delta\}\right)}{t} = \frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s x\| \geq \delta|\alpha|^{-1}\}\right)}{t}.$$

Letting $t \to +\infty$, it follows that

$$\limsup_{t\to+\infty} \frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s y - T_s z\| < \delta\}\right)}{t} = 1 \quad \text{and}$$
$$\limsup_{t\to+\infty} \frac{\mu\left(\{s \in [0,t] \; ; \; \|T_s y - T_s z\| \geq \delta\}\right)}{t} = 1.$$

Hence, S is a distributionally δ-scrambled set for $\mathcal{T}$ for every $\delta > 0$.

⌟

Actually, the existence of a distributional irregular vector and distributional chaos for a semigroup are equivalent, as we will prove in a while. We first prove, in analogy to Theorem 2.9, that there is an equivalence between the continuous and the discrete case concerning the distributionally irregular vectors, which should also be compared with [CMP07] about hypercyclic vectors.

Proposition 2.15. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup of operators on X and let x be a vector on X. Then the following properties are equivalent:*

(i) *x is a distributionally irregular vector for $\mathcal{T}$.*

(ii) *there exist $A, B \subset \mathbb{R}^+$ with $\overline{\text{Dens}}(A) = 1 = \overline{\text{Dens}}(B)$ such that:*

$$\lim_{\substack{s\to+\infty \\ s\in A}} \|T_s x\| = 0 \quad \text{and} \quad \lim_{\substack{s\to+\infty \\ s\in B}} \|T_s x\| = \infty.$$

(iii) *x is a distributionally irregular vector for T_t for all $t > 0$.*

(iv) *There exists $t_0 > 0$ such that x is a distributionally irregular vector for T_{t_0}.*

The Proposition can be proved applying Lemma 2.6 for the equivalence of (i) and (ii) and using the proof of Lemma 2.7 to prove (ii) implies (iii) and (iv) implies (ii). Alternatively, we provide a constructive proof without reducing it.

Proof. (i) implies (ii). Assume that $x \in X$ is a distributionally irregular vector for $\mathcal{T}$. Let $(\varepsilon_k)_k \subset]0,1]$ be a sequence decreasing to 0 (i.e., $\varepsilon_k \to 0$). Define

$$A_{\varepsilon_k} := \{s \in \mathbb{R}^+ \ ; \ \|T_s x\| < \varepsilon_k\}.$$

Then $A_{\varepsilon_{k+1}} \subseteq A_{\varepsilon_k}$ and $\overline{\mathrm{dens}}(A_{\varepsilon_k}) = 1$ for all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$ there exists a sequence $(m_{l,k})_l$ such that

$$\frac{\mu(A_{\varepsilon_k} \cap [0, m_{l,k}])}{m_{l,k}} > 1 - \frac{1}{l}, \quad l \in \mathbb{N}. \tag{2.6}$$

Taking subsequences if necessary, we have that $m_{2,2} > 0$ and $m_{k,k} < m_{k+1,k+1}$ for all $k \in \mathbb{N}$. So, the sequence $(t_k)_k$ defined by

$$t_k := \begin{cases} 0 & \text{if } k = 1, \\ m_{k,k} & \text{if } k > 1 \end{cases}$$

is strictly increasing.
Next, set

$$A := \bigcup_{k \geq 1} \left(A_{\varepsilon_k} \cap [t_k, t_{k+1}[\right).$$

Observe that $\overline{\mathrm{Dens}}(A) = 1$, because by (2.6) we have, for all $j > 1$, that

$$\frac{\mu(A \cap [0, t_j])}{t_j} \geq \frac{\mu(A_{\varepsilon_j} \cap [0, t_j])}{t_j} > 1 - \frac{1}{j}.$$

For every $s \in A$, there exists k such that $s \in [t_k, t_{k+1}[$ and $\|T_s x\| < \varepsilon_k$. If we make s tend to infinity, then k tends to infinity and $\|T_s x\|$ to zero.

Now, let $(\delta_k)_k$ be a sequence increasing to ∞. Define $B_{\delta_k} := \{s \in \mathbb{R}^+ \colon \|T_s x\| > \delta_k\}$. Then $B_{\delta_{k+1}} \subset B_{\delta_k}$ and $\overline{\text{Dens}}(B_{\delta_k}) = 1$ for all $k \in \mathbb{N}$. Moreover, proceeding as before one shows that there exists a strictly increasing sequence $(r_k)_k$ of positive integers such that the set

$$B := \bigcup_{k\geq 1} (B_{\delta_k} \cap [r_k, r_{k+1}[) .$$

has $\overline{\text{Dens}}(B) = 1$ and that for every $s \in B$, $\|T_s x\| \to \infty$.

Therefore, we have obtained that

$$\lim_{\substack{s\to+\infty \\ s\in A}} \|T_s x\| = 0 \quad \text{ and } \quad \lim_{\substack{s\to+\infty \\ s\in B}} \|T_s x\| = \infty.$$

(ii) implies (i). For each $\delta > 0$ there exists $r \in \mathbb{R}^+$ such that for every $s \in A$ with $s > r$, we have that $\|T_s x\| < \delta$. It follows that

$$\overline{\text{Dens}}(\{s \in \mathbb{R}^+ \ ; \ \|T_s x\| < \delta\}) = 1.$$

Analogously we obtain that

$$\overline{\text{Dens}}(\{s \in \mathbb{R}^+ \ ; \ \|T_s x\| > \delta\}) = 1,$$

and hence x is a distributionally irregular vector for $\mathcal{T}$.

(ii) implies (iii). The proof is given only for $t = 1$ since the other cases are similar.
Define $J = \{j \in \mathbb{N} \ ; \ \exists s \in A \text{ s.t. } s \in [j-1, j[\}$ and $K = \{k \in \mathbb{N} \ ; \ \exists s \in B \text{ s.t. } s \in [k, k+1[\}$, since

$$\mu(A) \leq \mu\left(\bigcup_{j\in J} [j-1, j[\right) = |J|,$$

and there exists $(n_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \frac{\mu(A \cap [0, n_l])}{n_l} = 1, \text{ then } \lim_{l\to\infty} \frac{|J \cap [0, n_l]|}{n_l} = 1,$$

so $\overline{\text{dens}}(J) = 1$. In the same manner, we can see that $\overline{\text{dens}}(K) = 1$. For every $j \in J$ we have that $\|T_1^j x\| = \|T_{j-s} T_s x\| \leq C\|T_s x\|$ where $C = \sup\limits_{0\leq t\leq 1} \|T_t\|$, which is finite since $\mathcal{T}$ is locally equicontinuous and it's greater than 0 by the definition of semigroup. Therefore

$$\lim_{\substack{j\to+\infty \\ j\in J}} \|T_1^j x\| \leq \lim_{\substack{s\to+\infty \\ s\in A}} C\|T_s x\| = 0.$$

For every $k \in K$ we have that $\|T_s x\| = \|T_{s-k}T_k x\| \leq C\|T_1^k x\|$ and hence

$$\lim_{\substack{k\to+\infty \\ k\in K}} \|T_1^k x\| \geq \lim_{\substack{s\to+\infty \\ s\in B}} C^{-1}\|T_s x\| = \infty.$$

Since $J, K \subseteq \mathbb{N}$, we can write $J = (n_k)_k$ and $K = (m_k)_k$ with $(n_k)_k, (m_k)_k$ two suitable strictly increasing sequences of positive integers with upper density equal to 1 , with

$$\lim_{k\to+\infty} \|T_1^{n_k} x\| = 0 \quad \text{and} \quad \lim_{k\to+\infty} \|T_1^{m_k} x\| = \infty.$$

(iii) implies (iv). Trivial.

(iv) implies (ii). For the sake of simplicity we suppose that $t_0 = 1$. Let $x \in X$ be a distributionally irregular vector for T_1. Then there exist two sequences $J = (n_k)_k$, $K = (m_k)_k$ with $\overline{\text{dens}}(J) = \overline{\text{dens}}(K) = 1$ such that $\|T_1^{n_k} x\| \to 0$ and $\|T_1^{m_k} x\| \to +\infty$ as $k \to +\infty$.
Define the sets $A = \{s \in \mathbb{R}^+ \ ; \ \exists j \in J \colon s \in [j, j+1[\}$ and $B = \{s \in \mathbb{R}^+ \ ; \ \exists k \in K \colon s \in [k-1, k[\}$. Since

$$\mu(A) = \mu\left(\bigcup_{j\in J} [j, j+1[\right) = |J|,$$

and there exists $(n_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \frac{|J \cap [0, n_l]|}{n_l} = 1, \text{ then } \lim_{l\to\infty} \frac{\mu(A \cap [0, n_l + 1])}{n_l + 1} = 1$$

we obtain that A has upper density 1. A similar argument gives us that $\overline{\text{Dens}}(B) = 1$. For every $s \in A$ we have that $\|T_s x\| = \|T_{s-j}T_j x\| \leq C\|T_1^j x\|$ and therefore

$$\lim_{\substack{s\to+\infty \\ s\in A}} \|T_s x\| \leq \lim_{\substack{j\to+\infty \\ j\in J}} C\|T_1^j x\| = 0.$$

And for every $s \in B$ we have that $\|T_1^k x\| = \|T_{k-s}T_s x\| \leq C\|T_s x\|$ and hence

$$\lim_{\substack{s\to+\infty \\ s\in B}} \|T_s x\| \geq \lim_{\substack{k\to+\infty \\ k\in K}} C^{-1}\|T_1^k x\| = \infty,$$

which concludes the proof. ⌟

Remark 2.16. Note that we have thereby proved that the following are equivalent.

(i) $\mathcal{T}$ has a (dense) distributionally irregular manifold.

(ii) T_t has a (dense) distributionally irregular manifold for all $t > 0$.

(iii) There exists $t_0 > 0$ such that T_{t_0} has a (dense) distributionally irregular manifold.

We can now state and prove the analogue of Theorem 1.11 for C_0-semigroups.

Theorem 2.17. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$. Then the following statements are equivalent.*

(i) *$\mathcal{T}$ has a distributionally irregular vector.*

(ii) *$\mathcal{T}$ is distributionally chaotic.*

(iii) *$\mathcal{T}$ admits a distributionally chaotic pair.*

(iv) *$\mathcal{T}$ satisfies the (CDC) for semigroups.*

Proof. (i) implies (ii). Follows by Proposition 2.14.

(ii) implies (iii) is trivial by the definition of distributional chaos.

(iii) implies (iv). Assume that $\mathcal{T}$ admits a distributionally chaotic pair. Then, by Remark 2.10 there exists $t_0 > 0$ such that T_{t_0} admits a distributionally chaotic pair. By Theorem 1.11 this means that T_{t_0} satisfies the (CDC) for operators and hence, by Proposition 2.13, $\mathcal{T}$ satisfies the (CDC) for semigroups.

(iv) implies (i). Suppose that $\mathcal{T}$ satisfies the (CDC) for semigroups. Then, by Theorem 2.13, there exists $t_0 > 0$ such that T_{t_0} satisfies the (CDC) for operators. By Theorem 1.11 this means that T_{t_0} has a distributionally irregular vector and hence, by Proposition 2.15, $\mathcal{T}$ has also a distributionally irregular vector. $\lrcorner$

To summarize, the diagram in Figure 2.1 provides an overview of the results presented in this chapter.

For the sake of completeness, we will provide here a proof without using Theorem 1.11, but inspired by the proof from [BBMP12]. First we will need the following propositions about the abundance of vectors with distributionally unbounded orbits.

Definition 2.18. Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup, and take $x \in X$. We say that x has a *distributionally unbounded orbit* for $\mathcal{T}$ if there exists a subset $B \subset \mathbb{R}_0^+$ with $\overline{\mathrm{Dens}}(B) = 1$ such that $\lim\limits_{s\in B} \|T_s x\| = \infty$. The orbit of x is said to be *distributionally unbounded below* or away from 0 if there exists a subset $A \subset \mathbb{R}_0^+$ with $\overline{\mathrm{Dens}}(A) = 1$ such that $\lim\limits_{s\in A} \|T_s x\| = 0$.

This definition is analogous to the definition of a vector with distributionally unbounded (respectively, unbounded below) orbit for an operator from [BBMP12], replacing A and B for subsets of $\mathbb{N}$ with upper density 1.

Remark 2.19. Note that, as a matter of fact, in Proposition 2.15, we have proved that if x has a distributionally unbounded (respectively, unbounded below) orbit for $\mathcal{T}$, then x has a distributionally unbounded (resp., unbounded below) orbit for each operator T_t with $t > 0$ orbit, and if x has a distributionally unbounded (respectively, unbounded below) orbit for an operator of $\mathcal{T}$, then x has a distributionally unbounded (respectively, unbounded below) orbit for $\mathcal{T}$.

For every $t > 0$.

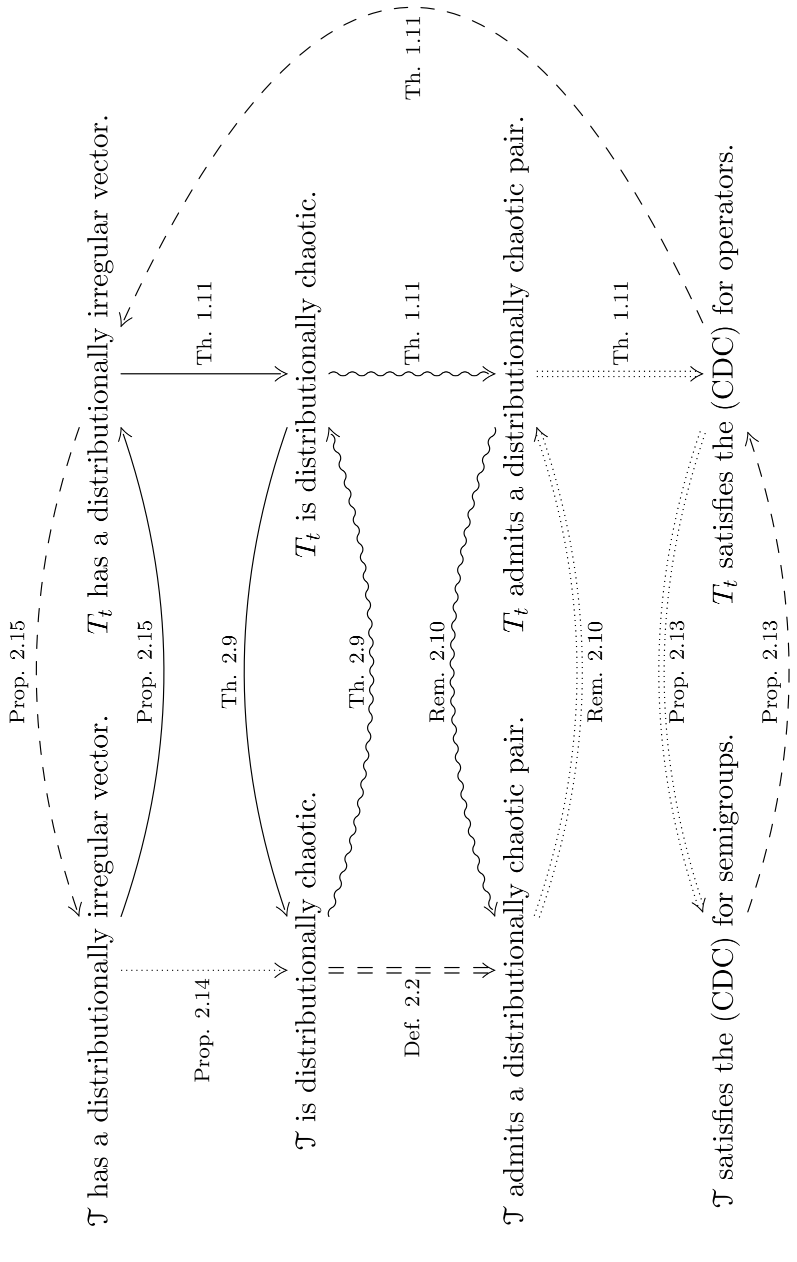

Figure 2.1: Overview of the results presented in this chapter

The following two propositions have been proved for operators in [BBMP12].

Proposition 2.20. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup and define $C := \sup_{0\leq t\leq 1} \|T_t\|$. The following statements are equivalent:*

(i) for every $k \in \mathbb{N}$ there exists $y_k \in X$ with $\|y_k\| = 1$ and $n_k > k$ such that

$$\mu\left(\{s < n_k \ ; \ \|T_s y_k\| > Ck\}\right) \geq n_k \left(1 - k^{-1}\right);$$

(ii) there exists $x \in X$ with distributionally unbounded orbit for $\mathcal{T}$;

(iii) the set of all $x \in X$ with distributionally unbounded orbit for $\mathcal{T}$ is residual in X.

Proof. (i) implies (iii). If (i) holds, by Lemma 2.7 it follows that

$$|(\{m \leq n_k \ ; \ \|T_1^m y_k\| > k\})| \geq n_k \left(1 - k^{-1}\right).$$

Therefore, by the analogous for operators, the set of all $x \in X$ with distributionally unbounded orbit for T_1 is residual in X. By Remark 2.19, we obtain (iii).
The implications (iii) implies (ii) and (ii) implies (i) are trivial. ⌟

Proposition 2.21. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup. Suppose that there exists a dense subset $X_0 \subset X$ such that each $x \in X_0$ has an orbit distributionally unbounded below. Then the set of all vectors with orbits distributionally unbounded below is residual.*

Proof. The proof follows directly from the counterpart for operators of this proposition and Remark 2.19. ⌟

We will also make us of the following remark about the (CDC).

Remark 2.22. The condition b) in the definition of the (CDC) for C_0-semigroups is equivalent in a Banach space to $(y_m)_m \subset \overline{\operatorname{span}\{x_k \colon k \in \mathbb{N}\}}$, $\|y_m\| = 1$ and there exist an increasing sequence of positive real numbers $(\rho_m)_m$ tending to ∞ with

$$\mu(\{s \in [0, \rho_m] \ ; \ \|T_s y_m\| > m\}) \geq \rho_m \left(1 - \frac{1}{m}\right) \text{ for every } m \in \mathbb{N}.$$

Proof of the remark. We start by proving that this condition implies condition b) of the (CDC). Take $(y_m)_m \subset \overline{\text{span}\{x_k \colon k \in \mathbb{N}\}}$, satisfying that $\|y_m\| = 1$ for every $m \in \mathbb{N}$. Define $z_m := \frac{y_m}{m}$, for $m \in \mathbb{N}$. It is clear that $z_m \in \overline{\text{span}\{x_k \colon k \in \mathbb{N}\}}$ and $\lim_{m\to\infty} \|z_m\| = 0$ for each $m \in \mathbb{N}$. Since

$$\|T_s z_m\| = \left\|T_s \frac{y_m}{m}\right\| = \frac{\|T_s y_m\|}{m} > \frac{m}{m} = 1,$$

it follows that $(z_m)_{m\in\mathbb{N}}$ satisfies condition b) from the definition of the (CDC) for $\delta = 1$.

Conversely, suppose that $(y_m)_m \subset \overline{\text{span}\{x_k \colon k \in \mathbb{N}\}}$, $\lim_{m\to\infty} y_m = 0$ and there exist $\delta > 0$ and an increasing sequence of positive real numbers $(\rho_m)_m$ tending to ∞ with

$$\mu(\{s \in [0, \rho_m] \ ; \ \|T_s y_m\| > \delta\}) \geq \rho_m \left(1 - \frac{1}{m}\right) \text{ for every } m \in \mathbb{N}.$$

We can assume that $\delta \geq 1$. Otherwise we would take $y'_m = \frac{y_m}{\delta}$. Passing to a subsequences if necessary, we have $\|y_m\| < \frac{1}{m}$, and define $z_m := \frac{y_m}{\|y_m\|}$. It is obvious that $(z_m)_{m\in\mathbb{N}} \subset \overline{\text{span}\{x_k \colon k \in \mathbb{N}\}}$ and $\|z_m\| = 1$ for every $m \in \mathbb{N}$. By the definition of z_m we get

$$\|T_s z_m\| = \left\|T_s \frac{y_m}{\|y_m\|}\right\| = \frac{\|T_s y_m\|}{\|y_m\|} > \frac{\delta}{m^{-1}} = m\delta \geq m$$

and therefore there exist an increasing sequence of positive real numbers $(\rho_m)_m$ tending to ∞ with

$$\mu(\{s \in [0, \rho_m] \ ; \ \|T_s z_m\| > m\}) \geq \rho_m \left(1 - \frac{1}{m}\right) \text{ for every } m \in \mathbb{N},$$

which is the desired conclusion. ⌟

Proof of Theorem 2.17. (ii) implies (iii) and (iii) implies (i) as before.
(iv) implies (ii): Let

$$X_0 = \left\{ x \in X \ ; \ \lim_{\substack{s\to\infty \\ s\in A}} \|T_s x\| = 0 \right\}.$$

Then X_0 is a subspace, $T(X_0) \subset X_0$, and $T(\overline{X_0}) \subset \overline{X_0}$. Moreover, $x_m \in X_0$ and $y_m \in \overline{X_0}$ for all $m \in \mathbb{N}$. By Proposition 2.21, the set of all vectors $x \in X_0$

with orbits distributionally unbounded below is residual in $\overline{X_0}$. By Proposition 2.20, the set of all vectors $x \in X_0$ with distributionally unbounded orbits is residual in $\overline{X_0}$. So the set of all distributionally irregular vectors is residual in $\overline{X_0}$, too. In particular, there exists a distributionally irregular vector.
(i) implies (iv): Let $f, g \in X$ be a distributionally chaotic pair, i.e., there exists $\delta > 0$ such that

$$\begin{aligned}
&\overline{\text{Dens}}(\{s \geq 0 \ ; \ \|T_s x - T_s y\| > \delta\}) = 1, \quad \text{and} \\
&\overline{\text{Dens}}(\{s \geq 0 \ ; \ \|T_s x - T_s y\| < \varepsilon\}) = 1.
\end{aligned}$$

for every $\varepsilon > 0$. Set $u = \dfrac{x-y}{\delta}$. Then

$$\overline{\text{Dens}}(\{s \geq 0 \ ; \ \|T_s u\| > 1\}) = 1, \quad \text{and} \quad \overline{\text{Dens}}(\{s \geq 0 \ ; \ \|T_s u\| < \varepsilon\}) = 1.$$

for each $\varepsilon > 0$. So for each $k \in \mathbb{N}$ there exists $a_k \in \mathbb{N}$ such that

$$\mu\left(\left\{s \leq a_k \ ; \ \|T_s u\| < \frac{1}{k}\right\}\right) \geq a_k \left(1 - \frac{1}{k}\right)$$

Define $A_k := \left\{s \leq a_k \ ; \ \|T_s u\| < \dfrac{1}{k}\right\}$ and $A := \bigcup_{k=1}^{\infty} A_k$. Then $\overline{\text{Dens}}(A) = 1$. Set $x_m = T_m u$ for each $m \in \mathbb{N}$. Clearly $\lim_{s \in A} \|T_s x_m\| = 0$ for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ choose s_m such that $\|T_{s_m} u\| < \dfrac{1}{m}$ and set $y_m = \dfrac{T_{s_m} u}{\|T_{s_m} u\|}$. Let $n_m \in \mathbb{N}$ satisfy $n_m > 2ms_m$ and

$$\mu\left(\{s \leq n_m \ ; \ \|T_s u\| > 1\}\right) \geq n_m \left(1 - \frac{1}{2m}\right).$$

Then

$$\mu\left(\{s_m \leq s \leq n_m \ ; \ \|T_s u\| > 1\}\right) \geq n_m \left(1 - \frac{1}{2m}\right) - s_m \geq n_m \left(1 - \frac{1}{m}\right).$$

If $s_m \leq s \leq n_m$ and $\|T_s u\| > 1$, then $\|T_{s-s_m} u\| > m$. Therefore

$$\mu\left(\{s \leq n_m \ ; \ \|T_s y_m\| > m\}\right) \geq n_m \left(1 - \frac{1}{m}\right)$$

and $\mathcal{T}$ satisfies the (CDC). ⌟

Propositions 2.20 and 2.21 and their counterparts for operators are really strong. Having a residual set of distributionally irregular vectors gives the curious case we address in the following example. One usual question formuled in linear dynamics is if certain sets of operators have common hypercyclic, frequently hypercyclic, irregular vectors, i.e. a vector that is hypercyclic, and so on for to different operators.

Example 2.23 (Rolewicz's operators). *On the spaces $X := \ell^p$, $1 \leq p < \infty$, or $X := c_0$ we consider the multiple*

$$T = \lambda B : X \to X, \quad (x_1, x_2, x_3, \dots) \to \quad (\lambda x_2, \lambda x_3, \lambda x_4, \dots)$$

of the backward shift, with $\lambda \in \mathbb{R}$. For any $c > 1$, the set of common distributionally irregular vectors of family $\{\lambda B\}_{|\lambda|\geq c}$, that is, $\bigcap_{|\lambda|\geq c} \mathcal{DI}(\lambda B)$ is residual in X.

Proof. Since the Rolewicz operators satisfy the (Godefroy-Shapiro criterion), we have that they admit a dense set $X_0 \subset X$ with $\lim_{n\to\infty} (\lambda B)^n x = 0$, for each $x \in X$ for every $|\lambda| > 1$. And

$$\sum_{n=0}^{\infty} \frac{1}{\|(\lambda B)^n\|} = \sum_{n=0}^{\infty} \frac{1}{|\lambda|^n} = \frac{1}{1 - \frac{1}{|\lambda|}} < \infty.$$

Therefore, by [BBMGP11, Corollary 30], λB admits a dense distributionally irregular manifold. Hence, by the version of Propositions 2.20 and 2.21 for operators, $\mathcal{DI}(\lambda B)$ is residual.

Let the sequence $(\lambda_i)_{i\in\mathbb{N}}$, with $|\lambda_i| > 1$ be such that $\bigcup_{i\in\mathbb{N}} [\lambda_i, \lambda_{i+1}[$ is a partition of $[c, \infty[$. For any pair λ_i, λ_{i+1} the set $\mathcal{DI}(\lambda_i B) \cap \mathcal{DI}(\lambda_{i+1} B)$ is residual. Take $x \in \mathcal{DI}(\lambda_i B) \cap \mathcal{DI}(\lambda_{i+1} B)$ and $\lambda_q \in]\lambda_i, \lambda_{i+1}[$. Then there exist two sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ with upper density 1 such that

$$\lim_{k\to\infty} \|(\lambda_{i+1} B)^{n_k} x\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|(\lambda_i B)^{m_k} x\| = \infty.$$

And hence,

$$\begin{aligned} \lim_{k\to\infty} \|(\lambda_q B)^{n_k} x\| &= \lim_{k\to\infty} \left\| \left(\frac{\lambda_q}{\lambda_{i+1}} \lambda_{i+1} B \right)^{n_k} x \right\| = \\ &= \lim_{k\to\infty} \left| \frac{\lambda_q}{\lambda_{i+1}} \right|^{n_k} \|(\lambda_{i+1} B)^{n_k} x\| = 0 \end{aligned} \tag{2.7}$$

and

$$\lim_{k\to\infty} \|(\lambda_q B)^{m_k} x\| = \lim_{k\to\infty} \left\| \left(\frac{\lambda_q}{\lambda_i} \lambda_i B \right)^{m_k} x \right\| = \lim_{k\to\infty} \left| \frac{\lambda_q}{\lambda_i} \right|^{m_k} \|(\lambda_i B)^{m_k} x\| = \infty. \tag{2.8}$$

Thus we have proved that

$$\bigcap_{\lambda_j \in [\lambda_i, \lambda_{i+1}]} \mathcal{DI}(\lambda_j B) = \mathcal{DI}(\lambda_i B) \cap \mathcal{DI}(\lambda_{i+1} B)$$

and therefore is residual.

Note that, by (2.7) and (2.8), $\mathcal{DI}(\lambda_r B) = \mathcal{DI}(\lambda_s B)$ if $|\lambda_r| = |\lambda_s|$. Since we have divided the set in a countable amount of intervals, and the countable intersection of residual sets is residual, the proof is finished. $\lrcorner$

Chapter 3

Sufficient conditions for distributional chaos

It is not always easy to verify whether a C_0-semigroup is distributionally chaotic or not, even with the help of the notion of Distributional irregular vectors and the Criterion for Distributional Chaos. For that reason we present some computable conditions for distributional chaos for operators and for semigroups. We recall first the following useful criterion:

Theorem 3.1. ([BBMGP11, Corollary 30]) *Let $T : X \to X$ be an operator such that there exist a dense subset $X_0 \subset X$ with $\lim_{n\to\infty} T^n x = 0$, for each $x \in X_0$, and an increasing sequence of integers $B = (m_k)_k$ with $\overline{\text{dens}}(B) = 1$ satisfying*

(i) *either* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\|T^{m_k}\|} < \infty$,

(ii) *or X is a complex Hilbert space and* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\|T^{m_k}\|^2} < \infty$.

Then T has a dense distributionally irregular manifold.

In order to prove the analogue of Theorem 3.1, we will need the following proposition.

Proposition 3.2. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$ and let $\alpha \geq 1$. Then the following statements are equivalent.*

(i) *There exists an increasing sequence of positive integers $B = (m_k)_k$ with $\overline{\text{dens}}(B) = 1$ such that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\|T_1^{m_k}\|^{\alpha}}$ is convergent.*

(ii) *There exists a Lebesgue measurable set $A \subseteq [0, \infty[$ with $\overline{\text{Dens}}(A) = 1$ such that the integral $\displaystyle\int_A \|T_t\|^{-\alpha} dt$ is convergent.*

Proof. Set $C := \sup_{t\in[0,1]} \|T_t\|$. Then $C < \infty$ by the local equicontinuity of $\mathcal{T}$. In particular, we observe that if $t \in [h-1, h]$ for some $k \in \mathbb{N}$, then we can write $h = t + (h - t)$ with $h - t \in [0, 1]$. Consequently, $\|T_h\| = \|T_{h-t}T_t\| \leq C\|T_t\|$ and so $\|T_t\|^{-1} \leq C\|T_h\|^{-1}$. On the other hand, if $t \in [h, h + 1]$ for some $h \in \mathbb{N}$, then we can write $t = h + (t - h)$ with $t - h \in [0, 1]$. It follows that $\|T_t\| = \|T_{t-h}T_h\| \leq C\|T_h\|$ and so $\|T_h\|^{-1} \leq C\|T_t\|^{-1}$.

(i) implies (ii). Set $A := \bigcup_{k\in\mathbb{N}} [m_k - 1, m_k]$. Then $\overline{\text{Dens}}(A) = \overline{\text{dens}}(B) = 1$ as it is easy to prove. Moreover, using the above inequalities, we have

$$\begin{aligned}\int_A \|T_t\|^{-\alpha} dt &= \sum_{k\in\mathbb{N}} \int_{m_k-1}^{m_k} \|T_t\|^{-\alpha}\, dt \leq C^{\alpha} \sum_{k\in\mathbb{N}} \int_{m_k-1}^{m_k} \|T_{m_k}\|^{-\alpha}\, dt \\ &= C^{\alpha} \sum_{k\in\mathbb{N}} \|T_1^{m_k}\|^{-\alpha} < \infty,\end{aligned}$$

i.e., (ii) is satisfied.

(ii) implies (i). Set $B := \{k \in \mathbb{N}\, ;\, \mu(A\cap[k, k+1]) \geq \frac{1}{2}\}$. Then $\overline{\text{dens}}(B) = 1$ by Lemma 2.8, and, using the above inequalities, we obtain

$$\begin{aligned}\infty > \int_A \|T_t\|^{-\alpha}\, dt &\geq \sum_{k\in B} \int_{A\cap[k,k+1]} \|T_t\|^{-\alpha}\, dt \\ \geq \sum_{k\in B} \int_{A\cap[k,k+1]} C^{-\alpha} \|T_k\|^{-\alpha}\, dt &\geq \frac{C^{-\alpha}}{2} \sum_{k\in B} \|T_1^k\|^{-\alpha},\end{aligned}$$

i.e., $\sum_{k\in B} \|T_1^k\|^{-\alpha} < \infty$. ⌟

We can now state the analogue of Theorem 3.1.

Theorem 3.3. ***Dense Distributionally Irregular Manifold Criterion*** *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a C_0-semigroup in $L(X)$ such that there exist a dense subset $X_0 \subset X$ such that $\lim_{t\to\infty} T_t x = 0$, for each $x \in X_0$, and a Lebesgue measurable set $A \subseteq [0,\infty)$ with $\overline{\mathrm{Dens}}(A) = 1$ satisfying*

(i) *either* $\displaystyle\int_A \frac{1}{\|T_t\|}\, dt < \infty$,

(ii) *or X is a complex Hilbert space and* $\displaystyle\int_A \frac{1}{\|T_t\|^2}\, dt < \infty$.

Then $\mathcal{T}$ has a dense distributionally irregular manifold.

Proof. The assumptions (i) and (ii) ensure that the operator T_1 satisfies condition (i) in Proposition 3.2. So, we can apply Theorem 3.1 to conclude that T_1 has a dense distributionally irregular manifold. Finally, by Proposition 2.15, $\mathcal{T}$ also has a dense distributionally irregular manifold. ⌟

We need a technical result on the power growth of an operator based on the eigenvectors associated to unimodular eigenvalues in the spirit of Ransford [Ran05].

Proposition 3.4. *Let X be a complex separable Banach space and let $T \in L(X)$. Assume that there exists a Borel probability measure m on $\mathbb{T}$ such that $m(\mathbb{T} \cap \sigma_p(T)) > 0$ and a bounded m–measurable function $f : \mathbb{T} \to X$ satisfying*

(i) $f(\lambda) \in \ker(\lambda I - T)$ *for all* $\lambda \in \mathbb{T}$, *and*

(ii) $f(\lambda) \neq 0$ *if* $\lambda \in \sigma_p(T) \cap \mathbb{T}$.

Then there exist $n_0 \in \mathbb{Z}$, $D > 0$ such that, for every $n \in \mathbb{N}$,

$$\frac{1}{\|T^n\|} \leq D \left\| \int_{\mathbb{T}} f(e^{i\theta}) e^{-i(n_0+n)\theta} dm(\theta) \right\|.$$

Proof. The proof is inspired by the proof of Lemma 4.1 in [Ran05]. For every $n \in \mathbb{Z}$ let

$$x_n = \int_{\mathbb{T}} f(e^{i\theta}) e^{-in\theta} dm(\theta).$$

Then there exists $n_0 \in \mathbb{Z}$ such that $x_{n_0} \neq 0$. Indeed, if $x_n = 0$ for all $n \in \mathbb{Z}$, then, given $\psi \in X'$, we have

$$\int_{\mathbb{T}} \psi\left(f(e^{i\theta})\right) e^{-in\theta} dm(\theta) = 0,$$

i.e., the Fourier coefficients of the measure $\psi\left(f(e^{i\theta})\right) dm(\theta)$ are all zero. It follows that $\psi\left(f(\lambda)\right) = 0$ for every $\lambda \in \mathbb{T} \cap \sigma_p(T) \setminus A_\psi$ with $m(A_\psi) = 0$. As X is separable, there exists a sequence $(\psi_k)_k \subseteq X'$ such that if $\psi_k(x) = 0$ for every $k \in \mathbb{N}$, then $x = 0$. Setting $A = \bigcup_k A_{\psi_k}$, we get that $m(A) = 0$ and $\psi_k\left(f(\lambda)\right) = 0$ for every $\lambda \in \mathbb{T} \cap \sigma_p(T) \setminus A$. Therefore $f(\lambda) = 0$ for every $\lambda \in \mathbb{T} \cap \sigma_p(T) \setminus A$, and this contradicts the assumption on f.
Observe now that for every $n \in \mathbb{Z}$, by the properties of f, $Tx_n = x_{n-1}$. In particular, $T^n(x_{n+n_0}) = x_{n_0}$ for every $n \in \mathbb{N}$. Hence

$$\frac{1}{\|T^n\|} \leq \frac{\|x_{n+n_0}\|}{\|x_{n_0}\|} \leq \frac{1}{\|x_{n_0}\|} \left\| \int_{\mathbb{T}} f(e^{i\theta}) e^{-i(n_0+n)\theta} dm(\theta) \right\|,$$

which is the desired conclusion. ⌟

Remark 3.5. Lemma 4.1 in [Ran05] ensures that if X is a separable Banach space and $T \in L(X)$, then there exists a function $f : \mathbb{T} \to X$, which is measurable with respect to every σ-finite Borel measure on $\mathbb{T}$, and satisfies conditions (i) and (ii) of Proposition 3.4.

The following notion was introduced by Bayart and Grivaux [BG05].

Definition 3.6. Let X be a complex separable infinite dimensional Banach space, m a probability measure on the unit circle $\mathbb{T}$ and $T \in L(X)$. A family $(E_j)_{j\in J}$ of m–measurable X–valued function defined on $\mathbb{T}$ is said to be a *spanning fields of unimodular eigenvectors of T with respect to m* if $E_j(\lambda) \in \ker(\lambda I - T)$ and span $\left\{\bigcup_{j\in J}\{E_j(\lambda)\ ;\ \lambda \in \mathbb{T} \setminus \Omega\}\right\}$ is dense in X for every m–measurable set $\Omega \subseteq \mathbb{T}$ with $m(\Omega) = 0$.

Proposition 3.7. *Let X be a complex separable infinite dimensional Banach space and $T \in L(X)$. If there exists a spanning eigenvector field $(E_j)_{j\in J}$ associated to unimodular eigenvalues of T with respect to a continuous probability measure m on $\mathbb{T}$, then there exists a set $X_0 \subseteq X$ such that X_0 is dense in X and $\lim_{n\to\infty} T^n x = 0$, for all $x \in X_0$.*

Proof. The proof can be found either in [GEPM11], Theorem 9.22 or in [BM09], Theorem 5.41. ⌟

Proposition 3.7 was recently strengthened by Grivaux in [Gri10].

Theorem 3.8. ([Gri10, Theorem 1.4]) *Let X be a complex separable infinite dimensional Banach space and $T \in L(X)$ be a bounded operator acting on X. If there exists a family $(E_j)_{j\in J}$ of spanning fields of unimodular eigenvectors of T with respect to a continuous probability measure m on $\mathbb{T}$, then T is frequently hypercyclic.*

For the final results of this chapter we need to introduce some extra notation. We recall that a space X has type p for some $p \in [1,2]$ if there exists a constant $M > 0$ such that for every choice $\{x_i\}_{i=1}^n$ of vector in X

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| dt \leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

where $(r_i)_i$ are the Rademacher functions. Every Banach space has type 1, while every Hilbert space has type 2. Moreover, if $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p \leq 2$, then the spaces L^p and L^q have type p.

Corollary 3.9. *Let X be a complex separable infinite dimensional Banach space and $T \in L(X)$. Assume that $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure and that there exists a family $(E_j)_{j\in\mathbb{N}}$ of spanning fields of unimodular eigenvectors of T with respect to the Lebesgue measure on $\mathbb{T}$. If*

(i) *X is a Hilbert space, or*

(ii) *X has type p for some $p > 1$ and one of the fields E_j is Lipschitz-continuous with $E_j(x) \neq 0$ for every $x \in \sigma_p(T) \cap \mathbb{T}$,*

then T is frequently hypercyclic and has a dense distributionally irregular manifold.

Proof. We first observe that the assumption that $(E_j)_{j\in\mathbb{N}}$ is a family of spanning fields of unimodular eigenvectors of T implies by Proposition 3.7 the existence of a dense subspace $X_0 \subset X$ such that $\lim_{n\to\infty} T^n x = 0$ for every $x \in X_0$. And by Theorem 3.8 T is frequently hypercyclic.

(i): Let X be a Hilbert space. By Proposition 3.4 and Remark 3.5 there exist a bounded Lebesgue measurable function $f\colon \mathbb{T} \to X$, $D > 0$ and $n_0 \in \mathbb{Z}$ such that

$$\frac{1}{\|T^n\|^2} \leq D^2 \left\| \int f(e^{i\theta}) e^{-i(n+n_0)\theta} d\theta \right\|^2 = D^2 \left\| \hat{f}(n+n_0) \right\|^2,$$

where, for every $k \in \mathbb{Z}$, $\widehat{f}(k)$ denotes the $k-th$ Fourier coefficient of f. Since $f \in L^2(\mathbb{T}, X)$, we have that $\sum_{n\in\mathbb{N}} \|\hat{f}(n)\|^2 < \infty$, so the assertion follows by Theorem 3.1.

(ii): Let X have type p for some $p > 1$ and $f = E_j$ be the Lipschitz continuous field. Then f satisfies the assumptions of Proposition 3.4 and by [Kön91, Theorem 1], $(\|\hat{f}(n)\|)_{n\in\mathbb{Z}} \in \ell^1$, thereby implying that

$$\sum_{n\in\mathbb{N}} \frac{1}{\|T^n\|} < \infty.$$

So, again by Corollary 3.1, we get the assertion. ⌟

At this point we can give a sufficient condition for distributional chaos of a semigroup involving the point spectrum of the generator of the semigroup. The importance of having this condition at hand is that, usually, one can compute more easily properties on the generator of the semigroup (especially, for generators associated with abstract Cauchy problems) than on the operators of the semigroup itself.

Corollary 3.10. *Let X be a complex separable infinite dimensional Banach space and $\mathfrak{T} = \{T_t\}_{t\geq 0}$ a C_0-semigroup in $L(X)$ with infinitesimal generator A. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded Lebesgue measurable functions $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $\sigma_p(A) \cap iI_j$ has positive Lebesgue measure, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and* $\operatorname{span}\{f_j(t) \ : \ j \in \Gamma, \ t \in I_j\}$ *is dense in X. If*

(i) *X is a Hilbert space, or*

(ii) *X has type p for some $p > 1$ and one of the functions f_j is Lipschitz-continuous,* *then $\mathcal{T}$ is frequently hypercyclic and has a dense distributional irregular manifold.*

Proof. We show that T_1 has a dense distributional irregular manifold.

Extend each f_j to $\mathbb{R}$ by setting $f_j = 0$ in $\mathbb{R} \setminus I_j$. For each $\theta \in [0, 2\pi]$, $j \in \Gamma$, $k \in \mathbb{Z}$, set $E_{j,k}(e^{i\theta}) := f_j(\theta + 2k\pi)$. It is easy to verify that $\sigma_p(T_1) \cap \mathbb{T}$ has positive Lebesgue measure and that $(E_{j,k})_{j\in\Gamma}$ is a family of spanning fields of unimodular eigenvectors of T_1 with respect to the Lebesgue measure. Therefore, by Theorem 3.8, T_1 is frequently hypercyclic and, by Corollary 3.9, if X is a Hilbert space, then T_1 has a dense distributional irregular manifold.

Assume that X has type p for some $p > 1$. Let $f = f_j$ be the Lipschitz-continuous function and suppose that $[0, 2\pi] \subseteq I_j$ (otherwise we can rescale). Let ϕ be a C_1-function such that supp$\phi = [0, 2\pi]$, with $\phi(0) = \phi(2\pi) = 0$. Define $g : \mathbb{T} \to X$ by setting $g(e^{i\theta}) := f_j(\theta)\phi(\theta)$. Then g is Lipschitz continuous (since $f\phi(0) = f\phi(2\pi)$) and the assertion follows as in Corollary 3.9. ⌟

The following example shows the applicability of Corollary 3.10.

Example 3.11. Consider the linear perturbation of the one-dimensional Ornstein-Uhlenbeck operator

$$\mathcal{A}_\alpha u = u'' + bxu' + \alpha u,$$

where $\alpha \in \mathbb{R}$, with domain

$$D(\mathcal{A}_\alpha) = \left\{ u \in L^2(\mathbb{R}) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}) \ ; \ \mathcal{A}_\alpha u \in L^2(\mathbb{R}) \right\}.$$

In [CM10; MP11], it was proved that if $\alpha > b/2 > 0$, then the semigroup generated by $\mathcal{A}_\alpha$ in $L^2(\mathbb{R})$ is chaotic and frequently hypercyclic. Actually, every operator of the semigroup is densely distributionally chaotic. Indeed, for every $\mu \in \mathbb{C}$, with $\Re\mu < -\frac{b}{2} + \alpha$ the functions u^1_μ and u^2_μ, whose Fourier transforms are

$$\widehat{u^1_\mu}(\xi) = e^{-\xi^2/2b}\xi|\xi|^{-(2+(\mu-\alpha)/b)}, \qquad \widehat{u^2_\mu}(\xi) = e^{-\xi^2/2b}|\xi|^{-(1+(\mu-\alpha)/b)},$$

are eigenfunctions of $\mathcal{A}_\alpha$ (see [CM10; Met01]).

For every $s \in \mathbb{R}$, consider the functions $f_1(s) = u^1_{is}$ and $f_2(s) = u^2_{is}$. For every $\phi \in X' = L^2(\mathbb{R})$ and $j = 1, 2$, by Parseval equality, we have

$$\langle \phi, f_j(s) \rangle = \int_{\mathbb{R}} \phi(x) u^j_{is}(x) dx = \int_{\mathbb{R}} \widehat{\phi}(x) \widehat{u}^j_{is}(x) dx \qquad s \in \mathbb{R}.$$

It is immediate to verify that $\langle \phi, f_j(\cdot) \rangle \in C(\mathbb{R})$ by Lebesgue theorem, hence the f_j are (weakly) measurable and locally bounded.

The argument of [CM10] shows that $\text{span}\{f_i(s) \; ; \; i = 1, 2, s \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$. Thus the assertion follows.

Chapter 4

Distributional chaos for translation C_0-semigroups

For linear discrete dynamical systems the shift operators on sequence spaces represent one of the most important classes of "test" operators. In the continuous case this role is played by the translation semigroup.

Firstly, let us introduce the weighted spaces of integrable functions where we are going to consider the translation C_0-semigroups. These spaces are denoted as $L^p_\rho([0, +\infty[)$, with $1 \leq p < \infty$ and ρ an admissible weight function.

Definition 4.1 ([DSW97])**.** By an *admissible weight* function on $\mathbb{R}_0^+$ we mean a measurable function $\rho : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

i) $\rho(\tau) > 0$ for all $\tau \in \mathbb{R}_0^+$,

ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(t + \tau)$ for all $\tau \in \mathbb{R}_0^+$ and all $t > 0$.

The following is a useful property for admissible weights.

Lemma 4.2 ([DSW97])**.** *Let ρ be an admissible weight function on $\mathbb{R}_0^+$. For each $l > 0$ there are constants $0 < m_1 < M_1$ (depending on ρ and l only)* *such that for each $\sigma \in \mathbb{R}_0^+$ and each $\tau \in [\sigma, \sigma + l]$, we have $m_1\rho(\sigma) < \rho(\tau) < M_1\rho(\sigma + l)$.*

Let $1 \leq p < \infty$, let ρ be an admissible weight function on $\mathbb{R}_0^+$ and let $\mathcal{M}([0, +\infty[)$ denote the space of measurable functions on the interval $[0, +\infty[$ in the sense of Lebesgue. We consider the separable Banach infinite dimensional space of p-integrable functions (in the Lebesgue sense) $L^p_\rho(\mathbb{R}^+)$ as

$$X := \{f \in \mathcal{M}([0, +\infty[) \; ; \; \|f\|_p < \infty\}, \text{ where } \|f\|_p = \left(\int_{[0,+\infty[} |f(s)|^p \rho(s) ds \right)^{1/p}.$$

The *translation semigroup* defined by $(T_t f)(x) = f(x + t)$, $t, x \geq 0$, is a well-defined C_0-semigroup by the definition of admissible weight.

This chapter is divided in three sections. In the first one we review some known results on the dynamics of the translation C_0-semigroup, later we state and prove some sufficient conditions for distributional chaos for this semigroup. Finally, in the third one, we establish a complete analogy between the study of distributional chaos for the translation C_0-semigroup and the corresponding one for backward shifts on weighted sequence spaces.

4.1 Existing results on the dynamics of translation C_0-semigroups

When studying linear dynamics on weighted spaces, the natural question that arises is the relevance of the weight. We have compiled in this section some characterizations of hypercyclic, mixing and Devaney chaotic translation C_0-semigroups in terms of the weight in order to be able to compare them later with our sufficient condition.

Theorem 4.3 (Desch et al.[DSW97]). *On $L^p_\rho(\mathbb{R}^+)$ with $1 \leq p < \infty$, the translation semigroup $\{T_t\}_{t\geq 0}$ is hypercyclic if and only if*

$$\liminf_{t\to\infty} \rho(t) = 0.$$

Theorem 4.4 (Desch et al.[DSW97]). *On $L^p_\rho(\mathbb{R})$ with $1 \leq p < \infty$, the translation semigroup $\{T_t\}_{t\geq 0}$ is hypercyclic if and only if for each $\theta \in \mathbb{R}$ there exist a sequence $(t_j)_{j\in\mathbb{N}}$ of positive real numbers tending to ∞ such that*

$$\lim_{j\to\infty} \rho(t_j + \theta) = \lim_{j\to\infty} \rho(-t_j + \theta) = 0.$$

Theorem 4.5 (Bermúdez et al. [BBCP05]). *We consider the translation C_0-semigroup $\{T_t\}_{t\geq 0}$ on the space $X = L^p_\rho([0, +\infty[)$, $1 \leq p < \infty$, for an admissible weight ρ. Then $\{T_t\}_{t\geq 0}$ is mixing if and only if*

$$\lim_{t\to\infty} \rho(t) = 0.$$

Theorem 4.6 (deLaubenfels, Emamirad [dE01]). *Let $X = L^p_\rho([0, +\infty[)$. The following are equivalent:*

i. *The translation C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is chaotic.*

ii. $\int_0^\infty \rho(s)ds < \infty$.

iii. $\sup\left\{\nu \in \mathbb{R}\ ;\ \int_0^\infty e^{\nu s}\rho(s)ds < \infty\right\} > 0$.

iv. *T_1 has a non-trivial periodic point.*

v. *T_1 is chaotic.*

Theorem 4.7 (Matsui et al. [MYT03]). *Let $X = L^p_\rho(\mathbb{R}^+)$. The translation C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is chaotic if, and only if, for every $\varepsilon, \theta > 0$ there exists $t > 0$ such that*

$$\sum_{k=1}^{\infty} \rho(\theta + kt) < \varepsilon.$$

4.2 Distributional chaos for translation C_0-semigroups

From now on our space X will be $L^p_\rho(\mathbb{R}^+)$ with $1 \leq p < \infty$, where ρ is an admissible weight, and $\mathcal{T} = \{T_t\}_{t\geq 0}$ will be the translation semigroup on X. And $d(x, y)$ will be the metric induced by the norm of the corresponding space.

The following result provides characterizations of distributional chaos for the translation semigroup.

Theorem 4.8. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be the translation semigroup on $X = L^p_\rho(\mathbb{R}^+)$. The following are equivalent:*

(1) There exist $f \in X$ and $\delta > 0$ such that

$$\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s f\|_p < \delta\} = 0.$$

(2) There exists $f \in X$ such that, for every $N > 0$,

$$\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s f\|_p < N\} = 0.$$

(3) $\mathcal{T}$ is densely distributionally chaotic.

Proof. If *(1)* holds, we can find an increasing sequence $(m_k)_k$ in $\mathbb{N}$ such that

$$\int_{[m_k,+\infty[} |f(t)|^p \rho(t) dt < \frac{1}{2^k}.$$

We define h by the formula

$$h(t) := \begin{cases} (1+k)f(t), & m_k < t < m_{k+1} \\ f(t), & 0 < t < m_1. \end{cases}$$

Let $m_0 := 0$. Note that $h \in X$, because

$$\|h(t)\|_p^p = \int_{\mathbb{R}^+} |h(t)|^p \rho(t) dt = \sum_{i\geq 0} \int_{[m_i,m_{i+1}]} |h(t)|^p \rho(t) dt$$
$$= \sum_{i\geq 0} (1+i)^p \int_{[m_i,m_{i+1}]} |f(t)|^p \rho(t) dt < \int_{[0,m_0]} |f(t)|^p \rho(t) dt + \sum_{k\geq 1} \frac{(1+k)^p}{2^k}.$$

Fix an arbitrary $N > \delta$ and $k_0 \in \mathbb{N}$ with $k_0 > N/\delta$. For all $t > m_{k_0}$, $|k_0 f(t)| \leq |h(t)|$. Then, since $k_0 \|T_s f\|_p \leq \|T_s h\|_p$ for all $s > m_{k_0}$, we have that

$$\underline{\mathrm{Dens}}\left(\{s \in \mathbb{R}^+ \ ; \ \|T_s h\|_p < N\}\right) \leq \underline{\mathrm{Dens}}\left(\{s \in \mathbb{R}^+ \ ; \ k_0\|T_s f\|_p < k_0\delta\}\right) = 0.$$

Hence we get that *(1)⇒(2)*.

If *(2)* holds, we can find an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that, for every $k \in \mathbb{N}$,

$$\mu(\{s \leq n_k : \|T_s f\|_p < 2N\}) < \frac{n_k}{k} \quad \text{for every } N > 0.$$

We can find a sufficiently fast increasing sequence $(q_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for h defined by the formula

$$h(t) := \begin{cases} f(t), & q_{2k-1} < t < q_{2k}; \\ 0, & otherwise; \end{cases}$$

on the one hand we have that each $k \in \mathbb{N}$ admits a $j = j(k)$, with $k \leq j$, so that $kq_{2k-1} < n_j < q_{2k}$ and that for every $N > 0$, there exists a $k_0 \in \mathbb{N}$ such that

$$\|T_s h - T_s f\|_p < N \quad \text{for every } s \in [q_{2k-1}, n_{j(k)}],$$

with $k > k_0$. For instance, it suffices to have, for $k > k_0$,

$$\int_{q_{2k}}^{\infty} |(f-h)(t)|^p \rho(t)dt < \frac{N^p}{Me^{\omega n_{j(k)}}}.$$

From this, and taking $M > 1$ and $\omega > 0$ satisfying that $\rho(\tau) \leq Me^{\omega t}\rho(t+\tau)$ for all $\tau \in \mathbb{R}_0^+$ and all $t > 0$, we see that

$$\begin{aligned}
\|T_s f - T_s h\|_p^p = & \int_0^{\infty} |(f-h)(t+s)|^p \rho(t)dt = \int_s^{\infty} |(f-h)(r)|^p \rho(r-s)dr \leq \\
\leq & Me^{\omega s} \int_s^{\infty} |(f-h)(r)|^p \rho(r)dr \\
\leq & Me^{\omega n_{j(k)}} \left(\int_{q_{2k-1}}^{q_{2k}} |(f-h)(r)|^p \rho(r)dr + \int_{q_{2k}}^{\infty} |(f-h)(r)|^p \rho(r)dr \right) \\
\leq & Me^{\omega n_{j(k)}} \left(0 + \int_{q_{2k}}^{\infty} |(f-h)(r)|^p \rho(r)dr \right) < \frac{N^p Me^{\omega n_{j(k)}}}{Me^{\omega n_{j(k)}}} \\
\leq & N^p,
\end{aligned}$$

for every $s \in [q_{2k-1}, n_{j(k)}]$ with $k > k_0$.

Therefore, for each $s \in [q_{2k-1}, n_{j(k)}]$ with k big enough, if $\|T_s h\|_p < N$, then we have that

$$\|T_s f\|_p = \|T_s f - T_s h + T_s h\|_p \leq \|T_s f - T_s h\|_p + \|T_s h\|_p < N + N = 2N.$$

Thus we obtain

$$\mu\left(\{s \in [q_{2k-1}, n_j] \; ; \; \|T_s h\|_p < N\}\right) \leq \mu\left(\{s \in [q_{2k-1}, n_j] \; ; \; \|T_s f\|_p < 2N\}\right).$$

Then,

$$\begin{aligned}\mu\left(\{s \le n_j \ ; \ \|T_s h\|_p < N\}\right) &\le q_{2k-1} + \mu\left(\{s \in [q_{2k-1}, n_j] \ ; \ \|T_s h\|_p < N\}\right)\\ &< q_{2k-1} + \frac{n_j}{j}.\end{aligned}$$

And hence,

$$\begin{aligned}\underline{\text{Dens}}\left\{s \in \mathbb{R}^+ \ ; \ \|T_s h\|_p < N\right\} &\le \lim_{k\to\infty} \frac{\mu\left(\left\{s \le n_{j(k)} \ ; \ \|T_s h\|_p < N\right\}\right)}{n_{j(k)}} \le \\ &\le \lim_{k\to\infty} \frac{q_{2k-1} + \frac{n_{j(k)}}{j(k)}}{n_{j(k)}} \le \lim_{k\to\infty} \frac{2}{j(k)} \\ &= 0.\end{aligned} \tag{4.1}$$

On the other hand, we have introduced in h sufficiently large intervals of 0's so that $\overline{\text{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s h\|_p < \varepsilon\} = 1$ for all $\varepsilon > 0$. For instance it suffices that $q_{2k+1} > kq_{2k}$ and that

$$\int_{q_{2k+1}}^{\infty} |h(t)|^p \rho(t)dt < \frac{1}{Me^{\omega k^2 q_{2k}}} \text{ for } k \text{ big enough.}$$

Then, there exist $k_0 \in \mathbb{N}$ such that for every $k > k_0$ and for each $s \in [q_{2k}, kq_{2k}]$ we have that

$$\begin{aligned}\|T_s h\|_p^p &= \int_0^{\infty} |h(t+s)|^p \rho(t)dt = \int_s^{\infty} |h(r)|^p \rho(r-s)dr \\ &\le Me^{\omega s} \int_s^{\infty} |h(r)|^p \rho(r)dr \\ &\le Me^{\omega k q_{2k}} \left(\int_{q_{2k}}^{q_{2k+1}} |h(r)|^p \rho(r)dr + \int_{q_{2k+1}}^{\infty} |h(r)|^p \rho(r)dr \right) \\ &\le Me^{\omega k q_{2k}} \left(0 + \int_{q_{2k+1}}^{\infty} |h(r)|^p \rho(r)dr \right) < \frac{Me^{\omega k q_{2k}}}{Me^{\omega k^2 q_{2k}}} \\ &\le e^{-\omega k q_{2k}(k-1)}.\end{aligned}$$

Therefore, since $\displaystyle\lim_{k\to\infty} \frac{kq_{2k} - q_{2k}}{kq_{2k}} = 1$, we have obtained

$$\overline{\text{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s h\|_p < \varepsilon\} = 1, \text{ for all } \varepsilon > 0.$$

We fix a dense sequence $(y_n)_n$ in X of functions with compact support, and we define

$$S = \bigcup_{n\in\mathbb{N}} \left(y_n + \left\{ \alpha h \ ; \ \frac{1}{n+1} < \alpha < \frac{1}{n} \right\} \right).$$

It is clear that S is a dense subset of X. We will show that it is a distributionally δ'-scrambled set for the translation semigroup.

Let $x, x' \in S$ with $x \neq x'$. W.l.o.g., $x = y_m + \alpha h$ and $x' = y_n + \beta h$ with $\alpha < \beta < 1$. Since y_n and y_m are functions with compact support, we have that

$$\overline{\mathrm{Dens}}(\{s \in \mathbb{R}^+ \ ; \ d(T_s x, T_s x') < \varepsilon\}) = \overline{\mathrm{Dens}}(\{s \in \mathbb{R}^+ \ ; \ (\beta-\alpha) \left\|T_s h\right\|_p < \varepsilon\}) = 1.$$

It only remains to show that

$$\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ (\beta - \alpha) \left\|T_s h\right\|_p < \delta'\} = 0,$$

which is an easy consequence of equation (4.1).

For *(3)⇒(1)*, we just take $g, h \in S$ with $g \neq h$ where S is a scrambled set for $\mathcal{T}$. By definition of distributional chaos we have that exists $\delta > 0$ such that

$$\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ d(T_s g, T_s h) < \delta\} = 0. \tag{4.2}$$

Define $f := g - h$, then $\|T_s f\| = d(T_s g, T_s h)$ and therefore

$$\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s f\|_p < \delta\} = 0.$$

This completes the proof. ⌟

The following result is the continuous version of a result for backward shifts given in [MGOP09].

Theorem 4.9. *The translation semigroup $\{T_t\}_{t\geq 0}$ is densely distributionally chaotic on X if we can find a measurable subset $A \subset \mathbb{R}^+$ such that $\overline{\mathrm{Dens}}(A) = 1$ and $\int_A \rho(s)ds < \infty$.*

Proof. Define f as follows

$$f(t) := \begin{cases} 1, & t \in A, \\ 0, & \textit{otherwise}. \end{cases}$$

We know that $f \in X$ because $\|f(t)\|_p^p = \int_A 1^p \rho(s)ds < \infty$. Since $\rho > 0$, by the admissibility of the weight, let $\delta > 0$ be such that $\rho(t) > 2\delta$ for every $t \in [0,2]$. Define the set $\mathcal{K} = \{k \in \mathbb{N} \ ; \ \mu(A \cap [k, k+1]) > \frac{1}{2}\}$. By Lemma 2.8, we know that $\overline{\mathrm{dens}}(\mathcal{K}) = 1$. If we now define the set $A' = \bigcup_{k\in\mathcal{K}}[k-1,k]$, then $\overline{\mathrm{Dens}}(A') = \overline{\mathrm{dens}}(\mathcal{K}) = 1$.
Hence if $s \in A'$, then

$$\|T_s f\|_p^p > \int_{[0,2]\cap A} |T_s f(t)|^p \rho(t)dt > \int_{[s,s+2]\cap A} f(t) 2\delta dt > 2\delta\mu([s,s+2]\cap A) > \delta.$$

Therefore $\underline{\mathrm{Dens}}\{s \in \mathbb{R}^+ \ ; \ \|T_s f\|_p < \delta^{1/p}\} = 0$, and by Theorem 4.8 the proof is finished. ⌟

As a consequence of Theorem 4.9 and the characterization of Devaney chaos for the translation semigroup given in Theorem 4.6, we obtain the analogous implication of the discrete version given in [MGOP09].

Corollary 4.10. *Let $X = L^p_\rho(\mathbb{R}^+)$. If the translation C_0-semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic on X, then it is distributionally chaotic.*

Despite the above result, Devaney chaos is far from being a characterization of distributional chaos for the translation C_0-semigroup, as the following example shows.

Example 4.11. *Let $(n_k)_k$ be a sequence in $\mathbb{N}$ with $n_{k+1} > kn_k$, $k \in \mathbb{N}$. We define the admissible weight by $\rho(t) = e^{n_k - t}$, $t \in [n_{k-1}, n_k[$, $k \in \mathbb{N}$, where $n_0 := 0$. Let*

$$f(t) := \begin{cases} \frac{1}{k^2}, & t \in [n_k - 1, n_k], \quad k \in \mathbb{N}, \\ 0, & \textit{otherwise}. \end{cases}$$

The function $f \in L^p_\rho(\mathbb{R}^+)$ since

$$\int_{\mathbb{R}^+} |f(t)|^p \rho(t)dt = \sum_{k\in\mathbb{N}} \int_{n_k-1}^{n_k} \frac{\rho(t)}{k^{2p}} dt \leq \sum_{k\in\mathbb{N}} \frac{e}{k^{2p}} < \infty.$$

Moreover,

$$\|T_s f\|_p^p \geq \int_{n_{k+1}-s-1}^{n_{k+1}-s} |f(s+t)|^p \rho(t) dt > \frac{e^k}{(k+1)^{2p}},$$

for any $s \in [k, k^2]$, $k \in \mathbb{N}$. Therefore, Theorem 4.8 yields that the translation semigroup is distributionally chaotic. On the other hand, it cannot be hypercyclic since $\rho(t) \geq 1$ for all $t \in \mathbb{R}^+$.

4.3 Backward shifts and translation C_0-semigroups

The final part of the chapter will be devoted to the interplay between the discrete case (backward shifts) and the continuous case (translation C_0-semigroup). We note that Theorem 2.9 was a result of this kind for distributional chaos in a general framework. During the recent years several results showing equivalences between analogous behaviour in the discrete and continuous cases have been obtained (see [CMP07] for hypercyclicity and [BBCP05] for the mixing property). In contrast, this equivalence does not necessarily hold neither for Devaney chaos [BB09], nor for hypercyclicity if we change the index semigroup [CMP07].

Proposition 4.12. *Let ρ be an admissible weight. There exists $\mathcal{K} \subset \mathbb{N}$ such that $\overline{\text{dens}}(\mathcal{K}) = 1$ and $\sum_{k \in \mathcal{K}} \rho(k) < \infty$, if and only if we can find a measurable $A \subset \mathbb{R}^+$ such that $\overline{\text{Dens}}(A) = 1$ and $\int_A \rho(t) dt < \infty$.*

Proof. Suppose that there exists $\mathcal{K} \subset \mathbb{N}$ with $\overline{\text{dens}}(\mathcal{K}) = 1$ and $\sum_{k \in \mathcal{K}} \rho(k) < \infty$. If we define $A := \bigcup_{k \in \mathcal{K}} [k-1, k]$, then $\overline{\text{Dens}}(A) = \overline{\text{dens}}(\mathcal{K}) = 1$, and by taking $l = 1$ in Lemma 4.2 we have that

$$\begin{aligned}\int_A \rho(t) dt &= \sum_{k \in \mathcal{K}} \int_{t \in [k-1,k]} \rho(t) dt \leq \sum_{k \in \mathcal{K}} \int_{t \in [k-1,k]} M_1 \rho(k) dt \\ &= \sum_{k \in \mathcal{K}} M_1 \rho(k) \int_{t \in [k-1,k]} dt = M_1 \sum_{k \in \mathcal{K}} \rho(k) < \infty.\end{aligned}$$

Conversely, if we can find an $A \subset \mathbb{R}^+$ such that $\overline{\text{Dens}}(A) = 1$ and $\int_A \rho(t) dt <$

∞, we define the set $\mathcal{K} = \{k \in \mathbb{N}\ ;\ \mu(A \cap [k, k+1]) > \frac{1}{2}\}$ and

$$\infty > \int_A \rho(t)dt \geq \sum_{k\in\mathcal{K}} \int_{A\cap[k,k+1]} \rho(t)dt$$
$$\geq \sum_{k\in\mathcal{K}} \int_{A\cap[k,k+1]} m_1\rho(k_n)dt = \sum_{k\in\mathcal{K}} m_1\rho(k) \int_{A\cap[k,k+1]} dt \geq \frac{m_1}{2} \sum_{k\in\mathcal{K}} \rho(k).$$

Therefore $\frac{m_1}{2} \sum_{k\in\mathcal{K}} \rho(k) < \infty$ and $\sum_{k\in\mathcal{K}} \rho(k) < \infty$.
By Lemma 2.8 we obtain that $\overline{\text{dens}}(\mathcal{K}) = 1$, which concludes the proof. ⌟

Corollary 4.13. *If we can find a subset $\mathcal{K} \subset \mathbb{N}$ such that $\overline{\text{dens}}(\mathcal{K}) = 1$ and $\sum_{k\in\mathcal{K}} \rho(k) < \infty$, then the translation semigroup $\{T_t\}_{t\geq 0}$ is densely distributionally chaotic.*

Proof. By the Proposition 4.12 if we have such a $\mathcal{K}$, we can find an $A \subset \mathbb{R}$ such that $\overline{\text{Dens}}(A) = 1$ and $\int_A \rho(t)dt < \infty$. Then we use Theorem 4.9 to finish the proof. ⌟

We define the weighted sequence space $\ell^p(v)$ as follows.

$$\ell^p(v) := \left\{ x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\ ;\ \|x\|_p := \sum_{n\in\mathbb{N}} |x_n|^p v_n < \infty \right\},$$

where the weight sequence $(v_n)_{n\in\mathbb{N}}$ is such that for every $n \in \mathbb{N}$, $\dfrac{v_n}{v_{n+1}} < \infty$.

Theorem 4.14. *Let $\rho : [0, +\infty[\longrightarrow \mathbb{R}^+$ be an admissible weight function such that the translation C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is distributionally chaotic on $L^p_\rho(\mathbb{R}^+)$. Then, for every sequence of weights $v = (v_n)_{n\in\mathbb{N}}$ such that there exist $0 < a < A < \infty$ with $a\rho(n-1) \leq v_n \leq A\rho(n)$, $n \in \mathbb{N}$, the backward shift B is distributionally chaotic on $\ell^p(v)$.*

Proof. Let S' be a distributionally scrambled set for $\mathcal{T}$. We pick $f \neq g \in S'$ and we define $x_n = \left(\int_n^{n+1} |f(t) - g(t)|^p dt\right)^{\frac{1}{p}}$. Let $x = (x_0, x_1, \ldots)$. Since ρ is an admissible weight function, there exist $m_1, M_1 > 0$ such that $m_1\rho(n) \leq \rho(t) \leq M_1\rho(n+1)$, for every $t \in [n, n+1]$, $n \geq 0$. Then,

$$\|x\|_p^p = \sum_{n=0}^{\infty} |x_n|^p v_n \leq \sum_{n=0}^{\infty} \left(\int_n^{n+1} |f(t) - g(t)|^p dt\right) A\rho(n)$$
$$\leq \frac{A}{m_1} \int_0^{\infty} |f(t) - g(t)|^p \rho(t)dt \leq \frac{A}{m_1} d(f, g)^p < \infty\,.$$

So we have that $x \in \ell^p(v)$. Since $\{T_t\}_{t\geq 0}$ is distributionally chaotic, T_1 is distributionally chaotic by Theorem 2.9. Then there exists $\delta > 0$ such that $\underline{\mathrm{dens}}\{k \leq n \ ; \ d(T_1^k f, T_1^k g) < \delta\} = 0$. We also have for each $k \in \mathbb{N}$ that

$$\begin{aligned}
\frac{a}{M_1} d(T_1^{k+2} f, T_1^{k+2} g)^p &= \frac{a}{M_1} \sum_{n=0}^{\infty} \int_n^{n+1} |f(t+k+2) - g(t+k+2)|^p \rho(t) dt \\
&\leq a \sum_{n=0}^{\infty} \left(\int_n^{n+1} |f(t+k+2) - g(t+k+2)|^p dt \right) \rho(n+1) \\
&= a \sum_{n=1}^{\infty} \left(\int_n^{n+1} |f(t+k+1) - g(t+k+1)|^p dt \right) \rho(n) \\
&\leq \sum_{n=1}^{\infty} \left(\int_n^{n+1} |f(t+k+1) - g(t+k+1)|^p dt \right) v_{n+1} \\
&= \sum_{n=2}^{\infty} \left(\int_{n+k}^{n+k+1} |f(s) - g(s)|^p ds \right) v_n \\
&= \sum_{n=2}^{\infty} |x_{n+k}|^p v_n \leq \|B^k x\|_p^p \,.
\end{aligned}$$

So that

$$\left| \left\{ k \leq n \ ; \ \|B^k x\|_p^p < \frac{a \cdot \delta^p}{M_1} \right\} \right| \leq \left| \left\{ k \leq n+2 \ ; \ d(T_1^k f, T_1^k g) < \delta \right\} \right| \,.$$

Therefore there exists $\delta' := \dfrac{\delta \cdot a^{1/p}}{M_1^{1/p}} > 0$, and $x \neq 0$ such that

$$\liminf_{n\to\infty} \frac{|\{k \leq n : \|B^k x\|_p < \delta'\}|}{n} = 0,$$

and B is distributionally chaotic by [MGOP09, Theorem 5]. ⌟

Remark 4.15. The scrambled set for the operator B in the previous theorem can be obtained as it is shown in the proof of [MGOP09, Theorem 5]. First of all, we can find increasing sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\left(\sum_{j=m_k}^{\infty} |x_j|^p v_j \right)^{1/p} < \frac{1}{2^k}, \quad \frac{m_k}{n_k} < \frac{1}{k}, \quad |\{s \leq n_k : \|B^s x\|_p < \delta\}| < \frac{n_k}{k}. \quad (4.3)$$

Now, let us define z as follows:

$$z_j := \begin{cases} (2+k)x_j, & m_k \leq j < m_{k+1}, \\ x_j, & 0 < j < m_0. \end{cases}$$

Using (4.3) we obtain that $z \in \ell^p(v)$. Given z, we can find a sufficiently fast increasing sequence $(q_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for $\bar{z}$ defined as

$$\bar{z}_j := \begin{cases} z_j, & q_{2k-1} \leq j < q_{2k},\ k \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

we have that each $k \in \mathbb{N}$ admits $j = j(k)$ so that $[m_j, n_j] \subset [q_{2k-1}, q_{2k}]$. These elements z and $\bar{z}$ verify that $d(B^s\bar{z}, B^s z) < \frac{\delta}{2}$, for all $s \in [m_j, n_j]$.

Therefore the scrambled set S for the backward shift in the previous theorem can be chosen as $S := \{z_\alpha := \alpha\bar{z} \text{ and } \alpha \in [0,1]\}$.

Theorem 4.16. *Let $v = (v_n)_{n\in\mathbb{N}}$ be a sequence of positive weights such that the backward shift B is distributionally chaotic on $\ell^p(v)$. Then for every admissible weight function ρ for which there are $0 < a < A < \infty$ satisfying $av_n \leq \rho(t) \leq Av_{n+1}$ for every $t \in [n, n+1]$, the translation C_0-semigroup is distributionally chaotic on $L^p_\rho(\mathbb{R}^+)$.*

Proof. Let S be the scrambled set for B. For every $x = (x_0, x_1, x_2, \ldots) \in S$ we can associate a function $f_x = \sum_{n=0}^\infty x_{n+2}\chi_{[n,n+1[}$, which verifies that T_1 acts on f_x as the backward shift does on the sequence x. Clearly, this function f_x is in $L^p_\rho(\mathbb{R}^+)$, since

$$\int_0^\infty |f(t)|^p \rho(t)dt \leq A\sum_{n=0}^\infty \int_n^{n+1} |x_{n+2}|^p v_{n+1} dt \leq AM\sum_{n=0}^\infty |x_{n+2}|^p v_{n+2} < \infty,$$

where $\dfrac{v_n}{v_{n+1}} \leq M < \infty$ for all $n \in \mathbb{N}$ by the definition of positive weight.

Let $x, y \in S$ and f_x, f_y be the corresponding elements in $L^p_\rho(\mathbb{R}^+)$. We have

$$\begin{aligned}
d(B^{k+2}x, B^{k+2}y)^p &= \sum_{n=0}^{\infty} |x_{n+k+2} - y_{n+k+2}|^p v_n \\
&= \sum_{n=0}^{\infty} \int_n^{n+1} |f_x(t+k) - f_y(t+k)|^p v_n dt \\
&\le \frac{1}{a} \sum_{n=0}^{\infty} \int_n^{n+1} |\mathcal{T}_1^k f_x(t) - \mathcal{T}_1^k f_y(t)|^p \rho(t) dt \\
&= \frac{1}{a} d(T_1^k f_x, T_1^k f_y)^p \\
&= \frac{1}{a} \sum_{n=0}^{\infty} \int_n^{n+1} |f_x(t+k) - f_y(t+k)|^p \rho(t) dt \\
&\le \frac{A}{a} \sum_{n=0}^{\infty} \int_n^{n+1} |f_x(t+k) - f_y(t+k)|^p dt \; v_{n+1} \\
&= \frac{A}{a} \sum_{n=0}^{\infty} |x_{n+k+2} - y_{n+k+2}| v_{n+1} \\
&\le \frac{A}{a} d(B^{k+1}x, B^{k+1}y)^p .
\end{aligned}$$

Reasoning as in the proof of Theorem 4.14, we obtain that there exists a $\delta > 0$ such that for every $f_x, f_y \in \{f_x \; ; \; x \in S\}$ with $f_x \neq f_y$ we have

$$\underline{\text{dens}}(\{k \in \mathbb{N} \; ; \; d(T_1^k f_x, T_1^k f_y) < \delta\}) = 0.$$

Then, since

$$|\{k \le n+1 \; ; \; d(B^k x, B^k y) < \varepsilon\}| \le |\{k \le n \; ; \; d(T_1^k f_x, T_1^k f_y) < A^{1/p}\varepsilon\}|,$$

and B is distributionally chaotic with respect to the scrambled set S, we get that

$$\overline{\text{dens}}(\{k \in \mathbb{N} \; ; \; d(T_1^k f_x, T_1^k f_y) < \varepsilon\}) = 1.$$

and therefore we conclude that T_1 is distributionally chaotic with a scrambled set $\{f_x \; ; \; x \in S\}$. ⌟

Remark 4.17. Obviously there exist weights as the ones in Theorems 4.14 and 4.16. In Theorem 4.14 we can define the weights $v_n := \rho(n)$ for each $n \in \mathbb{N}$. In Theorem 4.16 we can take for instance the polygonal formed by the sequence v as an admissible weight function.

Chapter 5

Completely distributionally chaotic C_0-semigroups

In [MGOP12], Martínez-Giménez et al. give an example of a completely distributionally chaotic operator, consequently proving that there exist cases in which the space X is a scrambled set. They also provide an example of a non hypercyclic completely distributionally chaotic operator, as well as sufficient conditions for the bilateral forward and backward shifts for being completely distributionally chaotic.

In this chapter, we will broaden the results from [MGOP12], exporting them to the C_0-semigroup setting, providing a detailed proof. With this motivation, we begin by looking for the existence of completely distributionally irregular C_0-semigroups .

Theorem 5.1. *Let $\rho : \mathbb{R} \to \mathbb{R}$ be an admissible weight in the sense of [DSW97] that satisfies additionally the following conditions:*

(i) there are sequences of integers $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, such that the supremum of the slope of ρ outside the interval $[m_{-k}, m_{k-1}]$ satisfies (for every $k \in \mathbb{N}$):

$$S_k := \sup\left\{ \frac{\rho(t)}{\rho(t-1)} \ ; \ t \notin [m_{-k}+1, m_{k-1}] \right\} \in]1, +\infty[,$$

(ii) there exists $D > 1$ such that $D\rho(m_{-j}) \geq \rho(s)$ for every $s \in [m_{-j}, m_{j-1}]$ and that, for every $\varepsilon > 0$, we find $k \in \mathbb{N}$ with $\rho(n_k) < \varepsilon$ and

$$S_k^{k(n_k - m_{-k})} \leq \min\left\{D, \frac{\min\{\rho(s);\ m_{-k} \leq s \leq m_{k-1}\}}{\rho(n_k)}\right\},$$

(iii) for every $N \in \mathbb{N}$, we can find $k \in \mathbb{N}$ such that $\rho(s) > N$, for $k \leq s \leq Nk$.

Then the forward translation semigroup $\mathcal{F} = (F_t)_{t\geq 0}$ on $L^p_\rho(\mathbb{R})$, which is defined by $(F_t f)(x) = f(x-t)$, $t \geq 0$, $x \in \mathbb{R}$, is completely distributionally irregular.

Proof. We first outline the proof. We will show that for every non-zero $f \in L^p_\rho(\mathbb{R})$, $\delta > 0$, and $l \in [n_k - m_{-k}, k(n_k - m_{-k})]$, with $k \in \mathbb{N}$ satisfying condition (ii), we obtain $\|F_l x\|^p < \delta$ (Steps 1 and 2). Later, we will construct the sets of upper density 1, A and B, that appear in the statement of Proposition 2.15, thus proving that f is distributionally irregular for $(F_t)_{t\geq 0}$ (Step 3).

Let $f \in L^p_\rho(\mathbb{R})$ be an arbitrary non-zero vector. Let $M, \omega \in \mathbb{R}$, with $M \geq 1$, be such that $\rho(t) \leq Me^{\omega\tau}\rho(t+\tau)$. Define $C := M\max\{1, e^\omega\}$. Given $D > 1$ satisfying condition *(ii)*, and an arbitrary $\delta > 0$, we fix $m \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus[-m,m]} |f(x)|^p \rho(x)dx < \frac{\delta}{2CD^2}.$$

Next we take $k \in \mathbb{N}$ satisfying condition *(ii)* with $\varepsilon := \delta\dfrac{\min\{\rho(s);\ |s| \leq m\}}{2CD\,\|f\|_p^p}$, and such that $[-m, m] \subset]m_{-k}, m_{k-1}[$.

Step 1. For any $l \in [n_k - m_{-k}, k(n_k - m_{-k})] \cap \mathbb{N}$ we have

$$\begin{aligned}\|F_l f\|_p^p &= \int_{\mathbb{R}} |f(y-l)|^p \rho(y)dy = \int_{\mathbb{R}} |f(x)|^p \rho(x+l)dx \\ &= \int_{\mathbb{R}} |f(x)|^p \rho(x)\frac{\rho(x+l)}{\rho(x)}dx,\end{aligned}$$

which can be split into the following three integrals:

$$\begin{aligned}&\int\limits_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p\rho(x)\frac{\rho(x+l)}{\rho(x)}dx + \int\limits_{x\notin[m_{-k},m_{k-1}]} |f(x)|^p\rho(x)\frac{\rho(x+l)}{\rho(x)}dx \\ &+ \int\limits_{x\in[-m,m]} |f(x)|^p\rho(x)\frac{\rho(x+l)}{\rho(x)}dx.\end{aligned} \tag{5.1}$$

Step 1.1. We start with the first integral from (5.1),

$$\int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p \rho(x)\frac{\rho(x+l)}{\rho(x)}dx \leq$$

$$\leq \int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p \rho(x)dx \max\left\{\frac{\rho(x+l)}{\rho(x)};\ x\in[m_{-k},m_{k-1}]\right\} = (*).$$

It is clear that the second factor from $(*)$ can be bounded by the following inequality

$$\max\left\{\frac{\rho(x+l)}{\rho(x)};\ x\in[m_{-k},m_{k-1}]\right\} \leq \frac{\max\{\rho(x+l);\ x\in[m_{-k},m_{k-1}]\}}{\min\{\rho(x);\ x\in[m_{-k},m_{k-1}]\}}.$$

For every $x\in[m_{-k},m_{k-1}]$ we have that

$$\rho(x+l) = \rho(n_k)\frac{\rho(x+l)}{\rho(n_k)} = \rho(n_k)\frac{\rho(x+l)}{\rho([x+l]+1)}\prod_{i=n_k}^{[x+l]}\frac{\rho(i+1)}{\rho(i)}.$$

Therefore

$$(*) \leq \frac{CS_k^l}{\frac{\min\{\rho(x);\ x\in[m_{-k},m_{k-1}]\}}{\rho(n_k)}} \int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p\rho(x)dx \leq$$

$$\leq C\int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p\rho(x)dx. \tag{5.2}$$

Step 1.2. For the second integral from (5.1) we will decompose the complementary of $[m_{-k},m_{k-1}]$ as follows:

$$\mathbb{R}\setminus[m_{-k},m_{k-1}] =$$
$$=]-\infty,m_{-k}-l[\cup[m_{-k}-l,m_{k-1}-l[\cup[m_{k-1}-l,m_{-k}[\cup]m_{k-1},\infty[.$$

For x in the intervals $]-\infty,m_{-k}-l[$ and $]m_{k-1},\infty[$, $x+l\notin]m_{-k},m_{k-1}]$, therefore, by condition *(i)* we obtain

$$\int_{\substack{]m_{k-1},\infty[\cup\\ ]-\infty,m_{-k}-l[}} |f(x)|^p\rho(x)\frac{\rho(x+l)}{\rho(x)}dx \leq$$

$$\leq \int_{\substack{]m_{k-1},\infty[\cup\\ ]-\infty,m_{-k}-l[}} |f(x)|^p\rho(x)\prod_{i=1}^{l}\frac{\rho(x+i)}{\rho(x+i-1)}dx \leq \int_{\substack{]m_{k-1},\infty[\cup\\ ]-\infty,m_{-k}-l[}} |f(x)|^p\rho(x)S_k^l.$$

For x in the interval $[m_{-k}-l, m_{k-1}-l[$, it follows from $x+l \in [m_{-k}, m_{k-1}[$ that condition *(ii)* yields $\dfrac{\rho(x+l)}{\rho(m_{-k})} \leq D$. From the election of l, we have that x is at most $m_{-k}+m_{k-1}-n_k$, so $[x] \leq x < m_{-k}$. Consequently, by condition *(i)*,

$$\frac{\rho(m_{-k})}{\rho([x])} = \prod_{i=1}^{m_{-k}-[x]} \frac{\rho([x]+i)}{\rho([x]+i-1)} \leq S_k^{m_{-k}-[x]} < S_k^l.$$

Thus, by the definition of C we get

$$\begin{aligned}\int_{m_{-k}-l}^{m_{k-1}-l} & |f(x)|^p \rho(x) \frac{\rho(x+l)}{\rho(x)} dx = \\ &= \int_{m_{-k}-l}^{m_{k-1}-l} |f(x)|^p \rho(x) \frac{\rho([x])}{\rho(x)} \frac{\rho(m_{-k})}{\rho([x])} \frac{\rho(x+l)}{\rho(m_{-k})} dx \\ &\leq \int_{m_{-k}-l}^{m_{k-1}-l} |f(x)|^p \rho(x) C S_k^l D dx,\end{aligned}$$

We can proceed analogously to bound the integral on the interval $[m_{k-1}-l, m_{-k}[$. With the following decomposition

$$\frac{\rho(x+l)}{\rho(x)} = \frac{\rho([x])}{\rho(x)} \frac{\rho(m_{-k})}{\rho([x])} \frac{\rho(x+m_{k-1}-([x]+1))}{\rho(m_{-k})} \frac{\rho(x+l)}{\rho(x+m_{k-1}-([x]+1))}$$

it is easily seen that

$$\begin{aligned}\int_{m_{k-1}-l}^{m_{-k}} & |f(x)|^p \rho(x) \frac{\rho(x+l)}{\rho(x)} dx \leq \\ &\leq \int_{m_{k-1}-l}^{m_{-k}} |f(x)|^p \rho(x) C S_k^{m_{-k}-[x]} D S_k^{l-m_{k-1}+[x]+1} dx \\ &\leq \int_{m_{k-1}-l}^{m_{-k}} |f(x)|^p \rho(x) dx \left(CDS_k^l\right).\end{aligned}$$

In consequence, since by condition *(ii)*, $S_k^l \leq D$ and, as $D, C \geq 1$, we have

$$\int\limits_{x \notin [m_{-k}, m_{k-1}]} |f(x)|^p \rho(x) \frac{\rho(x+l)}{\rho(x)} dx \leq \int\limits_{x \notin [m_{-k}, m_{k-1}]} |f(x)|^p \rho(x) dx \left(CD^2\right). \tag{5.3}$$

Step 1.3. To bound the third integral from (5.1), using condition *(ii)* and by the definition of ε, we get

$$\begin{aligned}\int_{-m}^{m} |f(x)|^p \rho(x)\frac{\rho(x+l)}{\rho(x)}dx &\leq \int_{-m}^{m} |f(x)|^p \rho(x)\frac{\rho(n_k) C S_k^{k(n_k - m_{-k})}}{\min\{\rho(s);\ |s| \leq m\}}dx \\ &< \frac{\|f\|_p^p C D}{\min\{\rho(s);\ |s| \leq m\}}\varepsilon \leq \frac{\delta}{2}.\end{aligned}$$

Step 1.4. We combine (5.2) and (5.3), thus bounding the first two integrals from (5.1). Since $\mathbb{R} \setminus [-m, m] = ([m_{-k}, m_{k-1}] \setminus [-m, m]) \cup (\mathbb{R} \setminus [m_{-k}, m_{k-1}])$, and we see that

$$\begin{aligned}&\int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p \rho(x)\frac{\rho(x+l)}{\rho(x)}dx + \int_{x\notin[m_{-k},m_{k-1}]} |f(x)|^p \rho(x)\frac{\rho(x+l)}{\rho(x)}dx \\ &\quad\leq C\int_{\substack{x\in[m_{-k},m_{k-1}]\\ x\notin[-m,m]}} |f(x)|^p \rho(x)dx + CD^2\int_{x\notin[m_{-k},m_{k-1}]} |f(x)|^p \rho(x)dx \\ &\quad\leq C\int_{x\notin[-m,m]} |f(x)|^p \rho(x)dx + (D^2-1)C\int_{x\notin[m_{-k},m_{k-1}]} |f(x)|^p \rho(x)dx \\ &\qquad\qquad \leq CD^2\int_{x\notin[-m,m]} |f(x)|^p \rho(x)dx < \frac{\delta}{2},\end{aligned}$$

by the selection of m.

So that, with the estimation obtained in Step 1.3, it follows that

$$\|F_l f\|_p^p = \int_{\mathbb{R}} |f(y-l)|^p \rho(y)dy < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Step 2. If $l \in [n_k - m_{-k}, k(n_k - m_{-k})] \setminus \mathbb{N}$, then the definition of C, which is nothing more than local equicontinuity, gives

$$\begin{aligned}\|F_l f\|_p^p &= \int_{\mathbb{R}} |f(y-l)|^p \rho(y)dy = \int_{\mathbb{R}} |f(x)|^p \rho(x+l)dx = \\ &= C\int_{\mathbb{R}} |f(x)|^p \rho(x+[l]+1)dx.\end{aligned}$$

As $[l]+1 \in [n_k - m_{-k}, k(n_k - m_{-k})] \cap \mathbb{N}$, we can proceed in the same manner as Step 1, and so $\|F_l f\|_p^p < C\delta$.

Step 3. Now we are going to construct the sets A and B of upper density 1 that appear in the statement of Proposition 2.15, thus getting that f is a distributionally irregular vector for $\mathcal{F}$.

Step 3.1 Since we have proved that for any arbitrary $\delta > 0$ we can construct a sequence decreasing to zero $(\delta_j)_{j\in\mathbb{N}}$ such that $\|F_l f\|_p^p < \delta_j$ for every $j \in \mathbb{N}$, and we define the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ by $\varepsilon_j := \delta_j \dfrac{\min\{\rho(s);\ |s| \le m\}}{2CD\,\|f\|_p^p}$.
By condition *(ii)*, we obtain a sequence of $(k_j)_{j\in\mathbb{N}}$ such that $\rho(n_{k_j}) < \varepsilon_j$.
If necessary, we can consider this sequence to be strictly increasing. We define the set $A \subset \mathbb{R}_0^+$ as follows:

$$A := \bigcup_{i\in\mathbb{N}} [n_{k_i} - m_{-k_i}, k_i(n_{k_i} - m_{-k_i})].$$

If we set $(t_i)_{i\in\mathbb{N}} := (k_i(n_{k_i} - m_{-k_i}))_{i\in\mathbb{N}}$, we have that for any $i \in \mathbb{N}$

$$\begin{aligned}\frac{\mu(A \cap [0, k_i(n_{k_i} - m_{-k_i})])}{t_i} &\ge \frac{\mu([n_{k_i} - m_{-k_i}, k_i(n_{k_i} - m_{-k_i})])}{t_i}\\ &= \frac{(k_i - 1)(n_{k_i} - m_{-k_i})}{k_i(n_{k_i} - m_{-k_i})} = \frac{(k_i - 1)}{k_i}.\end{aligned}$$

Consequently,

$$\lim_{\substack{i\to\infty\\ i\in\mathbb{N}}} \frac{\mu(A \cap [0, t_i])}{t_i} \ge \lim_{\substack{k_i\to\infty\\ i\in\mathbb{N}}} \frac{k_i - 1}{k_i} = 1.$$

Therefore we have that $\overline{\mathrm{Dens}}(A) = 1$ and $\lim\limits_{\substack{s\to\infty\\ s\in A}} \|F_s f\|_p = 0$.

Step 3.2 Now, since $f \neq 0$, there are $i_0 \in \mathbb{R}$, $\tau \in \mathbb{R}^+$ such that $\int_{i_0}^{i_0+\tau} |f(s)|^p ds \neq 0$.
By condition *(iii)*, given any $N \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $\rho(s) > N$ for all $s \in [k, Nk]$. Since $\sup_{s\in[k,Nk]} \rho(s) < \infty$, without loss of generality we may assume that $k > i_0$. By the arbitrarity of N we can construct sequences $(N_j)_{j\in\mathbb{N}}$, $(k_j)_{j\in\mathbb{N}}$ tending to infinity such that $\rho(s) > N_j$ for all $s \in [k_j, N_j k_j]$.
We define the set $B = \bigcup_{j\in\mathbb{N}} [k_j - i_0, k_j N_j - (i_0 + \tau)] \subset \mathbb{R}$. Let us see that $\overline{\mathrm{Dens}}(B) = 1$: Take the sequence $(t_j)_{j\in\mathbb{N}} := (k_j N_j - (i_0 + \tau))_{j\in\mathbb{N}}$.
For every $j \in \mathbb{N}$ we have

$$\begin{aligned}\frac{\mu(B \cap [0, k_j N_j - (i_0 + \tau)]}{t_j} &\ge \frac{\mu([k_j - i_0, k_j N_j - (i_0 + \tau)]}{t_j}\\ &= \frac{k_j\left(N_j - \dfrac{\tau}{k_j} - 1\right)}{k_j N_j - (i_0 + \tau)} = \frac{N_j - \dfrac{\tau}{k_j} - 1}{N_j - \dfrac{(i_0 + \tau)}{k_j}},\end{aligned}$$

and hence

$$\lim_{j\in\mathbb{N}} \frac{\mu(B\cap[0,t_j)])}{t_j} \geq \lim_{j\in\mathbb{N}} \frac{N_j - \dfrac{\tau}{k_j} - 1}{N_j - \dfrac{(i_0+\tau)}{k_j}} = 1.$$

Therefore $\lim\limits_{\substack{s\to\infty\\ s\in B}} \|F_s f\|_p = \infty$ and hence f is distributionally irregular for $(F_t)_{t\geq 0}$. ⌟

Remark 5.2. We can obtain an analogous result for the translation semigroup considering the isometry $\varphi : L^p_\rho(\mathbb{R}) \longrightarrow L^p_{\varphi(\rho)}(\mathbb{R})$, where $\varphi(f(x)) = f(-x)$. Note that with φ, the following diagram commutes:

$$\begin{tikzcd} L^p_\rho(\mathbb{R}) \arrow[r, "T_t"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & L^p_\rho(\mathbb{R}) \arrow[d, "\varphi"] \\ L^p_{\varphi(\rho)}(\mathbb{R}) \arrow[r, "F_t"'] & L^p_{\varphi(\rho)}(\mathbb{R}), \end{tikzcd} \tag{5.4}$$

therefore φ conjugates $\mathcal{F}$ to $\mathcal{T}$.

Corollary 5.3. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be and admissible weight that satisfies the following conditions:*

(i) There are sequences of integers $(u_j)_{j\in\mathbb{Z}}$ and $(v_j)_{j\in\mathbb{Z}}$ with $v_j < u_j < v_{j+1}$, $j \in \mathbb{Z}$, such that the infimum of the slope of ρ outside the interval $[u_{-k}, u_{k-1}]$ satisfies (for every $k \in \mathbb{N}$):

$$s_k := \inf\left\{\frac{\psi(t+1)}{\psi(t)}\ ;\ t \notin [u_{-k}, u_{k-1}-1]\right\} \in]0,1[,$$

(ii) there exists $D > 1$ such that $D\psi(u_{j-1}) \geq \psi(r)$ for every $r \in [u_{-j}, u_{j-1}]$ and that, for every $\varepsilon > 0$, we find $k \in \mathbb{N}$ with $\psi(v_{-k}) < \varepsilon$ and

$$s_k^{k(v_{-k}-u_{k-1})} \leq \min\left\{D, \frac{\min\{\psi(r);\ u_{-k} \leq r \leq u_{k-1}\}}{\psi(v_{-k})}\right\}, \tag{5.5}$$

(iii) for every $N \in \mathbb{N}$, we can find $k \in \mathbb{N}$ such that $\psi(r) > N$, for $-Nk \leq r \leq -k$.

Then the translation semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ on $L^p_\psi(\mathbb{R})$ is completely distributionally irregular.

Proof. Note that $\varphi^{-1} = \varphi$. Take $\rho := \varphi(\psi)$, $u_k = -m_{(-k)-1}$ and $v_k = -n_{-k}$ for every $k \in \mathbb{Z}$. Then $n_j < m_j < n_{j+1}$, for $j \in \mathbb{Z}$ and, since

$$\begin{aligned}
\inf\left\{\frac{\psi(t+1)}{\psi(t)}\ ;\ t \notin [u_{-k}, u_{k-1}-1]\right\} = \\
&= \inf\left\{\frac{\psi(t+1)}{\psi(t)}\ ;\ t \notin [-m_{k-1}, -m_{-k}-1]\right\} \\
&= \inf\left\{\frac{\psi(-r+1)}{\psi(-r)}\ ;\ -r \notin [-m_{k-1}, -m_{-k}-1]\right\} \\
&= \inf\left\{\frac{\psi(-r+1)}{\psi(-r)}\ ;\ r \notin [m_{-k}+1, m_{k-1}]\right\} \\
&= \inf\left\{\frac{\rho(r-1)}{\rho(r)}\ ;\ r \notin [m_{-k}+1, m_{k-1}]\right\} \\
&= \inf\left\{\frac{1}{\frac{\rho(r)}{\rho(r-1)}}\ ;\ r \notin [m_{-k}+1, m_{k-1}]\right\} \\
&= \frac{1}{\sup\left\{\frac{\rho(r)}{\rho(r-1)}\ ;\ r \notin [m_{-k}+1, m_{k-1}]\right\}},
\end{aligned}$$

we have that $s_k = S_k^{-1}$ and if $s_k \in]0,1[$, then $S^k \in]1, +\infty[$.

By assumption there exists $D > 1$ such that $D\psi(u_{j-1}) \geq \psi(r)$ for every $r \in [u_{-j}, u_{j-1}]$. By the definition of ρ, we have that $D\rho(-u_{j-1}) \geq \rho(-r)$ for every $-r \in [-u_{j-1}, -u_{-j}]$. Replacing u_j by $-m_{(-j)-1}$ and r by $-s$, we obtain that $D\rho(m_{-k}) \geq \rho(s)$ for every $s \in [m_{-j}, m_{j-1}]$.
Obviously, $\rho(n_k) = \psi(v_{-k})$ and therefore for every ε there is a k such that $\rho(n_k) < \varepsilon$. We can repeat the same substitutions in (5.5). First we take $\rho := \varphi(\psi)$, $u_k = -m_{(-k)-1}$ and $v_k = -n_{-k}$ for every $k \in \mathbb{Z}$, so from

$$s_k^{k(v_{-k}-u_{k-1})} \leq \min\left\{D, \frac{\min\{\psi(r);\ u_{-k} \leq r \leq u_{k-1}\}}{\psi(v_{-k})}\right\},$$

we obtain

$$s_k^{k(-n_k+m_{-k})} \leq \min\left\{D, \frac{\min\{\rho(-r);\ -m_{k-1} \leq -r \leq -m_{-k}\}}{\rho(n_k)}\right\}.$$

Then we replace r by $-s$, thus getting

$$s_k^{-k(n_k-m_{-k})} \leq \min\left\{D, \frac{\min\{\rho(s);\ m_{-k} \leq s \leq m_{k-1}\}}{\rho(n_k)}\right\}.$$

Finally, since $s_k = S_k^{-1}$ we conclude that

$$S_k^{k(n_k-m_{-k})} \leq \min\left\{D, \frac{\min\{\rho(s);\ m_{-k} \leq s \leq m_{k-1}\}}{\rho(n_k)}\right\}.$$

It suffices to make the following observation. If for every $N \in \mathbb{N}$, we can find $k \in \mathbb{N}$ such that $\psi(r) > N$, for $-Nk \leq r \leq -k$, the definition of ρ and taking $s = -r$ yield that $\rho(s) > N$ for $k \leq s \leq Nk$; which completes the proof since ρ satisfies the conditions of Theorem 5.1. ⌟

For the following example, the sequences $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$ are such that for every $t \in \mathbb{R}$ we have:

$$\rho(t-1) \leq \rho(t) \quad \text{when } n_k < t \leq m_k, \quad \text{and} \quad \rho(t-1) \geq \rho(t) \ \text{ when } m_k < t \leq n_{k+1}.$$

Example 5.4. *We will choose ρ such that $\mathcal{F}$ and $\mathcal{T}$ are completely distributionally irregular on $L^p_\rho(\mathbb{R})$ but $\mathcal{F}$ is not hypercyclic. First, we put some general conditions which will lead to inductively construct sequences of integers $(m_k)_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$ with the desired properties. We will require that sequences $(m_k)_{k\in\mathbb{Z}}$, $(n_k)_{k\in\mathbb{Z}}$ increase fast enough so that they satisfy the following conditions (Fig. 5.1):*

a) $m_0 = 1$, $n_1 = e^2$, $\rho(m_0) = e$, $\rho(n_1) = e^{-2}$,

b) $\rho(n_k) = e^{-2k}$, $\rho(m_k) = e^{2k+1}$, $k \in \mathbb{N}$, $\rho(s)/\rho(s-1) = \rho(t)/\rho(t-1)$ *if* $s, t \in [n_k + 1, m_k]$, *or if* $s, t \in [m_{k-1} + 1, n_k]$, $k \in \mathbb{N}$,

c) $m_k - n_k > 2(m_{k-1} - n_{k-1})$, $n_{k+1} - m_k > 2(n_k - m_{k-1})$, $k \in \mathbb{N}$, *and*

d) $\rho(-t) = \rho(t)^{-1}$, $t \in \mathbb{R}^+$, $m_k = -n_{-k}$, $k \in \mathbb{Z}$.

We will define our weight function as follows:

$$\rho(t) := \begin{cases} e^{-2k} e^{\frac{(t-n_k)(4k+1)}{m_k-n_k}}, & \text{for } t \in [n_k, m_k]; \\ e^{2k+1} e^{\frac{(t-m_k)(4k+3)}{n_{k+1}-m_k}}, & \text{for } t \in [m_k, n_{k+1}]. \end{cases}$$

Now we have two goals: to check if this weight function satisfies the conditions in Theorem 5.1 and Corollary 5.3, and to be able of construct the sequences $(m_k)_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$.

Observe that condition (b) yields that, for every $k \in \mathbb{Z}$, $\rho(n_k)$ will be a local minimum, $\rho(m_k)$ will be a local maximum, and therefore, $\rho(t)$ will increase from n_k to m_k and decrease from m_k to n_{k+1}. Then (d) gives that

$$\frac{\min\{\rho(s);\ m_{-k} \leq s \leq m_{k-1}\}}{\rho(n_k)} = \frac{\rho(n_{-k+1})}{\rho(n_k)} = e$$

for every $k \in \mathbb{N}$.

By condition (c) the slope of ρ in the interval $[n_{k+1}, m_{k+1}]$ is

$$\left(\frac{\rho(m_{k+1})}{\rho(n_{k+1})}\right)^{\frac{t-n_{k+1}}{m_{k+1}-n_{k+1}}} = e^{\frac{4k+5}{m_{k+1}-n_{k+1}}} < e^{\frac{4k+5}{2(m_k-n_k)}} < e^{\frac{4k+1}{m_k-n_k}}.$$

that is the slope in the interval $[n_k, m_k]$. So the supremum of the slope of ρ outside the interval $[m_{-k}, m_{k-1}]$ is $S_k = \rho(t)/\rho(t-1)$ for any $n_k + 1 \leq t \leq m_k$, $k \in \mathbb{N}$. In order to fulfil conditions (i) and (ii) in Theorem 5.1, we set $D = e$ and $S_k = e^{1/k(n_k - m_{-k})} = e^{1/2kn_k}$, $k \in \mathbb{N}$. Consequently we get $S_k^{m_k - n_k} = \rho(m_k)/\rho(n_k) = e^{4k+1}$, which implies $m_k = (8k^2 + 2k + 1)n_k$, $k \in \mathbb{N}$. Analogously, $\dfrac{\min\{\rho(s);\ m_{-k} \leq s \leq m_k\}}{\rho(n_{-k})} = \dfrac{\rho(n_k)}{\rho(n_{-k})} = e$ for every $k \in \mathbb{N}$, and the infimum of the slope outside the interval $[m_{-k}, m_{k-1}]$ is $s_k = \dfrac{\rho(t)}{\rho(t-1)} = e$ for every $m_k + 1 \leq t \leq n_{k+1}$, $k \in \mathbb{N}$. Again to have condition (i) and (ii) in Corollary 5.3, we set $D = e$, $s_k = e^{1/k(n_{-k} - m_k)} = e^{1/(2km_k)}$, $k \in \mathbb{N}$. Thus $s_k^{n_{k+1} - m_k} = \dfrac{\rho(n_{k+1})}{\rho(m_k)} = e^{-4k-3}$, and hence we get $n_{k+1} = (8k^2 + 6k + 1)m_k$, $k \in \mathbb{N}$. This allows us to construct inductively the sequences $(m_k)_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$. Now it is easy to check that taking $M, \omega = 1$, ρ is an admissible weight function.

To check condition (iii) in Theorem 5.1, we observe that

$$\rho(t) = e^{2k+1} s_k^{t-m_k} = e^{2k+1} e^{-\frac{t-m_k}{2km_k}} > e^{2k} \qquad \text{if } m_k \leq t \leq 2km_k.$$

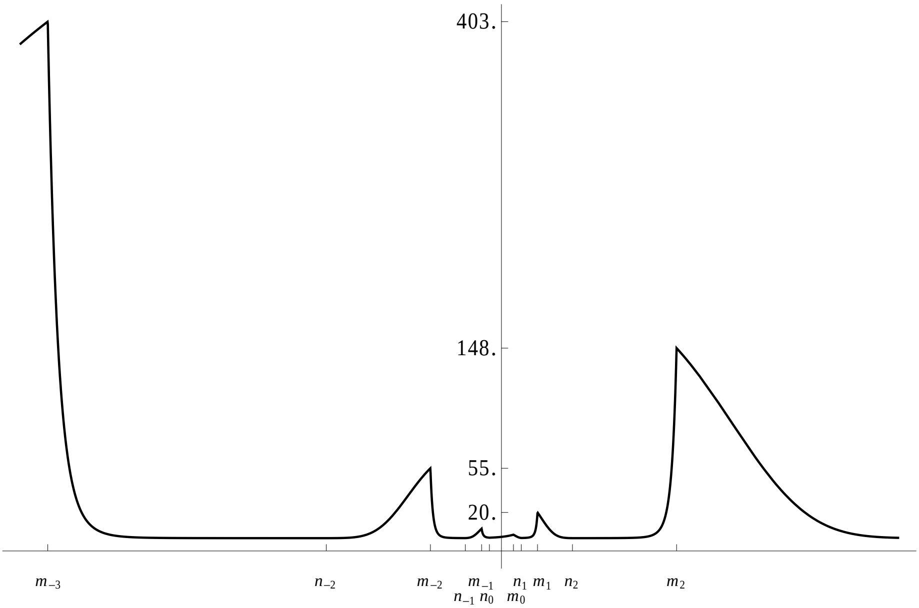

Figure 5.1: Example 5.4

To check condition (iii) in Corollary 5.3, we first note that if $t \in [n_{-k}, m_{-k}]$, then $\rho(t) = \rho(j)^{-1}$ for $-t = j \in [n_k, m_k]$, so

$$\rho(j) = e^{-2k} S_k^{j-n_k} = e^{-2k} e^{\frac{j-n_k}{2kn_k}} \leq e^{-2k} e^{\frac{(2k-1)n_k}{2kn_k}} \qquad \text{if } n_k \leq j \leq 2kn_k.$$

Therefore

$$\rho(t) \geq e^{2k-1} e^{\frac{1}{2k}} > e^{2k-1} \qquad \text{if } 2km_{-k} \leq t \leq m_{-k}.$$

This implies that all the conditions in Theorem 5.1 and Corollary 5.3 are satisfied and so $\mathcal{F}$ and $\mathcal{T}$ are completely distributionally irregular. Finally, since $\rho(t) = \rho(-t^{-1})$ for every $t \in \mathbb{R}^+$, there is no increasing sequence $(t_j)_{j \in \mathbb{N}}$ tending to ∞ such that

$$\lim_{j \to \infty} \rho(t_j) = \lim_{j \to \infty} \rho(-t_j) = 0.$$

Therefore it cannot be hypercyclic, by Theorem 4.4.

Chapter 6

Examples of distributionally chaotic C_0-semigroups associated to partial differential equations

6.1 Distributionally chaotic C_0-semigroups

In this section we consider several examples of C_0-semigroups that are already known to be Devaney chaotic and we will study when they exhibit distributional chaos. These examples will be considered on the following spaces:

$$L^p_\rho(I,\mathbb{C}) = \left\{ f \in \mathcal{M}(I,\mathbb{C}) \ ; \ \|f\|_{p,\rho} = \left(\int_I |f(s)|^p \rho(s) ds \right)^{1/p} < \infty \right\},$$

with $1 \leq p < \infty$, where I is an interval on $\mathbb{R}$ and ρ a weight function. If $\rho(x) = 1$, then we will simply denote it as $L^p(I,\mathbb{C})$, $1 \leq p < \infty$. The hypothesis on ρ may be different on each example.

In [Tak05] Takeo considered the following first order abstract Cauchy problem on $L^p(I,\mathbb{C})$, $1 \leq p < \infty$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \zeta(x) \dfrac{\partial u}{\partial x} + h(x)u, \\ u(0,x) = f(x), \quad x \in I, \end{cases} \tag{6.1}$$

where ζ and h are bounded continuous functions defined on I. This partial differential equation has been used to model the dynamics of a population of cells under simultaneous proliferation and maturation [LM94]. When $\zeta(x)$ is constant and equal to 1 and $I = \mathbb{R}_0^+$, the solution C_0-semigroup $\{T_t\}_{t\geq 0}$ of (6.1) is defined as

$$T_t f(x) = \exp\left(\int_x^{x+t} h(s)ds\right) f(x+t), \text{ for all } x, t \geq 0, f \in L^p(\mathbb{R}_0^+, \mathbb{C}). \quad (6.2)$$

Theorem 6.1. *If $h(x)$ is a real function and there is a measurable set $B \subset \mathbb{R}_0^+$ such that $\overline{\mathrm{Dens}}(B) = 1$ and $\int_B \exp\left(-p\int_0^x h(s)ds\right) dx < \infty$, then the C_0-semigroup $\{T_t\}_{t\geq 0}$ defined in (6.2) is distributionally chaotic on $L^p(\mathbb{R}_0^+, \mathbb{C})$, $1 \leq p < \infty$.*

Proof. If we define $\rho(x) = \exp(-p\int_0^x h(s)ds)$, then the operators of $\{T_t\}_{t\geq 0}$ can be rewritten as

$$T_t f(x) = (\rho(x)/\rho(x+t))^{1/p} f(x+t).$$

This function $\rho(x)$ is an admissible weight function in the sense of [DSW97, Def. 4.1], which ensures that the left translation semigroup $\{\tau_t\}_{t\geq 0}$ defined as

$$\tau_t f(x) = f(x+t), \qquad \text{for } x, t \geq 0, f \in L^p_\rho(\mathbb{R}_0^+, \mathbb{C}),$$

is a C_0-semigroup on $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$.

Let us define $\phi(f)(x) = (\rho(x))^{1/p} f(x)$ and consider the following commutative diagram:

$$\begin{array}{ccc} L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{\tau_t} & L^p_\rho(\mathbb{R}_0^+, \mathbb{C}) \\ \phi \big\downarrow & & \big\downarrow \phi \\ L^p(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{T_t} & L^p(\mathbb{R}_0^+, \mathbb{C}). \end{array} \quad (6.3)$$

The hypothesis on B let us conclude that $\{\tau_t\}_{t\geq 0}$ is distributionally chaotic on $L^p_\rho(\mathbb{R}_0^+, \mathbb{C})$, see [BP12, Th. 2.3]. Therefore, the conclusion is obtained since distributional chaos is preserved under uniform conjugacy by Corollary 2.11. ⌟

Remark 6.2. The previous result can be compared with the characterizations of hypercyclicity and Devaney chaos for the translation C_0-semigroup on the spaces $L^p_\rho(\mathbb{R}^+_0, \mathbb{C})$, $1 \leq p < \infty$: The translation C_0-semigroup $\{\tau_t\}_{t\geq 0}$ is hypercyclic on $L^p_\rho(\mathbb{R}^+_0, \mathbb{C})$ if, and only if, $\liminf_{x\to\infty} \rho(x) = 0$ [DSW97], and $\{\tau_t\}_{t\geq 0}$ is Devaney chaotic on it if, and only if, $\int_0^\infty \rho(x)dx < \infty$ [dE01; MYT03]. Using conjugacy, these results can be transferred to the C_0-semigroup $\{T_t\}_{t\geq 0}$ [GEPM11, Ex. 7.5.2].

On the one hand, if $h(x)$ is constant and equal to 1, then we have that $\{T_t\}_{t\geq 0}$ is Devaney chaotic and distributionally chaotic on $L^p(\mathbb{R}^+_0, \mathbb{C})$. On the other hand, taking

$$B = \bigcup_{n\in\mathbb{N}} \left] 10^{2^n-1}, 10^{2^{n+1}-2} \right],$$

and defining $h(x) = 1$ if $x \in B$ and $h(x) = -1$ elsewhere, we have $\overline{\mathrm{Dens}}(B) = 1$ and $\int_B \rho(x)dx < \infty$. Therefore $\{T_t\}_{t\geq 0}$ is distributionally chaotic on $L^p(\mathbb{R}^+_0, \mathbb{C})$. It is also hypercyclic since $\rho(10^{2^{n+1}-2}) < e^{-p9\cdot 10^{2^{n+1}-3}}$ for every $1 < n \in \mathbb{N}$, which yields that $\liminf\limits_{x\to\infty} \rho(x) = 0$ [Tak05, Th. 2.2]. However, it cannot be Devaney chaotic since $\int_{\mathbb{R}^+_0} \rho(x)dx = \infty$.

To sum up, we have an example of a C_0-semigroup that is hypercyclic, distributionally chaotic, but it is not Devaney chaotic. This example can be compared with the Example 4.11 of a distributionally chaotic translation C_0-semigroup that is neither hypercyclic nor Devaney chaotic.

Now, let us consider another example of a C_0-semigroup whose dynamical behaviour was already discussed in [Tak05]: Let $\rho : [0,1] \to \mathbb{R}^+$ be a continuous function such that there exist constants $M \geq 1, \omega \in \mathbb{R}$, and $\gamma < 0$ such that

$$\rho(x) \leq M e^{\omega t} \rho(e^{\gamma t} x), \qquad \text{for all } x \in [0,1], t > 0. \tag{6.4}$$

With such a function ρ, we can consider the spaces $L^p_\rho([0,1], \mathbb{C})$, for $1 \leq p < \infty$. The family of operators $\{S_t\}_{t\geq 0}$ with $S_t f(x) = f(e^{\gamma t}x)$, $t \geq 0$ defines a C_0-semigroup on them [Tak05].

Theorem 6.3. *If $\gamma < 0$, then the C_0-semigroup $\{S_t\}_{t\geq 0}$ is distributionally chaotic on $L^p_\rho([0,1], \mathbb{C})$, $1 \leq p < \infty$.*

Proof. Let us apply Theorem 3.3. Take $X_0 = \{f \in \mathcal{C}([0,1], \mathbb{C})\ ;\ f(0) = 0\}$. This set is dense in $L^p_\rho([0,1], \mathbb{C})$ and, clearly, $\lim_{t\to\infty} S_t f = 0$ for every $f \in X_0$, which fulfils condition (i) in Theorem 3.3.

Let us prove that $\int_0^\infty \|S_t\|_{p,\rho}^{-1} dt$ is finite: Fix $t > 0$ and a continuous function g on $[0,1]$ with $\|g\|_{p,\rho} = 1$, for instance $g(x) = 1/\rho(x)^{1/p}$.

There is some $t_0 > 0$ such that for $t > t_0$ we have $\left(\int_0^{e^{\gamma t}} \rho(x)dx\right)^{1/p} \leq t^{-2}/\|g\|_\infty$. For these $t > t_0$, define

$$g_t(x) = \begin{cases} g(e^{-\gamma t}x), & \text{if } 0 \leq x \leq e^{\gamma t}, \\ 0, & \text{elsewhere.} \end{cases} \tag{6.5}$$

Since $\|g_t\|_{p,\rho} \leq t^{-2}$ and $S_t g_t = g$, then $\|S_t\|_{p,\rho} \geq t^2$ for $t \geq t_0$. So that $\int_{t_0}^\infty \|S_t\|_{p,\rho}^{-1} dt$ is convergent, which yields the conclusion. ⌟

Remark 6.4. The assumption $\gamma < 0$ forces $w > 0$: if not, take any $x \in [0,1]$. Taking limits when $t \to \infty$ in the inequality $\frac{\rho(x)}{\rho(e^{\gamma t}x)} \leq Me^{\omega t}$ we have $\frac{\rho(x)}{\rho(0)} \leq 0$, which is a contradiction because ρ is a positive continuous function.

Remark 6.5. An alternative proof given by one of the referees from [BC12] is the following: If $\rho : [0,1] \to \mathbb{R}^+$ is a continuous weight function which is admissible in the sense of (6.4), then $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined as $\psi(x) := \rho(e^{\gamma x})e^{\gamma x}$ is an admissible weight function in the sense of [DSW97, Def. 4.1]. Therefore, taking $\phi : L^p_\psi(\mathbb{R}_0^+, \mathbb{C}) \to L^p_\rho([0,1], \mathbb{C})$ defined as $\phi(f)(x) := f\left(\frac{\log(x)}{\gamma}\right)$, we have the following commutative diagram:

$$\begin{array}{ccc} L^p_\psi(\mathbb{R}_0^+, \mathbb{C}) & \xrightarrow{\tau_t} & L^p_\psi(\mathbb{R}_0^+, \mathbb{C}) \\ \phi\big\downarrow & & \big\downarrow\phi \\ L^p_\rho([0,1], \mathbb{C}) & \xrightarrow{S_t} & L^p_\rho([0,1], \mathbb{C}). \end{array} \tag{6.6}$$

If $\gamma < 0$, then $\int_0^\infty \psi(x)dx < \infty$. So that, by uniform conjugacy, $\{S_t\}_{t\geq 0}$ is hypercyclic, Devaney chaotic and distributionally chaotic, see Remark 6.2.

We return to the initial value problem stated in (6.1). Consider the case when $I = [0,1]$, $\zeta(x) := \gamma x$, $\gamma < 0$, and $h \in \mathcal{C}([0,1], \mathbb{C})$. Under these hypothesis, the C_0-semigroup $\{\widetilde{T}_t\}_{t\geq 0}$ defined as

$$\widetilde{T}_t f(x) = \exp\left(\int_0^t h(e^{\gamma(t-r)}x)dr\right) f(e^{\gamma t}x) \text{ for } t \geq 0, x \in [0,1], \tag{6.7}$$

gives the solution C_0-semigroup to (6.1) on $L^p([0,1], \mathbb{C})$, $1 \leq p < \infty$ [Tak05, Th. 3.4]. The particular case when $\gamma = -1$ and $h(x) = -1/2$ was studied using the Wiener measure in [LM94].

Theorem 6.6. *If $\gamma < 0$ and $\min\{\Re(h(x))\ ;\ x \in [0,1]\} > \gamma/p$, then the C_0-semigroup $\{\widetilde{T}_t\}_{t\geq 0}$ defined in (6.7) is distributionally chaotic on $L^p([0,1],\mathbb{C})$, $1 \leq p < \infty$.*

Proof. We apply again Theorem 3.3: Condition 3.3.(i) holds in the same way as in the proof of Theorem 6.3 taking $X_0 = \{f \in \mathcal{C}([0,1]),\mathbb{C})\ ;\ f(0) = 0\}$.

In order to verify condition 3.3.(ii), let $\alpha \in \mathbb{R}$ be such that $\min\{\Re(h(x))\ ;\ x \in [0,1]\} > \alpha > \gamma/p$. For every $t > 0$, we define f_t as a function with $\|f_t\|_p = 1$ and $\mathrm{supp(f_t)} \subset [0, \mathrm{e^{\gamma t}}]$. Using it, we have the following estimations for $\|\widetilde{T}_t\|_p$:

$$\|\widetilde{T}_t\|_p \geq \|\widetilde{T}_t f_t\|_p \geq e^{\alpha t}\left(\int_0^1 |f_t(e^{\gamma t}x)|^p dx\right)^{1/p} = e^{(\alpha-\gamma/p)t}\left(\int_0^{e^{\gamma t}} |f_t(y)|^p dy\right)^{1/p}$$
$$= e^{(\alpha-\gamma/p)t}.$$

So that $\int_0^\infty \|\widetilde{T}_t\|_p^{-1} dt$ is finite, which yields the conclusion. ⌟

In addition, Brzeźniak and Dawidowicz also studied in [BD09] Devaney chaos for the case $\gamma = -1$ and $h(x) = \lambda \in \mathbb{R}$ in certain subspaces of Hölder continuous functions on $[0,1]$. For $\alpha \in]0,1], 0 < r \leq 1$, we define the space $C_r^\alpha([0,1])$ of functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,r} := \sup_{\substack{x,y\in[0,1] \\ 0<|x-y|<r}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

For $\alpha \in]0,1[$, let us consider $V_\alpha([0,1])$ the space of functions

$$\{f \in C_1^\alpha([0,1])\ ;\ \lim_{r\to 0^+} \|f\|_{\alpha,r} = 0 \text{ and } f(0) = 0\}.$$

In [BD09] it is shown that $V_\alpha([0,1])$ is a separable Banach space endowed with the norm $\|f\|_{\alpha,1}$. Furthermore, following a constructive approach, it is proved that if $\gamma = -1$ and $h(x) = \lambda > \alpha$, then $\{\widetilde{T}_t\}_{t\geq 0}$ is Devaney chaotic there and exponentially stable if $\lambda \leq \alpha$. We will prove that in this case $\{\widetilde{T}_t\}_{t\geq 0}$ is also distributionally chaotic.

Theorem 6.7. *If $\gamma = -1$ and $h(x) = \lambda > \alpha$, then the C_0-semigroup $\{\widetilde{T}_t\}_{t\geq 0}$ defined in (6.7) is distributionally chaotic on $V_\alpha([0,1]), \alpha \in]0,1[$.*

Proof. We will apply Theorem 3.3 again. Since in order to prove that $\{\widetilde{T}_t\}_{t\geq 0}$ is Devaney chaotic, Aroza and Mangino in [AM13] make use of the version of the Desch-Schappacher-Webb Criterion given by El Mourchid [El 06, Th 2.1], one can check that there is a dense set $X_0 \subset V_\alpha([0,1])$ such that $\lim_{t\to\infty} \widetilde{T}_t x = 0$ for all $x \in X_0$, and the first condition in Theorem 3.3 holds.

In order to verify condition 3.3.(i), take $0 < \varepsilon < (\lambda - \alpha)/2$ such that $\alpha + \varepsilon < 1$. Let us define $f_\varepsilon(x) = x^{\alpha+\varepsilon}$, $0 \leq x \leq 1$. Since $|x^{\alpha+\varepsilon} - y^{\alpha+\varepsilon}| \leq |(x-y)^{\alpha+\varepsilon}|$ for all $x, y \in [0,1]$, then we can easily see that $\|f_\varepsilon\|_{\alpha,1} = 1$ and $f_\varepsilon \in V_\alpha([0,1])$. We also get that $\|\widetilde{T}_t f_\varepsilon\|_{\alpha,1} = e^{(\lambda-\alpha-\varepsilon)t}$ and hence $\int_0^\infty dt/\|\widetilde{T}_t\|_{\alpha,1} \leq \int_0^\infty dt/\|\widetilde{T}_t f_\varepsilon\|_{\alpha,1} < \infty$. ⌟

6.2 The Desch-Schappacher-Web Criterion implies distributional chaos

Under the hypothesis of the last theorem, Takeo proved that $\{\widetilde{T}_t\}_{t\geq 0}$ is Devaney chaotic by applying the Desch-Schappacher-Webb Criterion [Tak05]. Independently, Brzeźniak and Dawidowicz also proved that $\{\widetilde{T}_t\}_{t\geq 0}$ is Devaney chaotic when $\gamma = -1$ and $h(x) = \lambda \in \mathbb{R}$ with $\lambda > -1/p$, that is known as the von Foerster-Lasota equation [BD09, Th. 8.3 & 8.4]. Furthermore, they also showed that for $\lambda \leq -1/p$ the orbits of all elements tend to 0, which makes chaos disappear. Therefore, we can affirm that Devaney chaos coincides exactly with distributional chaos for the same values of λ. As we will see later, this is due to the fact that Devaney chaos can be obtained here from the Desch-Schappacher-Webb Criterion. This can be easily seen if we reformulate Theorem 3.3 in terms of the infinitesimal generator of the C_0-semigroup. The following result is a continuous version of [BBMGP11, Cor. 31].

Theorem 6.8. *Let X be a complex Banach space and let $\mathcal{T}$ be a C_0-semigroup in $L(X)$ with infinitesimal generator $(A, D(A))$. If the following conditions hold:*

(i) there is a dense subset $X_0 \subset X$ with $\lim_{t\to\infty} T_t x = 0$, for each $x \in X_0$, and

(ii) there is some $\lambda \in \sigma_p(A)$ with $\Re(\lambda) > 0$,

then $\mathcal{T}$ has a dense distributionally irregular manifold. In particular, $\mathcal{T}$ is distributionally chaotic.

Proof. Fix $t > 0$. On the one hand, if condition (i) holds, then we have $\lim_{n\to\infty} T_t^n x = 0$ for every $x \in X_0$. On the other hand, by the point spectral mapping theorem for C_0-semigroups, since $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T_t)$. Therefore $r(T_t) \geq |e^{\lambda t}| > 1$ and, by [BBMGP11, Cor. 31], T_t admits a dense distributionally irregular manifold. By [ABMP13, Rem. 2], this is equivalent to say that $\mathcal{T}$ admits a dense distributionally irregular manifold. Furthermore, $\mathcal{T}$ is distributionally chaotic [ABMP13, Prop. 2]. ⌟

Remark 6.9. Clearly, the conditions in Theorem 6.8 hold whenever the Desch-Schappacher-Webb Criterion can be applied. Since this criterion can be applied in the following examples, we not only obtain Devaney chaos, but also distributional chaos (and the existence of a dense distributionally irregular manifold):

◦ 1) [BL01, Th. 1]:
In $X = \ell^p$, $1 \leq p < \infty$, or c_0, the C_0-semigroup generated by the system

$$\frac{df_n}{dt} = (Lf)_n = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \in \mathbb{N}_0.$$

with $(f_n)_{n\in\mathbb{N}_0} \in X$, when the sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ satisfy the following condition for every $n \in \mathbb{N}_0$:

- $0 < \alpha_n < \beta_n$,
- $\alpha_n = \alpha + a'_n$, for some $\alpha \geq 0$, with $\lim\limits_{n\to\infty} a'_n = 0$ and there is $q < 1$ such that $\left|\dfrac{a'_n}{\beta}\right| \leq q^{n+1}$, and
- $\beta_n = \beta b_n$, for some $\beta > \alpha$ and with $\lim\limits_{n\to\infty} b_n = 1$.

◦ 2) [DSW97, Ex. 4.12]:
In $X = L^2([0,\infty[,\mathbb{C})$ the solution C_0-semigroup to the following partial differential equation

$$\begin{cases} u_t(x,t) = au_{xx}(x,t) + bu_x(x,t) + cu(x,t), & \\ u(0,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = f(x) & \text{for } x \geq 0 \text{ with some } f \in X, \end{cases}$$

when $a, b, c > 0$ and $c < \frac{b^2}{2a} < 1$.

- 3) [CM10, Th. 3.1]:
  The C_0-semigroup by the perturbation of the one-dimensional Orsntein-Uhlenbeck operator
  $$\mathcal{A}_\alpha = u'' + bxu' + \alpha u$$
  with domain
  $$D(\mathcal{A}_\alpha) = \left\{ u \in L^2(\mathbb{R}) \cap W^{2,2}_{\text{loc}}(\mathbb{R}) \ ; \ \mathcal{A}_\alpha u \in L^2(\mathbb{R}) \right\},$$
  when $b > 0$ and $\dfrac{b}{2} < \alpha \in \mathbb{R}$.

- 4) [CPT10, Th. 2.1]: The solution semigroup to the hyperbolic heat transfer equation (HHTE) in absence of internal heat sources,
  $$\begin{cases} \tau u_{tt} + u_t = \alpha u_{xx}, & \\ u(0,x) = \phi_1(x), & x \in \mathbb{R}, \\ u_t(0,x) = \phi_2(x), & x \in \mathbb{R}; \end{cases}$$
  has
  $$A = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\partial_{xx} & \frac{-1}{\tau} I \end{pmatrix}$$
  as infinitesimal generator. This C_0-semigroup was already known to be Devaney chaotic (by application of the Desch-Schappacher-Webb Criterion) on the space $X_\rho \oplus X_\rho$ where
  $$X_\rho := \left\{ f(x) = \sum_{n \geq 0} \frac{a_n}{n!} (\rho x)^n \ ; \ (a_n)_n \in c_0(\mathbb{N}_0) \right\},$$
  for some $\rho > 0$, endowed with the norm
  $$\|f\| := \sup_{n \in N_0} \sup_{x \in \mathbb{R}} \rho^{-n} e^{\rho|x|} |\phi^{(n)}(x)|.$$
  Here, $(c_0(\mathbb{N}_0), \|\cdot\|_\infty)$ is the Banach space of all complex sequences tending to 0, endowed with the maximum norm.

See also [GEPM11, Ch. 7] for an improved version of the proof of this last example.

6.3 Distributional chaos for birth-and-death processes with proliferations

In [BM11], the authors have analysed the Devaney chaos in the problem of the exponential decay of the drug resistant population of cells.

Let us denote by f_n, $n \geq 1$, the number of copies of the drug resistant gene in a population of cells. The matrix of the process has constant coefficients and it is obtained from the following infinite system of equations.

$$\begin{aligned} f_1' &= af_1 + df_2, \\ f_n' &= af_n + bf_{n-1} + df_{n+1}, n \geq 2. \end{aligned}$$

We consider the so-called sub-critical case when $0 < b < d$.

The usual setting will be ℓ_1, nevertheless the space ℓ_s^1 of summable sequences with the weights s^n, $n \geq 0$, will be also considered. By $(e_m)_m$ we denote the canonical basis of ℓ_1.

Let us consider the operator $L_s = aI + C_s$ in ℓ_1 where C_s is an operator defined on ℓ_1 by

$$C_s = \begin{pmatrix} 0 & d/s & & & \\ sb & 0 & d/s & & \\ & sb & 0 & d/s & \\ & & sb & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Lemma 6.10 (See Lemma 1 in [BM11]). *We have*

$$(C_s^k \boldsymbol{f})_n = \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s}\right)^i f_{n-k+2i}, \tag{6.8}$$

where $\boldsymbol{f} = (f_1, f_2, \ldots)$, $f_i = 0$ for $i \leq 0$ and the Newton symbol is also 0 for negative entries.

We will consider the following restrictions:

$$0 < b < d, \tag{6.9}$$

$$|a| \leq 2\sqrt{bd}. \tag{6.10}$$

As in [BK99], we compute the ℓ^1-norm of C^k acting over $(f_n)_{n\geq 1} = e_m$ where $m > k$ then,

$$\|C_s^k \mathbf{f}\|_{\ell^1} = \sum_{n=0}^{\infty} \left| \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s}\right)^i \delta_{n-k+2i,m} \right|.$$

Since $\delta_{n-k+2i,m} = 0$ for $n < m-k$ or $n > m+k$, then,

$$\|C_s^k \mathbf{f}\|_{\ell^1} = \sum_{n=m-k}^{m+k} \left| \sum_{i=0}^{k} \left[\binom{k}{i} - \binom{k}{k-(n+i)} \right] (sb)^{k-i} \left(\frac{d}{s}\right)^i \delta_{n-k+2i,m} \right|.$$

With the change $j = k - i$ we get

$$\sum_{n=m-k}^{m+k} \left| \sum_{j=0}^{k} \left[\binom{k}{j} - \binom{k}{j-n} \right] (sb)^j \left(\frac{d}{s}\right)^{k-j} \delta_{n+k-2j,m} \right|.$$

Changing also $n' = n + k - m$, we have

$$\sum_{n'=0}^{2k} \left| \sum_{j=0}^{k} \left[\binom{k}{j} - \binom{k}{j+k-n'-m} \right] (sb)^j \left(\frac{d}{s}\right)^{k-j} \delta_{n'+m-2j,m} \right|.$$

If n' is odd, $\delta_{n'+m-2j,m} = 0$, then we are left with the even terms, getting

$$\sum_{j=0}^{k} \left| \left[\binom{k}{j} - \binom{k}{k-j-m} \right] (sb)^j \left(\frac{d}{s}\right)^{k-j} \right|.$$

Since $m > k$, we only have

$$\sum_{j=0}^{k} \binom{k}{j} \left| (sb)^j \left(\frac{d}{s}\right)^{k-j} \right|,$$

which is $\left(sb + \dfrac{d}{s}\right)^k$. Therefore, $\|C_s^k\| \geq \left(sb + \dfrac{d}{s}\right)^k$.

With these estimations, we can also estimate the norm of e^{tC_s} in $L(\ell_1)$.

$$\|e^{tC_s}\| = \left\| \sum_{k=0}^{\infty} \frac{(tC_s)^k}{k!} \right\|$$

Since C_s is a positive operator, for every $m > 0$ we have

$$\left\|\sum_{k=0}^{\infty}\frac{(tC_s)^k}{k!}\right\| \geq \left\|\sum_{k=0}^{m-1}\frac{(tC_s)^k}{k!}\right\| \geq \left\|\sum_{k=0}^{m-1}\frac{(tC_s)^k}{k!}e_m\right\| = \sum_{k=0}^{\infty}\frac{t^k\left(sb+\frac{d}{s}\right)^k}{k!}.$$

Therefore, taking supremum on m we get $\|e^{tC_s}\| \geq e^{t(sb+\frac{d}{s})}$, and hence

$$\frac{1}{\|e^{tL_s}\|} \leq \frac{1}{e^{t(a+sb+\frac{d}{s})}}. \tag{6.11}$$

Now we proceed to compute the sign of $a + sb + \frac{d}{s}$. On the one hand, if $a \geq 0$, then it is always positive for $s > 0$. On the other hand, if $a < 0$ but $|a| < 2\sqrt{bd}$, this is always positive for any $s > 0$. In both cases we have that $\frac{1}{\|e^{tL_s}\|}$ is integrable on $\mathbb{R}^+$ with respect to t.

Theorem 6.11. *$\{e^{tL_s}\}_{t\geq 0}$ is distributionally chaotic and admits a dense distributionally irregular manifold provided that a, b, d satisfy* (6.9) *and* (6.10)*.*

Proof. Since a, b, d satisfy (6.9) and (6.10), $\{e^{tL_s}\}_{t\geq 0}$ is hypercyclic by the Godefroy-Shapiro Criterion, as it can be seen in the proof of Theorem 4, [BM11]. By 6.11 we have that

$$\int_{\mathbb{R}^+}\frac{1}{\|e^{tL_s}\|} < \infty. \tag{6.12}$$

Therefore, by the Dense Distributionally Irregular Manifold Criterion (Theorem 3.3) the theorem holds. ⌟

This can be compared with the following result.

Theorem 6.12 (See Theorem 4, [BM11])**.** *If $0 < |b| < |d|$ and $|a| < |b + d|$ hold, then $\{e^{tL_s}\}_{t\geq 0}$ is Devaney chaotic.*

6.4 Further research lines

Consider the initial value problem of (6.1) on $L^1(\mathbb{R}_0^+, \mathbb{C})$ with $\zeta(x) = 1$ and $h(x) = \frac{kx^{k-1}}{1+x^k}$. Here, the solution C_0-semigroup $\{T_t\}_{t\geq 0}$ is defined as

$$T_t f(x) = \frac{1 + (x+t)^k}{1 + x^k} f(x + t), \quad x, t \geq 0. \tag{6.13}$$

The C_0-semigroup $\{T_t\}_{t\geq 0}$ defined in (6.13) is distributionally chaotic on $L^1(\mathbb{R}_0^+,\mathbb{C})$ by Th. 6.1. The hypercyclicity of this C_0-semigroup for $k = 2$ was obtained by El Mourchid in [El 06] and the Devaney chaos by Grosse-Erdmann and Peris in [GEPM11, Prop. 7.34]. In this case, the point spectrum of the infinitesimal generator is the closed left half plane. This inhibits the Desch-Schappacher-Webb Criterion to be applied in the way it has been formulated. Nevertheless, El Mourchid observed that the hypercyclic behaviour of this C_0-semigroup is essentially due to the imaginary eigenvalues of its infinitesimal generator [El 06], see also [GEPM11, Ex. 7.5.1]. In fact, the Desch-Schappacher-Webb Criterion can be strengthened and reformulated as follows:

Theorem 6.13. *[El 06, Th. 2.1] & [GEPM11, Th. 7.31] Let X be a complex separable Banach space, and let $\mathcal{T}$ be a C_0-semigroup on X with infinitesimal generator $(A, D(A))$. If there are $a < b$ and continuous functions $f_j : [a, b] \to X, j \in J$, with*

i. $f_j(s) \in \ker(isI - A)$ *for every* $s \in [a, b], j \in J$, *and*

ii. $\operatorname{span}\{f_j(s) \ : \ s \in [a, b], j \in J\}$ *is dense in* X,

then the semigroup $\mathcal{T}$ is Devaney chaotic.

To sum up, we have seen that even when we apply this stronger version of the Desch-Schappacher-Webb Criterion for the C_0-semigroup in (6.13), asking only for an abundance of eigenvalues of real part equal to zero, then we can also prove that there is a dense distributionally irregular manifold. Therefore we can pose the following problem:

Problem 6.14. *Do the hypothesis in Theorem 6.13 imply the existence of a dense distributionally irregular manifold for $\mathcal{T}$? If not, is there at least a distributionally irregular vector for $\mathcal{T}$?*

By the equivalence between a C_0-semigroup with a distributionally irregular vector and a distributionally chaotic C_0-semigroup, proved in Theorem 2.17, the former problem can also be presented as follows:

Problem 6.15. *Do the hypothesis in Theorem 6.13 imply that $\mathcal{T}$ is distributionally chaotic?*

These questions could have a positive answer, but it is still unknown whether Devaney chaos implies distributional chaos on C_0-semigroups.

Problem 6.16. *Are there examples of Devaney chaotic C_0-semigroups which are not distributionally chaotic?*

A C_0-semigroup is said to be *frequently hypercyclic* if there exists some $x \in X$ such that for every non-empty open set $U \subset X$, the set $U_x := \{s \geq 0 \ ; \ T_s x \in U\}$ has positive lower density, that is $\liminf_{t\to\infty}(1/t)\mu(U_x \cap [0,t])$ is positive. In [MP11] Mangino and Peris observed that with the same arguments used in [DSW97; El 06] one can show that the Desch-Schappacher-Webb Criterion implies frequent hypercyclicity. They also provide the Frequent Hypercyclicity Criterion for C_0-semigroups [MP11, Th. 2.2]. So that, one can raise the following question:

Problem 6.17. *Do the hypothesis of the Frequent Hypercyclicity Criterion for C_0-semigroups imply distributional chaos?*

The hypothesis in Theorem 6.13 also yield the mixing property for the C_0-semigroup $\mathcal{T}$, see [GEPM11]. Clearly, topological mixing implies transitivity (i.e. hypercyclicity), but it is strictly stronger than it.

On the one hand, Example 4.11 provides an example of a distributionally chaotic C_0-semigroup that it is not topologically mixing. On the other hand, in [MGOP12], there is an example of a backward shift operator on a weighted sequence space $\ell^p(v)$, $1 \leq p < \infty$, that is topologically mixing but it is not distributionally chaotic. This operator will provide us an analogous counterexample in the frame of C_0-semigroups.

Example 6.18. *Consider the sequence $(n_k)_k$ defined as $n_k = (k!)^3, k \in \mathbb{N}$, and define the function $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as $\rho(t) = 1$ if $0 \leq t < n_1$ and $\rho(t) = k^{-1}$ if $n_k \leq t < n_{k+1}, k \in \mathbb{N}$. This function is an admissible weight in the sense of [DSW97, Def. 4.1] and makes the translation semigroup $\{\tau_t\}_{t\geq 0}$ to be a*

C_0-semigroup. On the one hand, since $\lim_{t\to\infty} \rho(t) = 0$, then the translation C_0-semigroup is topologically mixing. On the other hand, if the translation C_0-semigroup was distributionally chaotic, by [BP12, Th. 2.10], the backward shift operator, defined as $B(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$, would be distributionally chaotic on the space $\ell_1(v) := \{(x_n)_n \,;\, \sum_{j\in\mathbb{N}} |x_j| v_j < \infty\}$ with $(v_n)_n = (\rho(n))_n$, which is a contradiction as it is indicated in [MGOP12].

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