Defez Candel, E. (2013). A Rodrigues-type formula for Gegenbauer matrix polynomials. Applied Mathematics Letters. 26:899-903. doi:10.1016/j.aml.2013.04.001.

A Rodrigues-type formula for Gegenbauer matrix polynomials[1]

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Abstract

This paper centers on the derivation of a Rodrigues-type formula for Gegenbauer matrix polynomial. A connection between Gegenbauer and Jacobi matrix polynomials is given.

Keywords: Gegenbauer matrix polynomials, Jacobi matrix polynomials, Rodrigues-type formula.

1. Introduction and notation

The Gegenbauer (so called *ultraspherical*) polynomials $C_n^{\lambda}(x)$ can be defined by the formula

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda + (1/2))_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x), (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)}, n \geq 0, \qquad (1)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial, $(c)_n$ is the Pochhammer symbol or shifted factorials. Here, $C_n^{\lambda}(x)$ satisfies the Rodrigues formula:

$$\left(1-x^2\right)^{\lambda-1/2} C_n^{\lambda}(x) = \frac{(-2)^n\,(\lambda)_n}{n!(n+2\lambda)_n} D^n\left[\left(1-x^2\right)^{n+\lambda-1/2}\right], \qquad (2)$$

see [1, p.303] or [2] for details. The extension of this classical family of polynomials to the matrix framework has been proposed in [12]. In fact, orthogonal matrix polynomials emerge in various important areas of applied mathematics, see [11, 6, 8, 9, 10]. Only very recently, different applications of matrix polynomials have been pointed out in the literature, *e.g.* dealing with the solution of matrix differential equations, finding approximations of inverse Laplace transforms, and calculating the matrix exponential approximation [5, 17, 16, 18].

[1] **Acknowledgements.** This research has been supported by the Universitat Politècnica de València under grant PAID-06-11-2020

The aim of this work is to obtain a Rodrigues-type formula for the Gegenbauer matrix polynomials defined in Ref. [12]. Using this formula, we find a connection between Gegenbauer matrix polynomials and Jacobi matrix polynomials, as introduced in Ref. [3]. This relation is similar to that between Laguerre's and Hermite matrix polynomials obtained in Ref. [15].

Throughout this paper, $\mathrm{Re}(z)$ denotes the real part of the complex number z, and I the identity matrix in $\mathbb{C}^{r\times r}$. A matrix polynomial of degree n is an expression of the form $P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$, where $x \in \mathbb{R}$, and $A_j \in \mathbb{C}^{r\times r}$ represents a complex square matrix for $0 \leq j \leq n$. The set of all matrix polynomials in $\mathbb{C}^{r\times r}$, for all $n \geq 0$, will be given by $\mathcal{P}[x]$. Let $f(z)$ and $g(z)$ be holomorphic functions of the complex variable z, which are defined on an open set Ω in the complex plane. If C is a matrix in $\mathbb{C}^{r\times r}$ so that the set of all its eigenvalues, $\sigma(C)$, lies in Ω, then, from matrix functional calculus [7, p. 558], it follows that

$$f(C)g(C) = g(C)f(C). \tag{3}$$

If P is a matrix in $\mathbb{C}^{r\times r}$ such that $\mathrm{Re}(z) > 0$ for all eigenvalue z of P, then $\Gamma(P)$ is well defined as

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt, t^{P-I} = \exp\left((P-I)\log(t)\right).$$

The reciprocal scalar *Gamma* function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{r\times r}$, the Riesz-Dunford functional calculus [7] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence, if $C \in \mathbb{C}^{r\times r}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then we have the matrix analogue of formula (1):

$$(C)_n = \Gamma(C+nI)\Gamma^{-1}(C), n \geq 0. \tag{4}$$

If we take into account the scalar factorial function, denoted by $(z)_n$ with $(z)_0 = 1$ and

$$(z)_n = z(z+1)\ldots(z+n-1), n \geq 1,$$

then by application of the matrix functional calculus, for any matrix $C \in \mathbb{C}^{r\times r}$ it holds

$$(C)_n = C(C+I)\ldots(C+(n-1)I), n \geq 1, (C)_0 = I. \tag{5}$$

If matrices $D, F \in \mathbb{C}^{r\times r}$ satisfy the spectral condition

$$\mathrm{Re}(z) > -1, \forall z \in \sigma(D), \mathrm{Re}(t) > -1, \forall t \in \sigma(F), \tag{6}$$

then

$$\int_{-1}^{1} (1+x)^D (1-x)^F dx = 2^{D+I} B\left(D+I, F+I\right) 2^F, \tag{7}$$

where $B(P,Q)$ is the Beta matrix function [14], defined by

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I} dt, \mathrm{Re}(z) > 0, \forall z \in \sigma(P), \mathrm{Re}(s) > 0, \forall s \in \sigma(Q).$$

From Theorem 2 of [13], if P, Q are commuting matrices in $\mathbb{C}^{r\times r}$ such that for all integer $n \geq 0$, the following condition holds

$$P + nI, Q + nI, P + Q + nI \text{ are invertible,} \tag{8}$$

then

$$B(P, Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P + Q). \tag{9}$$

For $k = 0, 1, 2, \ldots$, we denote $D^k(f(x)) = \frac{d^k}{dx^k}(f(x))$, and thus, for an arbitrary matrix $A \in \mathbb{C}^{r\times r}$, $D^k\left[t^{A+mI}\right] = (A + I)_m\left[(A + I)_{m-k}\right]^{-1} t^{A+(m-k)I}$.

The organization of the paper is as follows: In Section 2, we recall the definition and some properties of Gegenbauer matrix polynomials which will be used. In Section 3 we derive the Rodrigues-type formula for this class of orthogonal matrix polynomials. Finally, a connection between Gegenbauer matrix polynomials and Jacobi matrix polynomials, introduced in [3], is given.

2. Gegenbauer matrix polynomials

Let $D \in \mathbb{C}^{r\times r}$ such that

$$k \notin \sigma(D), \text{for every integer } k \geq -1. \tag{10}$$

The Gegenbauer matrix polynomial $P_n(x, D)$ is defined by formula (70) in Ref. [12, p. 281], and satisfies the following three-term recurrence relation:

$$\begin{aligned}&(n+1)P_{n+1}(x, D) - x\left[(2n-1)I - D\right]P_n(x, D) + \left[(n-2)I - D\right]P_{n-1}(x, D) = 0, n \geq 1,\\&P_0(x, D) = I, P_1(x, D) = -(I + D)x.\end{aligned} \tag{11}$$

If matrix D satisfies

$$\mathrm{Re}(z) < -1, \forall z \in \sigma(D), \tag{12}$$

then the Gegenbauer matrix polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_k(x, D)P_n(x, D)W(x)dx = \frac{\sqrt{\pi}(-D-I)_n\Gamma\left(\frac{-D}{2}\right)\Gamma^{-1}\left(\frac{-(I+D)}{2}\right)\left(\left(n-\frac{1}{2}\right)I-\frac{D}{2}\right)^{-1}\delta_{kn}}{n!}, \tag{13}$$

where δ_{kn} is the Kronecker delta and $W(x)$ is the matrix function [12].

$$W(x) = \left(1 - x^2\right)^{-\frac{D}{2}-I}. \tag{14}$$

Of course, for the scalar case ($r = 1$ and $D = d \in \mathbb{R}$), the Gegenbauer matrix polynomial $P_n(x, D)$ coincide with the Gegenbauer polynomial $C_n^{\lambda}(x)$ taking $\lambda = -\frac{d+1}{2}$.

3. A Rodrigues-type formula for Gegenbauer matrix polynomials

Suppose that $n \geq 1$ and let D be a matrix in $\mathbb{C}^{r\times r}$ which satisfies (10) and (12). Let us consider

$$P_n(x,D) = K_n^{-1}\,(W(x))^{-1}\,D^n\left[(1-x^2)^n W(x)\right] \tag{15}$$

where $W(x)$, defined by (14), is integrable on interval $(-1,1)$ and K_n is an invertible matrix to be determined. Let I_{nn} be defined by

$$I_{nn} = \int_{-1}^{1} x^n P_n(x,D) W(x)\,dx. \tag{16}$$

Replacing (15) and taking into account (3), we obtain

$$\begin{aligned} I_{nn} &= \int_{-1}^{1} x^n P_n(x,D)W(x)\,dx = \int_{-1}^{1} x^n K_n^{-1}\,(W(x))^{-1}\,D^n\left[(1-x^2)^n W(x)\right] W(x)\,dx \\ &= K_n^{-1}\int_{-1}^{1} x^n D^n\left[(1-x^2)^n W(x)\right]\,dx. \end{aligned}$$

Integrating by parts once

$$\begin{aligned} I_{nn} &= K_n^{-1}\int_{-1}^{1} x^n D^n\left[(1-x^2)^n W(x)\right]\,dx \\ &= K_n^{-1}\left(x^n D^{n-1}\left[(1-x^2)^n W(x)\right]\Big|_{-1}^{1} - n\int_{-1}^{1} x^{n-1}D^{n-1}\left[(1-x^2)^n W(x)\right]\,dx\right) \\ &= K_n^{-1}(-1)n\int_{-1}^{1} x^{n-1}D^{n-1}\left[(1-x^2)^n W(x)\right]\,dx, \end{aligned}$$

and then integrating by parts n times again, we finally arrive at

$$I_{nn} = K_n^{-1}(-1)^n n!\int_{-1}^{1}(1-x^2)^n W(x)\,dx. \tag{17}$$

From (17), one obtains

$$\begin{aligned} \int_{-1}^{1}(1-x^2)^n W(x)\,dx &= \int_{-1}^{1}(1-x^2)^n\left(1-x^2\right)^{-\frac{D}{2}-I}\,dx \\ &= \int_{-1}^{1}(1+x)^{-\frac{D}{2}+(n-1)I}\,(1-x)^{-\frac{D}{2}+(n-1)I}\,dx. \end{aligned}$$

As (12) holds, by the spectral mapping theorem [7], it follows that $\mathrm{Re}(z) > 1/2\ \forall z \in \sigma(-D/2)$, $\mathrm{Re}(z) > -1\ \forall z \in \sigma\left(-\frac{D}{2}+(n-1)I\right)$. We now apply (7), (9) and (3) to derive

$$\int_{-1}^{1}(1-x^2)^n W(x)\,dx = 2^{-\frac{D}{2}+nI}B\left(-\frac{D}{2}+nI, -\frac{D}{2}+nI\right)2^{-\frac{D}{2}+(n-1)I}$$

$$\begin{aligned} &= 2^{-\frac{D}{2}+nI}\Gamma^2\left(-\frac{D}{2}+nI\right)\Gamma^{-1}\left(-D+2nI\right)2^{-\frac{D}{2}+(n-1)I} \\ &= 2^{-D}2^{2n-1}\Gamma^2\left(-\frac{D}{2}+nI\right)\Gamma^{-1}\left(-D+2nI\right). \end{aligned}$$

Finally, after taking into account (17), we conclude

$$I_{nn}K_n = (-1)^n n! 2^{-D}2^{2n-1}\Gamma^2\left(-\frac{D}{2}+nI\right)\Gamma^{-1}\left(-D+2nI\right). \tag{18}$$

Furthermore, it is easy to see that the leading coefficient of each matrix polynomial $P_n(x, D)$ is given by the matrix

$$\frac{\left(-\frac{1}{2}\left(I+D\right)\right)_n 2^n}{n!}, \tag{19}$$

which under spectral condition (12) is nonsingular, see [12, p.281]. Applying now the Lemma 2.1 of Ref. [4], we can rewrite the matrix polynomial x^nI as a linear combination of Gegenbauer matrix polynomials, *i.e.*

$$x^nI = \sum_{k=0}^{n}\alpha_k P_k(x, D), \alpha_k \in \mathbb{C}^{r\times r}, k = 0, 1, \ldots, n. \tag{20}$$

Applying the recurrence relation (11) and (19), one finds

$$\begin{aligned} x^nI &= \sum_{k=0}^{n}\alpha_k P_k(x, D) = \alpha_n P_n(x, D) + \alpha_{n-1}P_{n-1}(x, D) + \cdots + \alpha_0 P_0(x, D) \\ &= \frac{\alpha_n}{n}\left((2n-3)I - D\right)xP_{n-1}(x, D) + R_{n-1}(x) \\ &= \alpha_n\left((2n-3)I - D\right)\frac{\left(-\frac{1}{2}\left(I+D\right)\right)_{n-1}2^{n-1}}{n!}x^n + R_{n-1}(x), \end{aligned}$$

where $R_{n-1}(x)$ is a matrix polynomial of degree $n-1$. Taking into account (12), matrices $((2n-3)I - D)$ and $\left(-\frac{1}{2}\left(I+D\right)\right)_{n-1}$ are nonsingular. Thus, in order to fulfill the above equality, we must impose

$$\alpha_n = \frac{n!}{2^{n-1}}\left((2n-3)I - D\right)^{-1}\left[\left(-\frac{1}{2}\left(I+D\right)\right)_{n-1}\right]^{-1}. \tag{21}$$

Replacing x^nI given by (20) in (16) and applying (3), we have

$$I_{nn} = \int_{-1}^{1} x^n P_n(x, D)W(x)\,dx = \sum_{k=0}^{n}\alpha_k\int_{-1}^{1}P_k(x, D)P_n(x, D)W(x)\,dx.$$

Eq. (13) serves to simplify I_{nn} and to derive the following form

$$I_{nn} = \sum_{k=0}^{n}\alpha_k\int_{-1}^{1}P_k(x, D)P_n(x, D)W(x)\,dx = \alpha_n\int_{-1}^{1}P_n^2(x, D)W(x)\,dx.$$

Theorem 4 of [12] immediately yields the final expression

$$I_{nn} = \alpha_n \frac{\pi^{\frac{1}{2}}(-D-I)_n \Gamma\left(-\frac{1}{2}D\right)\Gamma^{-1}\left(-\frac{1}{2}(I+D)\right)\left(-\frac{1}{2}D+\left(n-\frac{1}{2}\right)I\right)^{-1}}{n!}. \tag{22}$$

Because I_{nn} is nonsingular, we can substite (22) in (18), and obtain

$$K_n = (-1)^n n!\, I_{nn}^{-1} 2^{-D} 2^{2n-1} \Gamma^2\left(-\frac{D}{2}+nI\right)\Gamma^{-1}\left(-D+2nI\right).$$

Next, we simplify

$$K_n = \frac{(-1)^n n! 2^{3(n-1)}}{\sqrt{\pi}} 2^{-D}\left((2n-3)I-D\right)\left((2n-1)I-D\right)S_n, \tag{23}$$

where

$$\begin{aligned} S_n = {} & \Gamma^2\left(-\frac{D}{2}+nI\right)\Gamma^{-1}(-D+2nI)\left[\left(-\frac{(I+D)}{2}\right)_{n-1}\right] \\ & \times\ [(-D-I)_n]^{-1}\Gamma^{-1}\left(-\frac{D}{2}\right)\Gamma\left(-\frac{(I+D)}{2}\right)., \end{aligned}$$

and hence, substituting S_n in (15), we have the formula we were looking for:

$$\begin{aligned} K_n P_n(x,D) &= (W(x))^{-1} D^n\left[(1-x^2)^n W(x)\right] \\ &= \left(1-x^2\right)^{\frac{D}{2}+I} D^n\left[\left(1-x^2\right)^{-\frac{D}{2}+(n-1)I}\right], n \geq 1, \end{aligned} \tag{24}$$

where K_n is given by (23). If we take $K_0 = I$, formula (24) is also valid when $n = 0$. This result is summarized by

Theorem 3.1 (Rodrigues-type Formula). *Let $D \in \mathbb{C}^{r\times r}$ satisfy (10) and (12). Then, the Gegenbauer matrix polynomials $P_n(x,D)$ defined in formula (70) of [12, p. 281] may be expressed as*

$$K_n P_n(x,D) = \left(1-x^2\right)^{\frac{D}{2}+I} D^n\left[\left(1-x^2\right)^{-\frac{D}{2}+(n-1)I}\right],$$

for $n = 0, 1, 2, \ldots$, where $K_0 = I$ and K_n is given by (23) for $n \geq 1$.

We now consider Jacobi matrix polynomials which satisfy the Rodrigues' formula according to Theorem 4.1 of [3, p.795]:

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B} D^n\left[(1-x)^{(A+nI)}(1+x)^{(B+nI)}\right], n \geq 0, \tag{25}$$

where $A, B \in \mathbb{C}^{r\times r}$ satisfy

$$\mathrm{Re}(z) > -1 \text{ for } z \in \sigma(A),\ \mathrm{Re}(z) > -1 \text{ for } z \in \sigma(B) \text{ and } AB = BA.$$

As D satisfies (10) and (12), then matrix $-D/2-I$ satisfies $\mathrm{Re}(z) > -1/2$ for $z \in \sigma(-D/2-I)$. Taking $A = B = -D/2 - I$ in (25), one gets for $n \geq 1$:

$$\begin{aligned} & P_n^{(-\frac{D}{2}-I,-\frac{D}{2}-I)}(x) \\ &= \frac{(-1)^n}{2^n n!}(1-x)^{\frac{D}{2}+I}(1+x)^{\frac{D}{2}+I}D^n\Big[(1-x)^{(-\frac{D}{2}+(n-1)I)}(1+x)^{(-\frac{D}{2}+(n-1)I)}\Big] \\ &= \frac{(-1)^n}{2^n n!}(1-x^2)^{\frac{D}{2}+I}D^n\left[(1-x^2)^{(-\frac{D}{2}+(n-1)I)}\right], \end{aligned}$$

and using (24), we find

$$P_n^{(-\frac{D}{2}-I,-\frac{D}{2}-I)}(x) = \frac{(-1)^n}{2^n n!}K_n P_n(x, D), \tag{26}$$

which is the matricial traslation of formula (1). Note that formula (26) is also true for $n = 0$. Of course, formula (26) is reduced to the formula (1) for the scalar case ($r = 1$, $D = d \in \mathbb{R}$, $\lambda = -\frac{d+1}{2}$). Thus, a connection between Gegenbauer matrix polynomials and Jacobi matrix polynomials is established by formula (26).

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