Do the generalized polynomial chaos and Fröbenius methods retain the statistical moments of random differential equations?

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Abstract

The aim of this paper is to explore whether the generalized polynomial chaos (gPC) and random Fröbenius methods preserve the first three statistical moments of random differential equations. There exist exact solutions only for a few cases, so there is a need to use other techniques for validating the aforementioned methods in regards to their accuracy and convergence. Here we present a technique for indirectly study both methods. In order to highlight similarities and possible differences between both approaches, the study is performed by means of a simple but still illustrative test-example involving a random differential equation whose solution is highly oscillatory. This comparative study shows that the solutions of both methods agree very well when the gPC method is developed in terms of the optimal orthogonal polynomial basis selected according to the statistical distribution of the random input. Otherwise, we show that results provided by the gPC method deteriorate severely. A study of the convergence rates of both methods is also included.

Keywords: Random Fröbenius method, Generalized polynomial chaos, Statistical moments, Random differential equations
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1. Motivation

The representation of stochastic processes (s.p.'s) and random variables (r.v.'s) plays an important role in many scientific areas. In particular, such representations are very useful in obtaining their main statistical functions (such as average and variance) as well as in simulating them. These issues are of prime importance in dealing with mathematical models involving uncertainty in their formulation. Although it is desirable that these representations be exact, often, in practice, only approximate expressions are attainable. For instance, when solving random differential equations (r.d.e.'s), one obtains a representation of its solution, which only exceptionally, can be computed exactly.

The generalized polynomial chaos (gPC) and random Fröbenius constitute powerful methods to solve r.d.e.'s but, in general, just in an approximate manner. Indeed, both methods represent the solution s.p. through infinite series, say $x_Q^{\mathrm{PC}}(t)$ and $x_M^{\mathrm{F}}(t)$, that need to be truncated at orders Q and M, respectively, to be computationally feasible. It is important to establish the convergence and accuracy of both methods. In this paper, we devise a simple and reliable way to explore, indirectly, the ability of both techniques to preserve accurately the first statistical moments associated not with the solution s.p. but with the r.d.e. itself. To conduct our study, we have chosen the Airy r.d.e. [1]

$$\ddot{x}(t) + t\xi x(t) = 0, \tag{1}$$

because exact expressions for its first statistical moments are not available except by infinite series, therefore the previous observations are completely applicable. In addition, it is well-known that the solutions of the deterministic

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Airy differential equation are highly oscillatory, hence it is expected that in dealing with its stochastic counterpart, numerical solutions need to be calculated accurately so that differences, if any, between the gPC and Fröbenius methods are highlighted. To carry out the current study a key idea is to rewrite the r.d.e. (1) in the equivalent form

$$-t\xi = \frac{\ddot{x}(t)}{x(t)}. \tag{2}$$

The quality of the numerical approximations of the gPC and Fröbenius methods can be better assessed using r.d.e. (2) rather than (1). In fact, we will compare the statistical moments of order n of the left-hand side, which are exact, against the corresponding values of the right-hand side, which will be approximated:

$$(-1)^n t^n \mathrm{E}\left[\xi^n\right] \approx \begin{cases} \mathrm{E}\left[\left(\dfrac{\ddot{x}_Q^{\mathrm{PC}}(t)}{x_Q^{\mathrm{PC}}(t)}\right)^n\right], \\ \\ \mathrm{E}\left[\left(\dfrac{\ddot{x}_M^{\mathrm{F}}(t)}{x_M^{\mathrm{F}}(t)}\right)^n\right], \end{cases} \quad n \geq 0, \tag{3}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator and $x_Q^{\mathrm{PC}}(t)$ and $x_M^{\mathrm{F}}(t)$ are the approximations to the solution $x(t)$ using the gPC method of order Q and the Fröbenius method of order M, respectively. This study will be performed by comparing the time intervals over which the approximations given by the right-hand side of (3) are acceptable for $n = 1, 2, 3$. Notice that the approach previously proposed draws a strong similarity with the so-called method of the moments which is widely used in statistics [2]. Indeed, for each t, we can interpret the statistical moments of order n that appear in the right-hand side of expression (3) as approximations of the corresponding ones to the r.v.'s $-t\,\xi$, which are known exactly. If these values are close enough for every $n \geq 0$ and t, then, based on the equivalence between r.d.e. (1)–(2), the method of the moments shows that the approximations $x_Q^{\mathrm{PC}}(t)$ and $x_M^{\mathrm{F}}(t)$ are reliable.

This paper is organized as follows. In Section 2, we summarize the gPC and Fröbenius techniques focusing on r.d.e.'s with only one single input r.v. as is the case of (1). Section 3 is devoted to show the comparative study previously described through two illustrative examples. These examples show good convergent rates of both methods which also are fairly easy to implement. This section also includes our main conclusions.

2. Preliminaries

The gPC method is a technique that allows the representation of second-order r.v.'s and s.p.'s defined on a probability space $(\Omega, \mathfrak{F}, P)$, by orthogonal polynomial expansions $\{\Phi_i\}$. These polynomials come from the Wiener-Askey scheme and, in general, depend on a number of r.v.'s, $\zeta_1(\omega)$, $\zeta_2(\omega)$, ..., $\omega \in \Omega$, [3]. As is shown in references [4, 5], the gPC method has been shown to be a useful technique to solve r.d.e.'s of the form

$$\mathfrak{D}(t, \boldsymbol{\xi}(\omega); x) = f(t, \boldsymbol{\xi}(\omega)), \tag{4}$$

where $\mathfrak{D}$ denotes a differential operator; $\boldsymbol{\xi}(\omega) = \boldsymbol{\xi} = (\xi_1, \xi_2, \ldots)$ is a vector of r.v.'s $\xi_i = \xi_i(\omega)$, which dimension determines the so-called order of the chaos; $f(t, \boldsymbol{\xi}(\omega))$ is a forcing term and $x = x(t, \boldsymbol{\xi}(\omega))$ is the solution s.p. to be determined. For the sake of clarity in the presentation and, in accordance with model (1), throughout this paper we will focus on the simplest case where the order of the chaos is one, i.e., we will assume that there only is one single input r.v., say $\xi = \xi(\omega)$, involved in the r.d.e. (4). As a consequence the orthogonal polynomial expansions $\{\Phi_i\}$ will only depend on one single r.v. $\zeta = \zeta(\omega)$ as well, [3]. In order to solve the r.d.e. (4) and, based on the gPC method, one represents both, the input r.v. ξ and the unknown $x = x(t, \xi)$, as follows

$$\xi = \sum_{i=0}^{\infty} \xi_i \Phi_i(\zeta), \quad x^{\mathrm{PC}}(t, \zeta) = \sum_{i=0}^{\infty} x_i(t) \Phi_i(\zeta). \tag{5}$$

Notice that in accordance with (4), the solution s.p. of this r.d.e. formally depends on the input r.v. ξ, however using the gPC method it is represented in terms of the auxiliary r.v. ζ, which could be different from ξ. Bearing in mind this fact, in the sequel we will denote the solution s.p. by $x^{\mathrm{PC}}(t,\zeta)$ or $x(t,\xi)$ depending on the context. At this point, we note that, if $\{\Phi_i\}$ are the Hermite polynomials, then according to the Cameron-Martin theorem [6], for a fixed value of t, these expansions converge in the mean square (m.s.) sense in the Hilbert space $(L_2(\Omega), \langle\cdot\rangle)$. That is, they converge to any $L_2(\Omega)$ functional with respect to the norm inferred from the inner product $\langle X, Y\rangle = \mathrm{E}[XY]$. Notice that when $Y = 1$, $\langle\cdot\rangle$ represents the expectation operator as well. In (5) coefficients ξ_i are computed as follows

$$\xi_i = \frac{\langle \xi, \Phi_i(\zeta)\rangle}{\langle \Phi_i(\zeta), \Phi_i(\zeta)\rangle}, \quad i = 0, 1, 2, \ldots$$

In order to compute the solution s.p. $x(t)$ of r.d.e. (4), the coefficients $x_i(t)$, usually referred to as the modes of the solution, need to be calculated. To carry out this in practice, three main steps are followed. First, to be computationally feasible, one considers a truncation of order, say Q, of the infinite series (5)

$$\xi = \sum_{i=0}^{Q} \xi_i \Phi_i(\zeta), \quad x_Q^{\mathrm{PC}}(t,\zeta) = \sum_{i=0}^{Q} x_i(t)\Phi_i(\zeta). \tag{6}$$

The total number of expansion terms, i.e., $Q + 1$ is determined by $Q = P$ being P the highest degree of the orthogonal polynomials $\{\Phi_i\}$ (see [3] for further details). Once a truncation order Q is fixed, to construct the best approximation $x_Q^{\mathrm{PC}}(t,\zeta)$, a selection of the optimal basis $\{\Phi_i(\zeta)\}$ has to be made according to the type of random input ξ (see [3]). In the second step one substitutes representations (6) into (4)

$$\mathfrak{D}\left(t, \sum_{i=0}^{Q} \xi_i \Phi_i(\zeta); \sum_{i=0}^{Q} x_i(t)\Phi_i(\zeta)\right) = f\left(t, \sum_{i=0}^{Q} \xi_i \Phi_i(\zeta)\right),$$

then one multiplies successively this equation by the different orthogonal polynomials $\left\{\Phi_j\right\}$ and one takes the statistical average operator in order to simplify computations by taking advantage of orthogonality

$$\left\langle \mathfrak{D}\left(t, \sum_{i=0}^{Q} \xi_i \Phi_i(\zeta); \sum_{i=0}^{Q} x_i(t)\Phi_i(\zeta)\right), \Phi_j(\zeta)\right\rangle = \left\langle f\left(t, \sum_{i=0}^{Q} \xi_i \Phi_i(\zeta)\right), \Phi_j(\zeta)\right\rangle, \quad 0 \le j \le Q.$$

In this manner a set of $Q+1$ coupled (deterministic) ordinary differential equations (o.d.e.'s) preserving the linearity/non-linearity of the original operator $\mathfrak{D}$ is set.

The last step consists of solving this system whose unknowns are $x_i(t)$. Therefore the method relies on the ability of analytic and/or numerical techniques to solve systems of o.d.e.'s. The computation of modes $x_i(t)$ is important not only because it permits to obtain an approximate representation of the solution s.p. according to (6) but also its main statistical functions such as the average and variance

$$\mu_x(t) = \mathrm{E}[x(t,\xi)] = x_0(t), \qquad \sigma_x^2(t) = \mathrm{Var}[x(t,\xi)] \approx \sum_{i=1}^{Q} (x_i(t))^2\, \mathrm{E}\left[(\Phi_i(\zeta))^2\right].$$

Besides the gPC method, other useful techniques have been developed to solve r.d.e.'s. Here we are also specifically interested in the random Fröbenius method which is based on an extension to the random scenario of its deterministic counterpart. By assuming that time-dependent data are second-order m.s. analytic s.p.'s (notice that it includes the source term $f(t,\xi)$) and that every random input is of second-order too, this method seeks the solution s.p. to (4) as an infinite power series. This yields the following representation for the solution s.p. and the forcing term (as well as every involved s.p. coefficient, if any)

$$x^{\mathrm{F}}(t,\xi) = \sum_{i=0}^{\infty} x_i(\xi)t^i, \quad f(t,\xi) = \sum_{i=0}^{\infty} f_i(\xi)t^i.$$

In contrast with the gPC method, we notice that these representations depend directly on the input r.v. ξ rather than an auxiliary r.v. ζ. Next, these representations are substituted into the r.d.e. (4)

$$\mathfrak{D}\left(t, \xi; \sum_{i=0}^{\infty} x_i(\xi)t^i\right) = \sum_{i=0}^{\infty} f_i(\xi)t^i,$$

in order to obtain some sort of recurrence relationship between coefficients $x_i(\xi)$. Such recurrence permits to determine these coefficients, and then a formal solution s.p. can be defined. When applying this method, the point lies in the determination of the domain where the series is m.s. convergent as well as in the justification of the steps followed to built the formal power series solution. This usually requires the application of both, mean square and mean fourth operational calculus [7, 8, 9]. Once the infinite power series solution $x^{\mathrm{F}}(t,\xi)$ has been rigorously constructed, for the same reasons previously given for the gPC method, in practice it has to be truncated. Approximations for the expectation and the variance functions can be computed as follows

$$\mathrm{E}\left[x_M^{\mathrm{F}}(t,\xi)\right] = \sum_{i=0}^{M} \mathrm{E}\left[x_i(\xi)\right] t^i, \qquad \mathrm{Var}\left[x_M^{\mathrm{F}}(t,\xi)\right] = \mathrm{E}\left[\left(x_M^{\mathrm{F}}(t,\xi)\right)^2\right] - \left(\mathrm{E}\left[x_M^{\mathrm{F}}(t,\xi)\right]\right)^2, \tag{7}$$

where

$$\mathrm{E}\left[\left(x_M^{\mathrm{F}}(t,\xi)\right)^2\right] = \sum_{i=0}^{M} \mathrm{E}\left[(x_i(\xi))^2\right] t^{2i} + 2\sum_{i=1}^{M}\sum_{j=0}^{i-1} \mathrm{E}\left[x_i(\xi)x_j(\xi)\right] t^{i+j}. \tag{8}$$

Every average appearing in (7)–(8) is calculated taking the expectation operator on the recurrence relationship previously established for the coefficients $x_i(\xi)$ together with the operational properties of expectation.

3. Comparing the gPC and Fröbenius methods. Conclusions

As it was pointed out in Section 2, both the gPC and random Fröbenius methods have demonstrated to be, in general, powerful techniques to solve many types of r.d.e.'s, although, as some authors have already highlighted, they also have some shortcomings [10, 11]. In this section we deal with this issue by comparing both techniques following the approach described in Section 1. By the reasons previously mentioned, to conduct the study we have selected the r.d.e. (1) with deterministic initial conditions $x(0) = 1$, $\dot{x}(0) = 1$. To assess better this comparative study, we will consider two different distributions for the random input $\xi = \xi(\omega)$: first, it is assumed to be a uniform r.v. on the interval $[0,1]$, i.e., $\xi \sim \mathrm{Un}([0,1])$; second, a Gaussian distribution with the same mean and variance is assumed: $\xi \sim \mathrm{N}(1/2; 1/12)$. Since the treatment of both cases is similar, we just detail the first case, where (3) can be written as

$$h(t;n) = (-1)^n \frac{t^n}{n+1} = t^n \mathrm{E}\left[\xi^n\right] \approx \begin{cases} \mathrm{E}\left[\left(\dfrac{\ddot{x}_Q^{\mathrm{PC}}(t,\zeta)}{x_Q^{\mathrm{PC}}(t,\zeta)}\right)^n\right] = g^{\mathrm{PC}}(t,Q;n), \\ \mathrm{E}\left[\left(\dfrac{\ddot{x}_M^{\mathrm{F}}(t,\xi)}{x_M^{\mathrm{F}}(t,\xi)}\right)^n\right] = g^{\mathrm{F}}(t,M;n), \end{cases} \quad n \geq 0. \tag{9}$$

On one hand, since $\xi \sim \mathrm{Un}([0,1])$, then, in accordance with the gPC method, in the following computations of $g^{\mathrm{PC}}(t,Q;n)$, we will take as the (optimal) trial basis $\{\Phi_i(\zeta)\}$ the Legendre polynomials where $\zeta \sim \mathrm{Un}([-1,1])$ (see [3]). It will also be shown that if some other basis is chosen, the numerical results deteriorate. This will be illustrated by taking $\{\Phi_i(\zeta)\}$ the Hermite polynomials where $\zeta \sim \mathrm{N}(0;1)$, i.e., ζ is a standard Gaussian r.v. In the following, we introduce the notation $g^{\mathrm{PC-L}}(t,Q;n)$ and $g^{\mathrm{PC-H}}(t,Q;n)$ to distinguish in (9) between both computations. On the other hand, notice that in this case

$$g^{\mathrm{F}}(t,M;n) = \int_0^1 \left(\frac{\displaystyle\sum_{i=1}^{M} \frac{(-1)^i \xi^i (3i-2)!!!}{(3i-2)!} t^{3i-2} + \sum_{i=1}^{M} \frac{(-1)^i \xi^i (3i-1)!!!}{(3i-1)!} t^{3i-1}}{\displaystyle\sum_{i=0}^{M} \frac{(-1)^i \xi^i (3i-2)!!!}{(3i)!} t^{3i} + \sum_{i=0}^{M} \frac{(-1)^i \xi^i (3i-1)!!!}{(3i+1)!} t^{3i+1}} \right)^n \mathrm{d}\xi.$$

In order to carry out this comparative study and, taking into account (9), we define the following relative errors (with respect to the exact moment of order n given by $h(t;n)$) that correspond to the gPC method (using Legendre polynomials) and Fröbenius methods, respectively

$$\mathrm{e}^{\mathrm{PC-L}}(t,Q;n) = \frac{\left|g^{\mathrm{PC-L}}(t,Q;n) - h(t;n)\right|}{|h(t;n)|}, \; \mathrm{e}^{\mathrm{F}}(t,M;n) = \frac{\left|g^{\mathrm{F}}(t,M;n) - h(t;n)\right|}{|h(t;n)|}.$$

Similarly, we will denote by $e^{PC-H}(t, Q; n)$ the corresponding relative error of the gPC method using Hermite polynomials.

Table 1 shows the numerical results when using the gPC and Fröbenius methods in the sense previously detailed corresponding to $n = 1, 2, 3$ in (9). To be more specific, for each n, in the case of the gPC technique, we set the order P of the orthogonal polynomial basis $\{\Phi_i(\zeta)\}$ (which, as we said, in this case coincides with the total number of expansion terms Q of $x_Q^{PC}(t, \zeta)$ in (6)) and, for both, Legendre and Hermite polynomials, we have collected the numerical values $t = T^*$ such that the errors $e^{PC-L}(t, Q; n)$ and $e^{PC-H}(t, Q; n)$ are, respectively, less than 5% over the corresponding whole intervals $[0, T^*]$. With respect to the Fröbenius approach we have proceeded as follows: given n and M, in Table 1 we have collected the values of T^* such that the error $e^F(t, M; n)$ is less than 5% for each $t \in [0, T^*]$. As can be seen, the numerical values show that both, the gPC and Fröbenius methods, provide similar results only when the gPC method is expanded with respect to Legendre polynomials, which corresponds to the optimal basis. Otherwise, they deteriorate severely. Notice that the values shown in Table 1 are congruent: fixed n, the value of T^* increases as Q (or M) does, whereas, fixed Q (or M), the value of T^* decreases as n increases from $n = 1$ to $n = 3$.

T^*:	$e^{PC-L}(T^*, Q; n) < 0.05$	$e^{PC-H}(T^*, Q; n) < 0.05$	$e^F(T^*, M; n) < 0.05$
$n = 1$	$Q = 4 \rightarrow T^* = 4.1$	$Q = 4 \rightarrow T^* = 2.3$	$M = 20 \rightarrow T^* = 3.6$
	$Q = 6 \rightarrow T^* = 6.3$	$Q = 6 \rightarrow T^* = 3.7$	$M = 30 \rightarrow T^* = 5.0$
	$Q = 8 \rightarrow T^* = 7.0$	$Q = 8 \rightarrow T^* = 4.8$	$M = 40 \rightarrow T^* = 6.1$
$n = 2$	$Q = 4 \rightarrow T^* = 3.2$	$Q = 4 \rightarrow T^* = 1.6$	$M = 20 \rightarrow T^* = 3.5$
	$Q = 6 \rightarrow T^* = 5.3$	$Q = 6 \rightarrow T^* = 1.6$	$M = 30 \rightarrow T^* = 4.7$
	$Q = 8 \rightarrow T^* = 6.4$	$Q = 8 \rightarrow T^* = 2.3$	$M = 40 \rightarrow T^* = 5.5$
$n = 3$	$Q = 4 \rightarrow T^* = 3.1$	$Q = 4 \rightarrow T^* = 1.6$	$M = 20 \rightarrow T^* = 3.4$
	$Q = 6 \rightarrow T^* = 4.7$	$Q = 5 \rightarrow T^* = 1.6$	$M = 30 \rightarrow T^* = 4.3$
	$Q = 8 \rightarrow T^* = 6.0$	$Q = 6 \rightarrow T^* = 2.3$	$M = 40 \rightarrow T^* = 5.5$

Table 1: Comparative study of statistical moments preservation by the gPC and Fröbenius methods to r.d.e. (1) with initial conditions $x(0) = 1$, $\dot{x}(0) = 1$ and $\xi \sim \text{Un}([0, 1])$. Legendre (column e^{PC-L}) and Hermite (column e^{PC-H}) bases have been employed when applying the gPC method.

Below, we analyse the convergence of both methods PC-L (gPC with the adequate basis, Legendre) and Fröbenius through the relative errors $e^{PC-L}(T^*, Q; n)$ and $e^F(T^*, M; n)$, respectively. The analysis is made for each moment of order $n = 1, 2, 3$ and, the values $Q = 4, 6, 8$ for PC-L, and $M = 20, 30, 40$ for Fröbenius. In Table 2, for each n we have fixed T^* in such a way that for all Q, $e^{PC-L}(T^*, Q; n) < 0.05$ holds. Notice that it is fulfilled whether $Q = 4$. In Table 3, an analogous analysis has been performed for Fröbenius method: for each n, T^* has been chosen so that for all M, the condition $e^F(T^*, M; n) < 0.05$ is satisfied. In this case, it is true for $M = 20$. The results collected in Tables 2 and 3 show, through the errors $e^{PC-L}(T^*, Q; n)$ and $e^F(T^*, M; n)$, the convergence of PC-L and Fröbenius methods, respectively. From Table 2, we see that the convergence rate is at least linear in Q. Whereas in Table 3, it is roughly quadratic.

$e^{PC-L}(T^*, Q; n)$	$Q = 4$	$Q = 6$	$Q = 8$
$n = 1, T^* = 4.1$	0.000674996	3.93968×10^{-6}	1.06343×10^{-7}
$n = 2, T^* = 3.2$	0.0102613	2.07488×10^{-6}	7.34544×10^{-10}
$n = 3, T^* = 3.1$	0.0016891	9.58077×10^{-7}	2.80641×10^{-10}

Table 2: Relative errors, $e^{PC-L}(T^*, Q; n)$, with a fixed T^* for each moment of order n and different values of Q.

$e^F(T^*, M; n)$	$M = 20$	$M = 30$	$M = 40$
$n = 1, T^* = 3.6$	0.0299544	3.90632×10^{-6}	2.3783×10^{-11}
$n = 2, T^* = 3.5$	0.0171511	1.69584×10^{-6}	7.39231×10^{-12}
$n = 3, T^* = 3.4$	0.0100965	7.44309×10^{-7}	2.30691×10^{-12}

Table 3: Relative errors, $e^F(T^*, M; n)$, with a fixed T^* for each moment of order n and different values of M.

As was stated, to strengthen the conclusions drawn in the previous study, we present in Table 4 the corresponding results for the case when $\xi \sim \text{N}(1/2; 1/12)$. Now, the results provided by the gPC method agree with those ones computed by the Fröbenius method when the orthogonal polynomial basis $\{\Phi_i(\zeta)\}$ is constructed in terms of the Hermite polynomials since $\zeta \sim \text{N}(0; 1)$ (see column $e^{PC-H}(T^*, Q; n)$ in Table 4). Otherwise, they deteriorate.

T^*:	$e^{PC-L}(T^*, Q; n) < 0.05$	$e^{PC-H}(T^*, Q; n) < 0.05$	$e^F(T^*, M; n) < 0.05$
$n = 1$	$Q = 4 \rightarrow T^* = 2.6$	$Q = 4 \rightarrow T^* = 4.1$	$M = 20 \rightarrow T^* = 3.5$
	$Q = 6 \rightarrow T^* = 2.9$	$Q = 6 \rightarrow T^* = 4.9$	$M = 30 \rightarrow T^* = 4.9$
	$Q = 8 \rightarrow T^* = 3.5$	$Q = 8 \rightarrow T^* = 5.6$	$M = 40 \rightarrow T^* = 5.9$
$n = 2$	$Q = 4 \rightarrow T^* = 2.3$	$Q = 4 \rightarrow T^* = 3.0$	$M = 20 \rightarrow T^* = 3.3$
	$Q = 6 \rightarrow T^* = 2.3$	$Q = 6 \rightarrow T^* = 3.9$	$M = 30 \rightarrow T^* = 4.2$
	$Q = 8 \rightarrow T^* = 2.5$	$Q = 8 \rightarrow T^* = 4.6$	$M = 40 \rightarrow T^* = 5.0$
$n = 3$	$Q = 4 \rightarrow T^* = 2.2$	$Q = 4 \rightarrow T^* = 2.5$	$M = 20 \rightarrow T^* = 3.2$
	$Q = 6 \rightarrow T^* = 2.2$	$Q = 5 \rightarrow T^* = 3.4$	$M = 30 \rightarrow T^* = 3.6$
	$Q = 8 \rightarrow T^* = 2.2$	$Q = 6 \rightarrow T^* = 4.2$	$M = 40 \rightarrow T^* = 4.9$

Table 4: Comparative study of statistical moments preservation by the gPC and Fröbenius methods to r.d.e. (1) with initial conditions $x(0) = 1$, $\dot{x}(0) = 1$ and $\xi \sim N(1/2; 1/12)$. Legendre (column e^{PC-L}) and Hermite (column e^{PC-H}) bases have been employed when applying the gPC method.

In this paper we have compared the ability of the generalized polynomial chaos (gPC) and Fröbenius methods to preserve accurately the first statistical moments of right/left-hand sides of a random differential equation. To show similarities and highlight differences between both approaches, we have chosen the random Airy differential equation for two main reasons. First, it has a highly oscillatory solution that permits to contrast better the numerical results provided by both methods. Second, we can isolate the random input in one hand-side of the Airy differential equation and, therefore compare the exact computations with the approximations obtained by the gPC and Fröbenius methods. So we have established exactly the accuracy of the approximations and we have also studied their rate of convergence. Our study shows that both approaches agree very well whenever the gPC method is developed in terms of a suitable polynomial orthogonal basis in accordance with the type of statistical distribution of the random input. This contribution also reveals the great importance of developing the gPC method using the adequate orthogonal polynomial basis according to the type of probability distribution of the input r.v. ξ in order to obtain reliable results.

Finally, we emphasize that the gPC (using the suitable basis) and the Fröbenius techniques validate each other since they provide similar approximations. Although the nature of our approach has been empirical, the variety and representativeness of the situations analyzed together with highly oscillatory behaviour of the r.d.e. under consideration permits expect that the conclusions reached remain true in other cases. Based on previous comments, we think that both techniques are useful methods when dealing with r.d.e.'s.

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