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Solving the random Legendre differential equation: Mean square power series solution and its statistical functions

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Abstract

In this paper we construct, by means of random power series, the solution of second order linear differential equations of Legendre-type containing uncertainty through its coefficients and initial conditions. By assuming appropriate hypotheses on the data, we prove that the constructed random power series solution is mean square convergent. In addition, the main statistical functions of the approximate solution stochastic process generated by truncation of the exact power series solution are given. Finally, we apply the proposed method to some illustrative examples to compare the numerical results for the average and the variance with respect to those obtained by Monte Carlo approach.

Key words: Random differential equation, random power series solution, Mean square and mean fourth calculus

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1 Introduction

Deterministic Legendre differential equation as well as its polynomial solutions play an significant role in the solution of physical problems [1]. As only a few examples, we mention that they appear in solving Kepler equation to study the motion of the planets or in the solution of physical problems based on partial differential equations using spherical coordinates. Specific examples in this latter sense are: the resolution of Laplace equation to compute the potential of a conservative field such as the space gravitational potential, the resolution of Poisson equation to compute the potential of a non-conservative field such as the electrostatic potential of charged bodies, the resolution of D'Alembert equation to study the wave propagation in strings, membranes or solid bodies, etc, [1–6]. In practice, the involved data in these applied problems, such as coefficients, forcing terms and/or initial conditions, need to be fixed after careful measures that usually contain the error of the corresponding measuring instruments. The inherent complexity of the surrounding medium or materials involved in the mathematical modeling of previous physical problems makes more realistic to assume that the data are random variables or stochastic processes rather than deterministic constants or functions. As a consequence, it seems to be advisable to develop reliable methods to study the random counterpart of the deterministic Legendre differential equation.

In dealing with random differential equations and their applications to complex problems appearing in different scientific areas useful methods are available such as Monte Carlo [7,8], polynomial chaos [9,10], Wiener-Hermite technique [11,12], dishonest method [13], [14, p.144], Itô calculus [15,16], etc. In this paper we consider the so-called mean square and mean fourth calculus which constitute powerful approaches to deal with random differential equations [17,18].

The aim of this paper is to construct mean square power series solution of the random Legendre differential equation

$$(1-t^2)\ddot{X}(t) - 2t\dot{X}(t) + A(A+1)X(t) = 0, \quad |t| < 1, \tag{1}$$

where A is a non-negative random variable satisfying certain conditions to be specified later. This includes the computation of the main statistical functions of the solution stochastic process such as its average and variance functions. Some of the tools and techniques to reach this first goal are shared with those that some of the authors have recently presented to study random Airy differential equation [19]. We point out that an important difficulty to be overcome is the lack of sub-multiplicativity of the mean square norm (and hence also of the mean fourth norm) together with the necessity of bounding products of random variables that appear as coefficients of the constructed mean square power series solution.

The paper is organized as follows. Section 2 deals with some preliminaries about the mean square and mean fourth calculus that will be required throughout the paper. The concept of fundamental set of solution stochastic processes for equation (1) is introduced in Section 2. In addition, this section includes an important inequality related to the norm of the product of random variables which will play a key role in the next section since it manages satisfactorily the lack of submultiplicativity of the mean square and mean fourth norms. Section 3 deals with the construction of a mean square convergent power series solution to (1) in the case that A is a non-negative random variable satisfying certain conditions related to the exponential growth of its absolute moments with respect to the origin. Average and variance statistical functions of the truncated random power series solution are studied in Section 4. In Section 5 we show some illustrative examples where we compare numerical results for the average and variance obtained by random power series and Monte Carlo approaches, respectively. Finally, conclusions are presented in Section 6.

2 Preliminaries

For the sake of clarity in the presentation, we begin this section by introducing some concepts, notations and results that may be found in [17, chap.4], [20, part IV], [21, chap.1-3]. Let $(\Omega, \mathcal{F}, P)$ be a probability space. In this paper we will work in the set L_2 which elements are second order real random variables (2-r.v.'s), i.e., $X : \Omega \to \mathbb{R}$ such that $\mathrm{E}[X^2] < \infty$, where $\mathrm{E}[\cdot]$ denotes the expectation operator. One can demonstrate that L_2 endowed with the so-called 2-norm

$$\|X\|_2 = \left(\mathrm{E}\left[X^2\right]\right)^{1/2},$$

has a Banach space structure.

As it is usual, given a r.v. X, $\mathrm{E}\left[|X|^s\right]$, $s > 0$ will denote the s-th absolute moment with respect the origin. Note that, $\mathrm{E}\left[|X|^0\right] = 1$. It is easy to prove that if $\mathrm{E}\left[|X|^s\right] < \infty$ then there exists $\mathrm{E}\left[|X|^r\right]$ for all $r : 0 \le r \le s$. The following result, so-called c_s-inequality, is useful for bounding the absolute moments of a binomial expression in terms of the absolute moments of both summands [20, p.157]. Moreover it establishes that if s-th absolute moments of X and Y are finite then s-th absolute moment of $X+Y$ does

$$\mathrm{E}\left[|X+Y|^s\right] \le c_s \left(\mathrm{E}\left[|X|^s\right] + \mathrm{E}\left[|Y|^s\right]\right), \quad c_s = \begin{cases} 1 & \text{if } s \le 1, \\ 2^{s-1} & \text{if } s \ge 1. \end{cases}$$

We say that $\{X(t) : t \in \mathcal{T}\}$ is a second order stochastic process (2-s.p.), if the r.v. $X(t) \in L_2$ for each $t \in \mathcal{T}$, being $\mathcal{T}$ the so-called space of times. Throughout this paper we will assume that $\mathcal{T}$ is always a real interval. The expectation function of $X(t)$ provides a statistical measure of its average statistical behavior on the domain $\mathcal{T}$ and it will denoted by $\mathrm{E}[X(t)]$ or $\mu_X(t)$, while its covariance function $\mathrm{Cov}[X(t), X(s)]$ is defined by

$$\begin{aligned} \mathrm{Cov}[X(t), X(s)] &= \mathrm{E}[(X(t) - \mu_X(t))(X(s) - \mu_X(s))] \\ &= \mathrm{E}[X(t)X(s)] - \mu_X(t)\mu_X(s), \quad t, s \in \mathcal{T}. \end{aligned} \tag{2}$$

When $s = t$, this yields the variance function

$$\mathrm{Var}[X(t)] = \mathrm{Cov}[X(t), X(t)] = \mathrm{E}\left[(X(t))^2\right] - (\mu_X(t))^2, \quad t \in \mathcal{T}, \tag{3}$$

which gives us a measure of the fluctuation of the s.p. about its mean function on $\mathcal{T}$. The term $\Gamma_X(t,s) = \mathrm{E}[X(t)X(s)]$ appearing into (2) is called the correlation function and it plays an important role in the m.s. calculus because many important stochastic results can be characterized through this two-variables deterministic function (see, [17, chap.4]).

A sequence of 2-r.v.'s $\{X_n : n \geq 0\}$ is said to be mean square (m.s.) convergent to $X \in L_2$ if

$$\lim_{n\to\infty} \|X_n - X\|_2 = \lim_{n\to\infty} \left(\mathrm{E}\left[(X_n - X)^2\right]\right)^{1/2} = 0.$$

Later we will present a method to provide an approximate solution s.p. to random differential equation (1). The following properties will play a fundamental role when we are interested in computing the mean and the variance functions of such approximations as well as assuring that they are close to the correspondent exact values.

Lemma 1 *([17, p.88]) Let $\{X_n : n \geq 0\}$, $\{Y_n : n \geq 0\}$ be two sequences of 2-r.v.'s m.s. convergent to X and Y, respectively. Then*

$$\mathrm{E}[X_n Y_n] \xrightarrow[n\to\infty]{} \mathrm{E}[XY].$$

In particular,

$$\mathrm{E}[X_n] \xrightarrow[n\to\infty]{} \mathrm{E}[X], \quad \mathrm{Var}[X_n] \xrightarrow[n\to\infty]{} \mathrm{Var}[X].$$

Remark 2 *This result can be straightforwardly extended to a sequence of s.p.'s that suits better our interests. In this case note that if $\{X_n(t) : t \in \mathcal{T}\}$ m.s. converges to $X(t)$ for $t \in \widehat{\mathcal{T}} \subset \mathcal{T}$ then the domain of convergence of the average and variance is at least $\widehat{\mathcal{T}}$, but it could be even larger.*

A 2-s.p. $\{X(t) : t \in \mathcal{T}\}$ is said to be m.s. continuous in $\mathcal{T}$ if

$$\lim_{\tau\to 0} \|X(t+\tau) - X(t)\|_2 = 0,$$

for each $t \in \mathcal{T}$, such that $t+\tau \in \mathcal{T}$ (see Example 4 below). A 2-s.p. $\{X(t) : t \in \mathcal{T}\}$ is said to be m.s. differentiable at $t \in \mathcal{T}$ and $\dot{X}(t)$ denotes its m.s. derivative if

$$\lim_{\tau \to 0} \left\| \frac{X(t+\tau)-X(t)}{\tau} - \dot{X}(t) \right\|_2 = 0,$$

for all $t \in \mathcal{T}$, such that $t+\tau \in \mathcal{T}$. These two m.s. concepts kept the same relation that their deterministic counterparts, i.e., if $\{X(t) : t \in \mathcal{T}\}$ is m.s. derivable at t then it is also m.s. continuous at t [17, p.95]. The following example will be used in our subsequent development.

Example 3 *([19, examples 2 and 3]) Let $\{X_n : n \geq 1\}$ be a sequence of r.v.'s in L_2 and $t \in \mathcal{T}$ being $\mathcal{T}$ a real interval, then for each positive integer n_0, the 2-s.p. $\{X_{n_0} t^{n_0} : t \in \mathcal{T}\}$ is m.s. derivable and then m.s. continuous for all $t \in \mathcal{T}$. The m.s. derivative of this s.p. is given by: $n_0 X_{n_0} t^{n_0-1}$.*

If X and Y are 2-r.v.'s, Schwarz inequality establishes that

$$\mathrm{E}[\,|X\,Y|\,] \leq \left(\mathrm{E}\left[X^2\right]\right)^{1/2} \left(\mathrm{E}\left[Y^2\right]\right)^{1/2}.$$

A generalization of this result is the Holder inequality that will be required later [20, p.158].

$$\mathrm{E}\left[|XY|\right] \leq \left(\mathrm{E}\left[|X|^r\right]\right)^{1/r} \left(\mathrm{E}\left[|Y|^s\right]\right)^{1/s}, \quad \text{where } r>1 \text{ and } \frac{1}{r}+\frac{1}{s}=1. \tag{4}$$

Example 4 *Let $X(t) = A(A+1)/(1-t^2)$ be a s.p. defined on $D = \{t : |t| < 1\}$, where A is a 4-r.v. Let us denote $g(t) = 1/(1-t^2)$ which is continuous on D. Then applying Schwarz inequality one gets the m.s. continuity of s.p. $X(t)$ on D:*

$$\begin{aligned} \|X(t+\tau)-X(t)\|_2 &= \left(\mathrm{E}\left[A^2(A+1)^2\right]\right)^{1/2} |g(t+\tau)-g(t)| \\ &\leq \left(\mathrm{E}\left[A^4\right]\right)^{1/4} \left(\mathrm{E}\left[(A+1)^4\right]\right)^{1/4} |g(t+\tau)-g(t)| \xrightarrow[\tau \to 0]{} 0, \end{aligned}$$

since both expectations factors are finite because A is a 4-r.v.

Later we will require to use the following basic property

$$AX_n \xrightarrow[n\to\infty]{\text{m.s.}} AX, \tag{5}$$

which holds true if $A \in L_2$, $\{X_n : n \geq 0\}$ is a sequence of 2-r.v.'s such that $X_n \xrightarrow[n\to\infty]{\text{m.s.}} X$ and A, X_n are independent r.v.'s for each n. However, independence hypothesis cannot be assumed in many practical cases like those that we will consider below. This motivates the introduction of r.v.'s X such that $\mathrm{E}[X^4] < \infty$ which will be denoted by 4-r.v.'s. Note that a 4-r.v. is a 2-r.v.

The set L_4 of all 4-r.v.'s endowed with the norm

$$\|X\|_4 = \sqrt[4]{\mathrm{E}[X^4]}, \tag{6}$$

is a Banach space (see [22, p.9]). A stochastic processes $\{X(t)\colon t \in T\}$, where $\mathrm{E}\Big[(X(t))^4\Big] < \infty$ for all $t \in T$, will be called a 4-s.p. Applying Theorem 8, one can prove immediately that

$$\|XY\|_4 = \|X\|_4 \|Y\|_4. \tag{7}$$

provided that $X, Y \in L_4$ are independent r.v.'s. A sequence of 4-r.v.'s $\{X_n : n \geq 0\}$ is said to be mean fourth (m.f.) convergent to a 4-r.v. X if $\lim_{n\to\infty} \|X_n - X\|_4 = 0$. This type of convergence will be represented by $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$. By applying Schwarz inequality one can establish the link between m.s. and m.f. convergence.

Lemma 5 *([19]) Let $\{X_n : n \geq 0\}$ be a sequence of 4-r.v.'s and suppose that $X_n \xrightarrow[n\to\infty]{m.f.} X$. Then $X_n \xrightarrow[n\to\infty]{m.s.} X$.*

Following result is a consequence of Lemmas 1 and 5 and it provides sufficient conditions in order to property (5) holds true without assuming hypotheses based on independence.

Lemma 6 *([19]) Let A be a 4-r.v. and $\{X_n : n \geq 0\}$ a sequence of 4-r.v.'s such that $X_n \xrightarrow[n\to\infty]{m.f.} X$. Then $AX_n \xrightarrow[n\to\infty]{m.s.} AX$.*

Random linear differential equation (1) can be written in the form

$$\ddot{X}(t) + A_1(t)\dot{X}(t) + A_2(t)X(t) = 0, \quad t_1 < t < t_2, \tag{8}$$

where $A_1(t)$ and $A_2(t)$ are m.s. continuous s.p.'s (see Example 4). Analogously to deterministic framework, in order to describe its solution s.p. $X(t)$ one can first to determine a fundamental set of solutions.

Definition 7 *Let $A_1(t)$ and $A_2(t)$ be s.p.'s, and let $X_1(t)$ and $X_2(t)$ be two solutions of the second-order random differential equation (8). We say that $\{X_1(t), X_2(t)\}$ is a fundamental set of solution processes of (8) in $t_1 < t < t_2$, if any solution $X(t)$ of (8) admits a unique representation of the form*

$$X(t) = C_1 X_1(t) + C_2 X_2(t), \quad t \in (t_1, t_2), \tag{9}$$

where C_1 and C_2 are r.v.'s uniquely determined by $X(t)$.

The wronskian process $W_{\mathcal{S}}(t) = X_1(t)\dot{X}_2(t) - X_2(t)\dot{X}_1(t)$ plays a relevant role to provide a fundamental set of solutions. In fact, one can extended the deterministic proof to demonstrate that if there exists $t_0 \in (t_1, t_2)$ such that $W_{\mathcal{S}}(t_0) \neq 0$, then $\mathcal{S}$ is a fundamental set of solution processes to (8).

The following result will be useful to take advantage of property (7) to compute some bounds later.

Theorem 8 *([23, p.93]) Let X, Y be independent r.v.'s. and f, g measurable Borel functions of each r.v., respectively. Then $f(X)$ and $g(Y)$ are also independent r.v.'s.*

We close this section by establishing the following inequality that will play a prime role for bounding the 4-norm of a product of r.v.'s what will be required in the next section.

Proposition 9 *Let $\{Y_i\}_{i=1}^n$, $n \geq 1$ be r.v.'s such that $\mathrm{E}\left[(Y_i)^{4n}\right] < \infty$, $i = 1, 2, \ldots, n$, then*

$$\left\| \prod_{i=1}^{n} Y_i \right\|_4 \leq \prod_{i=1}^{n} \left(\| (Y_i)^n \|_4 \right)^{1/n}, \quad n \geq 1. \tag{10}$$

This result draws directly from the following one by taking $X_i = (Y_i)^4$ and considering the definition of 4-norm given by (6).

Proposition 10 *Let $\{X_i\}_{i=1}^n$, $n \geq 1$ be r.v.'s such that $\mathrm{E}\left[|X_i|^n\right] < \infty$, $i = 1, 2, \ldots, n$, then*

$$\mathrm{E}\left[\prod_{i=1}^{n} |X_i| \right] \leq \left(\prod_{i=1}^{n} \mathrm{E}\left[|X_i|^n\right] \right)^{1/n}, \quad n \geq 1. \tag{11}$$

Proof. We proceed by induction on the integer n. For $n = 1$, the result follows immediately and becomes and identity. Let us assume that (11) holds for n. Then we apply Holder inequality (4) for $X = \prod_{i=1}^{n} X_i$, $Y = X_{n+1}$, $r = (n+1)/n$ and $s = n + 1$:

$$\mathrm{E}\left[|X_1| \cdots |X_n| \, |X_{n+1}|\right] \leq \left(\mathrm{E}\left[(|X_1| \cdots |X_n|)^{(n+1)/n} \right] \right)^{n/(n+1)} \left(\mathrm{E}\left[(|X_{n+1}|)^{n+1} \right] \right)^{1/(n+1)}. \tag{12}$$

By induction hypothesis one gets

$$\left(\mathrm{E}\left[(|X_1| \cdots |X_n|)^{(n+1)/n} \right] \right)^n \leq \mathrm{E}\left[(|X_1|)^{n+1} \right] \cdots \mathrm{E}\left[(|X_n|)^{n+1} \right].$$

Then substituting this expression in (12) one obtains

$$\mathrm{E}\left[|X_1| \cdots |X_n| \, |X_{n+1}|\right] \leq \left(\mathrm{E}\left[\prod_{i=1}^{n+1} |X_i|^{n+1} \right] \right)^{1/(n+1)},$$

which proves the proposition. ☒

3 Solving the random Legendre differential equation

This section deals with the construction of a power series solution of the random differential equation (1) which is m.s. convergent in certain domain about $t = 0$ to be specified later. Hereinafter, we will assume that the absolute moments with respect to the origin of non-negative r.v. A appearing in (1) increase at the most exponentially, that is, there exist a nonnegative integer n_0 and positive constants H and M such that

$$\mathrm{E}\left[|A|^n\right] \le H\, M^n < +\infty, \quad \forall\, n \ge n_0. \tag{13}$$

Equivalently, we assume that $\mathrm{E}\left[|A|^n\right] = \mathcal{O}(M^n)$ for a positive constant M.

Let us seek a formal power series solution s.p. to problem (1)

$$X(t) = \sum_{n\ge 0} X_n t^n, \tag{14}$$

where coefficients X_n are 2-r.v.'s to be determined. Assuming that $X(t)$ is termwise m.s. differentiable, by applying Example 3, one gets

$$\dot{X}(t) = \sum_{n\ge 1} nX_n t^{n-1}, \;\; -2t\dot{X}(t) = \sum_{n\ge 1} -2nX_n t^n = -2X_1 t + \sum_{n\ge 2} -2nX_n t^n, \tag{15}$$

$$\ddot{X}(t) = \sum_{n\ge 2} n(n-1)X_n t^{n-2}, \tag{16}$$

$$(1-t^2)\ddot{X}(t) = 2X_2 + 6X_3 t + \sum_{n\ge 2}(n+2)(n+1)X_{n+2}t^n - \sum_{n\ge 2} n(n-1)X_n t^n. \tag{17}$$

By imposing that (14), (15) and (17) satisfy (1), one gets

$$\begin{aligned} &2X_2 + A(A+1)X_0 + \left\{\left[A(A+1)-2\right]X_1 + 6X_3\right\} t \\ &+ \sum_{n\ge 2}\left\{(n+2)(n+1)X_{n+2} + \left[-n(n-1) - 2n + A(A+1)\right]X_n\right\} t^n = 0. \end{aligned}$$

Therefore a candidate m.s. solution s.p. to problem (1) can be obtained by imposing

$$\left.\begin{aligned} 2X_2 + A(A+1)X_0 = 0, \quad \left[A(A+1)-2\right]X_1 + 6X_3 = 0, \\ (n+2)(n+1)X_{n+2} + \left[-n(n+1) + A(A+1)\right]X_n = 0,\;\; n \ge 2, \end{aligned}\right\}$$

i.e.,

$$X_{n+2} = -\frac{(A+n+1)(A-n)}{(n+2)(n+1)} X_n, \;\; n \ge 0,$$

where we have used that $-n(n+1)+A(A+1)=(A+n+1)(A-n)$. By a recursive reasoning, these coefficients X_n can be represented as follows

$$X_{2m}=\frac{(-1)^m}{(2m)!}X_0P_1(m),\quad P_1(m)=\prod_{k=1}^{m}(A-2k+2)(A+2k-1),\ \ m\geq 0,\ (18)$$

$$X_{2m+1}=\frac{(-1)^m}{(2m+1)!}X_1P_2(m),\quad P_2(m)=\prod_{k=1}^{m}(A-2k+1)(A+2k),\ \ m\geq 0, \tag{19}$$

where we agree $\prod_{k=u}^{v}f(k)=1$ if $v<u$, as usual. As a consequence, s.p. given by (14) can be represented as

$$X(t)=X_1(t)+X_2(t),\quad \text{where}\ \begin{cases} X_1(t)=\sum_{m\geq 0}X_{2m}t^{2m}, \\ X_2(t)=\sum_{m\geq 0}X_{2m+1}t^{2m+1}, \end{cases} \tag{20}$$

where coefficients X_{2m} and X_{2m+1} are given by (18)–(19).

Previous exposition has been addressed to obtain a *formal* power series solution of random Legendre differential equation (1). Note that we have implicitly applied the commutation between the r.v. A and the random infinite sum given by (14) that, according to Lemma 6, needs to be legitimized. Thus, we have to justify that m.f. convergence of random power series defined in (18)–(20). We shall do that for the first series $X_1(t)$ since for the second one we can proceed analogously. By assuming independence between initial condition X_0 and r.v. A, by (7) and Theorem 8 one gets

$$\sum_{m\geq 0}\|X_{2m}\|_4|t|^{2m}=\|X_0\|_4\sum_{m\geq 0}\frac{1}{(2m)!}\|P_1(m)\|_4|t|^{2m}. \tag{21}$$

Under hypotheses (13), we can apply inequality (9) for $n=2$ and then we get

$$\begin{aligned}\|P_1(m)\|_4&=\left\|\left(\prod_{k=1}^{m}(A-2k+2)\right)\left(\prod_{k=1}^{m}(A+2k-1)\right)\right\|_4\\ &\leq\left(\left\|\prod_{k=1}^{m}(A-2k+2)^2\right\|_4\right)^{1/2}\left(\left\|\prod_{k=1}^{m}(A+2k-1)^2\right\|_4\right)^{1/2}.\end{aligned} \tag{22}$$

Now we bound first factor of right-hand side of (22) applying firstly inequality (9) for $n=m$, secondly c_s-inequality for $X=A$, $Y=-2k+2$ and $s=8m$, and finally arithmetic-geometric inequality [24, p.29]:

$$\begin{aligned}
\left(\left\|\prod_{k=1}^{m}(A-2k+2)^2\right\|_4\right)^{1/2} &\leq \prod_{k=1}^{m}\left(\left\|(A-2k+2)^{2m}\right\|_4\right)^{1/(2m)} \\
&= \prod_{k=1}^{m}\left(\mathrm{E}\left[|(A-2k+2)|^{8m}\right]\right)^{1/(8m)} \\
&\leq \prod_{k=1}^{m}\left(2^{8m-1}\left\{\mathrm{E}\left[|A|^{8m}\right]+|-2k+2|^{8m}\right\}\right)^{1/(8m)} \\
&= 2^{1-\frac{1}{8m}}\left\{\left(\prod_{k=1}^{m}\mathrm{E}\left[|A|^{8m}\right]+(2(k-1))^{8m}\right)^{1/m}\right\}^{1/8} \\
&\leq 2^{1-\frac{1}{8m}}\left(\frac{1}{m}\sum_{k=1}^{m}\mathrm{E}\left[|A|^{8m}\right]+(2k)^{8m}\right)^{1/8} \\
&= 2^{1-\frac{1}{8m}}\left(\mathrm{E}\left[|A|^{8m}\right]+\frac{2^{8m}}{m}\sum_{k=1}^{m}k^{8m}\right)^{1/8} \\
&\leq 2^{1-\frac{1}{8m}}\left(\mathrm{E}\left[|A|^{8m}\right]+(2m)^{8m}\right)^{1/8}.
\end{aligned}$$

Taking into account hypothesis (13) we can further bound previous expression

$$\left(\left\|\prod_{k=1}^{m}(A-2k+2)^2\right\|_4\right)^{1/2} \leq 2^{1-\frac{1}{8m}}\left(MH^{8m}+(2m)^{8m}\right)^{1/8}, \ \forall m \geq m_0.$$

On the other hand, we always can choice an integer $m_1 \geq m_0 \geq 0$, large enough such that: $(2m)^{8m} \geq HM^{8m}$ for each $m \geq m_1$, then

$$\left(\left\|\prod_{k=1}^{m}(A-2k+2)^2\right\|_4\right)^{1/2} \leq 2^{2-\frac{1}{8m}}(2m)^m, \ \forall m \geq m_1.$$

Following an analogous reasoning, we can get just the same bound for the second factor appearing in the right-hand side of (22). As consequence we obtain

$$\|P_1(m)\|_4 \leq 2^{4-\frac{1}{4m}}(2m)^{2m}, \ \forall m \geq m_1,$$

what allow us to assures that the following deterministic series majorizes that given in (21):

$$\sum_{m\geq m_1}\alpha_m, \quad \text{where} \quad \alpha_m = \frac{1}{(2m)!}\|X_0\|_4\, 2^{4-\frac{1}{4m}}(2m)^{2m}|t|^{2m}.$$

By applying D'Alembert criterion, it is easy to check that this series is convergent in the domain $D = \{t \in \mathbb{R} : |t| < 1/e\}$, where $e = \exp(1)$ is the Euler constant. Therefore, random series $X_1(t)$ given by (18)–(20) is m.f. convergent in D, and so by Lemma 5 is also m.s. convergent. Following an analogous way, it is easy to establish the m.s. convergence of the second series $X_2(t)$ given by (19)–(20) in the same domain. Note that the reasoning above shows that both series solution $X_1(t)$ and $X_2(t)$ are m.s. uniformly convergent, therefore taking into account Example 3 and theorem 10 of [19], the formal differentiation considered in (15)–(16) is justified. On the other hand, taking $t_0 = 0$ and considering that $X_1(0) = 1$, $\dot{X}_1(0) = 0$, $X_2(0) = 0$ and $\dot{X}_2(0) = 1$, one gets that $W_{\mathcal{S}}(0) = 1 \neq 0$, then according to (9) the solution of random differential equation (1) with random initial conditions $X(0) = X_0$ and $\dot{X}(0) = X_1$ is given by

$$X(t) = X_0\widetilde{X_1}(t) + X_1\widetilde{X_2}(t), \quad t \in D = \{t \in \mathbb{R} : |t| < 1/e\}, \tag{23}$$

where $\widetilde{X_1}(t)$ and $\widetilde{X_2}(t)$ are defined by

$$\widetilde{X_1}(t) = \sum_{m\geq 0} \frac{(-1)^m}{(2m)!} P_1(m) t^{2m}, \quad \widetilde{X_2}(t) = \sum_{m\geq 0} \frac{(-1)^m}{(2m+1)!} P_2(m) t^{2m+1}, \tag{24}$$

where $P_1(m)$ and $P_2(m)$ are given by (18), (19), respectively. Summarizing the following result has been established:

Theorem 11 *Let us assume that r.v. A satisfies condition (13) and it is independent of r.v.'s X_0 and X_1. Then the differential equation (1) with initial conditions $X(0) = X_0$, $\dot{X}(0) = X_1$ admits a random power series solution of the form (23)–(24) and (18)–(19). Moreover the solution is m.s. convergent for each $t \in D = \{t \in \mathbb{R} : |t| < 1/e\}$.*

Remark 12 *Unlike its deterministic counterpart where the domain of convergence of the correspondent power series solution is $\mathcal{D} = \{t : |t| < 1\}$ [3, p.183], Theorem 11 just guarantees a smaller domain given by D. We point out that regarding the construction of the solution stochastic process to random differential equation (1) this is not an inconvenient of the previous approach. In fact, one the random power series solution $\widehat{X_1}(t)$ has been constructed on the domain $D = D_1 = (0, \widehat{t_1})$ with $0 < \widehat{t_1} < 1/e$, an extended solution, say $\widehat{X_2}(t)$, can be constructed on the domain $D_2 = (\widehat{t_1}, 2\widehat{t_1})$ by applying exactly the same argument but considering as initial conditions $\widehat{X_2}(\widehat{t_1}) = \widehat{X_1}(\widehat{t_1})$ and $\dot{\widehat{X}}_2(\widehat{t_1}) = \dot{\widehat{X}}_1(\widehat{t_1})$. This procedure can be repeated as many times as necessary up to fill $\mathcal{D}$. Note that taking $\widehat{t_1}$ close enough to $1/e$, the procedure will end just in three steps.*

From a practical point of view once a non-negative r.v. A has been set, we need to check that it satisfies condition (13) in order to guarantee that random power series given by (23)–(24) and (18)–(19) is m.s. convergent. Although, in general this condition is not useful because of the lack of explicit expressions for the absolute moments with respect to the origin of relevant r.v.'s such as Binomial, Poisson, etc. Nevertheless, if A is a r.v. having finite domain, say, $a_1 \leq A(\omega) \leq a_2$, for each $\omega \in \Omega$ then it satisfies condition (13). Indeed, without loss of generality let us assume that A is a continuous r.v. with probability density function $f_A(a)$, then

$$\mathrm{E}\left[|A|^n\right] = \int_{a_1}^{a_2} |a|^n f_A(a)\, da \leq H^n, \quad \text{where} \quad H = \max(|a_1|, |a_2|).$$

Substituting the integral for a sum, previous conclusion remains true if A is a discrete r.v. Note that important r.v.'s such as Binomial, Hypergeometric, Uniform or Beta have finite domain. As a consequence, we can take advantage of the so-called truncation method (see [20]) to deal with unbounded r.v.'s such as Exponential or Gaussian. In fact, given a r.v. with an unbounded domain it can be approximated by censuring adequately its domain and this approximation can be improved further by enlarging enough the truncated domain (see Example 14 below for details).

4 Approximate average and variance functions of the mean square random power series solution

This section is devoted to compute approximations of the average and the variance of the m.s. solution defined by (23)–(24) and (18)–(19). These approximations will be expressed in terms of the data $\mathrm{E}[X_0]$, $\mathrm{E}[X_1]$, $\mathrm{E}[X_0 X_1]$, $\mathrm{E}\left[(X_0)^2\right]$, $\mathrm{E}\left[(X_1)^2\right]$ and certain moments related to algebraic transformations of the random coefficient A that will be specified later. Note that the solution is an infinite series, then in practice we need to truncate it at finite terms, so we will consider the truncation of order M

$$X^M(t) = X_0 \widetilde{X}_1^M(t) + X_1 \widetilde{X}_2^M(t), \quad \text{where} \quad \begin{cases} \widetilde{X}_1^M(t) = \displaystyle\sum_{m=0}^{\left[\frac{M}{2}\right]} \frac{(-1)^m}{(2m)!} P_1(m) t^{2m}, \\ \widetilde{X}_2^M(t) = \displaystyle\sum_{m=0}^{\left[\frac{M-1}{2}\right]} \frac{(-1)^m}{(2m+1)!} P_2(m) t^{2m+1}, \end{cases} \tag{25}$$

which corresponds to a polynomial of degree M.

Since r.v. A is assumed to be independent of random initial conditions $X(0) = X_0$ and $\dot{X}(0) = X_1$, then taking the expectation operator in (25) one gets

$$\mu_{X^M}(t) = \mathrm{E}[X_0] \sum_{m=0}^{\left[\frac{M}{2}\right]} \frac{(-1)^m t^{2m}}{(2m)!} \mathrm{E}[P_1(m)] + \mathrm{E}[X_1] \sum_{m=0}^{\left[\frac{M-1}{2}\right]} \frac{(-1)^m t^{2m+1}}{(2m+1)!} \mathrm{E}[P_2(m)]. \tag{26}$$

Depending whether non-negative r.v. A is discrete, with probability mass function $p_A(a)$, or continuous, with probability density function $f_A(a)$, the expectation terms involved in (26) can be computed as follows

$$\mathrm{E}\left[P_1(m)\right] = \begin{cases} \displaystyle\sum_{a>0:p_A(a)>0} \prod_{j=1}^{m} (A-2j+2)(A+2j-1)p_A(a), \\ \displaystyle\int_0^{\infty} \prod_{j=1}^{m} (A-2j+2)(A+2j-1)f_A(a)\,da, \end{cases}$$

and

$$\mathrm{E}\left[P_2(m)\right] = \begin{cases} \displaystyle\sum_{a>0:p_A(a)>0} \prod_{j=1}^{m} (A-2j+1)(A+2j)p_A(a), \\ \displaystyle\int_0^{\infty} \prod_{j=1}^{m} (A-2j+1)(A+2j)f_A(a)\,da. \end{cases}$$

Taking into account the expression (3), for computing the variance of the truncated solution process, now we only require to calculate $\mathrm{E}\left[\left(X^M(t)\right)^2\right]$. Since $X(t) = X_1(t) + X_2(t)$, from (20) one gets

$$\begin{aligned} \mathrm{E}\left[\left(X^M(t)\right)^2\right] &= \mathrm{E}\left[\left(\sum_{m=0}^{\left[\frac{M}{2}\right]} X_{2m}t^{2m}\right)^2\right] + \mathrm{E}\left[\left(\sum_{m=0}^{\left[\frac{M-1}{2}\right]} X_{2m+1}t^{2m+1}\right)^2\right] \\ &+ 2\sum_{m=0}^{\left[\frac{M}{2}\right]} \sum_{n=0}^{\left[\frac{M-1}{2}\right]} \mathrm{E}\left[X_{2m}X_{2n+1}\right] t^{2(m+n)+1}, \end{aligned} \tag{27}$$

where $[\cdot]$ denotes the integer function. To compute the two first terms on the right-hand side we will use the following relationship

$$\mathrm{E}\left[\left(X^P(t)\right)^2\right] = \sum_{p=0}^{P} \mathrm{E}\left[(X_p)^2\right] t^{2p} + 2\sum_{p=1}^{P}\sum_{l=0}^{p-1} \mathrm{E}\left[X_p\,X_l\right] t^{p+l}.$$

Hence

$$\mathrm{E}\left[\left(\sum_{m=0}^{\left[\frac{M}{2}\right]} X_{2m}t^{2m}\right)^2\right] = \sum_{m=0}^{\left[\frac{M}{2}\right]} \mathrm{E}\left[(X_{2m})^2\right] t^{4m} + 2\sum_{m=1}^{\left[\frac{M}{2}\right]}\sum_{n=0}^{m-1} \mathrm{E}\left[X_{2m}X_{2n}\right] t^{2(m+n)},$$

$$\mathrm{E}\left[\left(\sum_{m=0}^{\left[\frac{M-1}{2}\right]} X_{2m+1}t^{2m+1}\right)^2\right] = \sum_{m=0}^{\left[\frac{M-1}{2}\right]} \mathrm{E}\left[(X_{2m+1})^2\right] t^{4m+2} + 2\sum_{m=1}^{\left[\frac{M-1}{2}\right]}\sum_{n=0}^{m-1} \mathrm{E}\left[X_{2m+1}X_{2n+1}\right] t^{2(m+n+1)}.$$

The expectations involved in these expressions together with those that are contained in the last term of the right-hand side of (27) can be computed as follows:

$$\mathrm{E}\left[X_{2m}X_{2n}\right] = \frac{(-1)^{n+m}}{(2m)!(2n)!}\mathrm{E}\left[(X_0)^2\right]\mathrm{E}\left[P_1(m)P_1(n)\right], \quad m,n = 0,1,2,\ldots,$$

$$\mathrm{E}\left[X_{2m+1}X_{2n+1}\right] = \frac{(-1)^{n+m}}{(2m+1)!(2n+1)!}\mathrm{E}\left[(X_1)^2\right]\mathrm{E}\left[P_2(m)P_2(n)\right], \quad m,n=0,1,2,\ldots,$$

$$\mathrm{E}\left[X_{2m}X_{2n+1}\right] = \frac{(-1)^{n+m}}{(2m)!(2n+1)!}\mathrm{E}\left[X_0X_1\right]\mathrm{E}\left[P_1(m)P_2(n)\right], \quad m,n=0,1,2,\ldots,$$

where expectations above can be computed as follows:

$$\mathrm{E}\left[P_1(m)P_1(n)\right] = \begin{cases} \displaystyle\sum_{a>0:p_A(a)>0}\prod_{j=0}^{m}(a-2j+2)(a+2j-1)\prod_{j=0}^{n}(a-2j+2)(a+2j-1)p_A(a), \\ \displaystyle\int_0^{\infty}\prod_{j=0}^{m}(a-2j+2)(a+2j-1)\prod_{j=0}^{n}(a-2j+2)(a+2j-1)f_A(a)\,da, \end{cases}$$

$$\mathrm{E}\left[P_1(m)P_2(n)\right] = \begin{cases} \displaystyle\sum_{a>0:p_A(a)>0}\prod_{j=0}^{m}(a-2j+2)(a+2j-1)\prod_{j=0}^{n}(a-2j+1)(a+2j)p_A(a), \\ \displaystyle\int_0^{\infty}\prod_{j=0}^{m}(a-2j+2)(a+2j-1)\prod_{j=0}^{n}(a-2j+1)(a+2j)f_A(a)\,da, \end{cases}$$

$$\mathrm{E}\left[P_2(m)P_2(n)\right] = \begin{cases} \displaystyle\sum_{a>0:p_A(a)>0}\prod_{j=0}^{m}(a-2j+1)(a+2j)\prod_{j=0}^{n}(a-2j+1)(a+2j)p_A(a), \\ \displaystyle\int_0^{\infty}\prod_{j=0}^{m}(a-2j+1)(a+2j)\prod_{j=0}^{n}(a-2j+1)(a+2j)f_A(a)\,da. \end{cases}$$

At this point, Lemma 1 plays a crucial role since it guarantees the convergence of the average and the variance of the truncated solution (25).

5 Examples

In this section we provide several illustrative examples. The results obtained to approximate the average and the variance by means of the series method presented in this paper are compared with respect to the corresponding ones provided by Monte Carlo approach that can be considered once of the most widespread methods to deals with random differential equations.

Example 13 *Let us consider the random differential equation (1) where A is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 3$, i.e., $A \sim Be(\alpha = 2; \beta = 3)$. Let X_0 and X_1 be initial conditions such that $\mathrm{E}[X_0] = 1$, $\mathrm{E}\left[(X_0)^2\right] = 2$, $\mathrm{E}[X_1] = 2$, $\mathrm{E}\left[(X_1)^2\right] = 5$. We also assume that X_0, X_1 and A are independent r.v.'s. Note that A satisfies condition (13) since it takes values on the bounded interval $[0,1]$. Then Theorem 11 guarantees that the m.s. series solution of problem (1) with initial conditions X_0 and X_1 is given by (23)–(24) and (18)–(19), and it is m.s. convergent on $[0,1/e]$ at least. Table 1 collects the expectation of the truncated solution s.p. for different values of the truncation order M (denoted by $\mu_{X^M}(t)$) at different values of the time parameter t.*

These numerical results are compared with respect to the corresponding ones obtained by Monte Carlo method ($\widetilde{\mu}_X^m(t)$) using m simulations. One observes

t	$\mu_{X^M}(t)$ $M=10$	$\mu_{X^M}(t)$ $M=20$	$\mu_{X^M}(t)$ $M=80$	$\widetilde{\mu}_X^m(t)$ $m=10^5$	$\widetilde{\mu}_X^m(t)$ $m=2\times10^5$	$\widetilde{\mu}_X^m(t)$ $m=5\times10^5$
0.0	1.00000	1.00000	1.00000	0.994659	0.999361	1.00316
0.1	1.19532	1.19532	1.19532	1.19002	1.19485	1.19843
0.2	1.38240	1.38240	1.38240	1.37719	1.38212	1.38545
0.3	1.56276	1.56276	1.56276	1.55770	1.56267	1.56572
0.4	1.73776	1.73776	1.73776	1.73290	1.73787	1.74060
0.5	1.90876	1.90879	1.90879	1.90421	1.90912	1.91145
0.6	2.07727	2.07750	2.07750	2.07327	2.07805	2.07993
0.7	2.24502	2.24631	2.24633	2.24259	2.24712	2.24845
0.8	2.41388	2.41971	2.42007	2.41702	2.42115	2.42176
0.9	2.58518	2.6077	2.61324	2.61134	2.61469	2.61419

Table 1
Comparison of the average using random truncated power series and Monte Carlo methods in Example 13

that for values of t near of the origin (where the initial conditions are established and the series solution s.p. is centered), the approximations obtained by the method proposed in this paper coincide for different truncation orders of the series solution. In fact, Table 1 shows that the approximations coincide in all their decimal digits for $M = 10$ from $t = 0$ to $t = 0.40$. The full stabilization of the five decimal digits showed from $t = 0$ to $t = 0.9$ is achieved for $M = 80$. Regarding approximations obtained by Monte Carlo method, they improve as the number m of simulations increases, as expected. However, it is worthwhile pointing out that, in general, they are worse than those obtained by truncated series method. Even more, the achievement of better numerical approximations using Monte Carlo entails an increase of the number of simulations, and therefore of the computational cost, which is higher than that required by the random truncated series method.Table 2 compares the values of variance for the truncation method with respect to Monte Carlo method. In order to show accurate approximations of the variance, $Var\left[X^M(t)\right]$, greater values of M are required. So, in Table 2 we have considered values of M that differ from those we have taken in Table 1. In this case, stabilization is achieved for $M = 110$.

Example 14 *In this example we take advantage of the truncation method (see [20]) to deal with a r.v. A that neither satisfy condition (13) nor has bounded domain. Let us consider model (1) where A is an Exponential r.v., $A \sim \mathrm{Exp}(\lambda = 0.25)$. We will assume that the initial conditions X_0 and X_1*

t	$\mathrm{Var}\left[X^M(t)\right]$ $M=20$	$\mathrm{Var}\left[X^M(t)\right]$ $M=80$	$\mathrm{Var}\left[X^M(t)\right]$ $M=110$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m=10^5$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m=2\times10^5$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m=5\times10^5$
0.0	1.00000	1.00000	1.00000	0.999229	0.998349	1.00186
0.1	1.00003	1.00003	1.00003	0.999857	0.998477	1.00179
0.2	1.00048	1.00048	1.00048	1.00089	0.999029	1.00205
0.3	1.00259	1.00259	1.00259	1.00358	1.00125	1.00388
0.4	1.00898	1.00898	1.00898	1.0105	1.00774	1.00989
0.5	1.02472	1.02472	1.02472	1.0267	1.02357	1.02515
0.6	1.05997	1.05997	1.05997	1.06229	1.05889	1.05978
0.7	1.13701	1.13703	1.13703	1.13944	1.13595	1.13608
0.8	1.31400	1.31477	1.31477	1.31667	1.31355	1.31286
0.9	1.79008	1.82113	1.82116	1.82076	1.81934	1.81806

Table 2
Comparison of the variance using random truncated power series and Monte Carlo methods in Example 13

together with A are independent r.v.'s such that $\mathrm{E}[X_0]=1.5$, $\mathrm{E}\left[(X_0)^2\right]=3$, $\mathrm{E}[X_1]=3$, $\mathrm{E}\left[(X_1)^2\right]=10$. Note that r.v. A has unbounded domain. Then, we will consider the truncation of this r.v. on intervals $[0,\widehat{a}]$ for $\widehat{a}=10,20,50$ that will contain all the values of A with probability $\int_0^{\widehat{a}} 0.25\exp(-0.25a)\,da$, that correspond to 0.917915, 0.999996 and ≈ 1, respectively. The probability density function associated to the new censured r.v., say B, is

$$f_B(b)=\frac{\exp(-0.25b)}{\displaystyle\int_0^{\widehat{a}}\exp(-0.25b)\,db},\quad 0\le b\le \widehat{a}.$$

As a consequence, B satisfies hypotheses of Theorem 11 since it takes values on a bounded interval. Tables 3 and 4 show approximations of the expectation and variance of the solution s.p. computed by the truncation series on the interval $[0,\widehat{a}]$ and truncation order M, and Monte Carlo methods. Numerical results show that both, average $\mu_{X^M_{[0,\widehat{a}]}}(t)$ and variance $Var\left[X^M_{[0,\widehat{a}]}(t)\right]$, obtained by truncation series method are close to those computed by Monte Carlo (for which no truncation on the r.v. A has been considered) as we enlarge the length of the censured interval. Except in the case of the variance on the interval $[0,50]$ where $M=200$, in both tables, we have taken $M=100$ as the order of truncation since it corresponds to numerical stabilization of the results while $m=2\times10^5$ and $m=5\times10^5$ simulations have been considered for Monte Carlo method.

t	$\mu_{X^M_{[0,\widehat{a}]}}(t)$ $M=100, \widehat{a}=10$	$\mu_{X^M_{[0,\widehat{a}]}}(t)$ $M=100, \widehat{a}=20$	$\mu_{X^M_{[0,\widehat{a}]}}(t)$ $M=100, \widehat{a}=50$	$\widetilde{\mu}_X^m(t)$ $m=2\times 10^5$	$\widetilde{\mu}_X^m(t)$ $m=5\times 10^5$
0.0	1.50000	1.50000	1.50000	1.50014	1.49866
0.1	1.65422	1.56799	1.55081	1.55058	1.54973
0.2	1.54492	1.34274	1.33376	1.33441	1.33554
0.3	1.28778	1.12846	1.12562	1.12854	1.13096
0.4	1.03944	0.996154	0.983988	0.988924	0.990145
0.5	0.909792	0.893783	0.891029	0.897502	0.89584
0.6	0.896828	0.832867	0.82659	0.8367	0.829465
0.7	0.895265	0.784002	0.77795	0.7866	0.779368
0.8	0.808709	0.741713	0.737478	0.748983	0.739216
0.9	0.719894	0.701313	0.701115	0.704677	0.700813

Table 3
Comparison of the average using random truncated power series and Monte Carlo methods in Example 14

t	$\mathrm{Var}\left[X^M_{[0,\widehat{a}]}(t)\right]$ $M=100, \widehat{a}=10$	$\mathrm{Var}\left[X^M_{[0,\widehat{a}]}(t)\right]$ $M=100, \widehat{a}=20$	$\mathrm{Var}\left[X^M_{[0,\widehat{a}]}(t)\right]$ $M=200, \widehat{a}=50$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m=2\times 10^5$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m=5\times 10^5$
0.0	0.75	0.75	0.75	0.748658	0.749314
0.1	0.637807	0.727614	0.77445	0.771705	0.77315
0.2	0.876669	1.37508	1.38233	1.38815	1.38445
0.3	1.77302	2.07496	2.05924	2.08024	2.06537
0.4	2.68231	2.63051	2.62557	2.65344	2.63869
0.5	3.21101	3.17763	3.15544	3.18115	3.16771
0.6	3.71169	3.72846	3.72451	3.7337	3.73241
0.7	4.28573	4.4122	4.41011	4.43339	4.42044
0.8	5.34813	5.41149	5.40622	5.43357	5.42255
0.9	7.2643	7.35957	7.35931	7.40313	7.37703

Table 4
Comparison of the variance using random truncated power series and Monte Carlo methods in Example 14

6 Conclusions

In this article we have constructed a power series solution to the random Legendre differential equation (1) with coefficients depend on a random variable A which has been assumed to be independent of the random initial conditions X_0 and X_1. This includes the computation of approximations of the average and variance functions to the random power series solution. These approximations not only agree but also improve those provided by Monte Carlo method as we have shown through different illustrative examples. In order to obtain a random power series solution to (1), we have assumed that random variable A satisfies condition (13) which is related to the exponential growth of its absolute moments with respect to the origin. This condition is satisfied by every random variable having bounded codomain, otherwise it has been shown that the method of truncation for random variables is an useful tool that allows us to take advantage of our approach in order to get reliable approximations, both for the mean and variance. The foundations of the theoretical results used in this paper have been based on the so-called mean square and mean fourth calculus which can be considered as promising and powerful theory to deal with other second-order linear random differential equations in our forthcoming work.

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