Numerical solution of random differential models

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Abstract

This paper deals with the construction of numerical solution of random initial value problems by means of random improved Euler method. Conditions for the mean square convergence of the proposed method is established. Finally, an illustrative example is included where the main statistics properties such as the mean and the variance of the stochastic approximation solution process are given.

Key words: Random differential equations, mean square calculus, numerical solution
MSC2010: 35R60, 60H35

1 Introduction

Random differential equations are powerful tools to model problems involving rates of changes of quantities representing variables under uncertainties or randomness, [1], [2], [3]. Many of these models are based on random differential equations of the form

$$\dot{\mathbf{X}}(t) = F\left(\mathbf{X}(t), t\right), \; t_0 \leq t \leq t_e, \quad \mathbf{X}(t_0) = \mathbf{X}_0, \tag{1.1}$$

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where $\mathbf{X}_0$ is a random vector and, the unknown, $\mathbf{X}(t)$, as well as, the right-hand side $F(\mathbf{X}(t), t)$ are vector stochastic processes. Reliable numerical solutions for problem (1.1) have been studied recently in [4], [5], [6]. In this paper, we present a random improved Euler method and we establish its mean square convergence in the fixed station sense. The proof of its convergence can be straightforward adapted to the extension of the random framework of others cases such as the so-called modified Euler and Runge-Kutta schemes [7], taking advantage of the approach here presented; comments are added to this issue. Apart from studying latter random scheme in order to obtain approximations of the solution stochastic process, we are also interested in providing approximations of the average and variance functions of the solution because they revel important information about the statistical behavior of the solution.

This paper is organized as follows. Section 2 deals with some preliminary definitions, results, notations and examples that clarify the presentation of the paper. Section 3 is addressed to the analysis of the mean square convergence of the numerical schemes here presented. An illustrative example is included in the last section.

2 Preliminaries

This section deals with some preliminary notations, results and examples that will clarify the presentation of the main results of this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space. In the following we are interested in second order real random variables (2-r.v.'s), $Y : \Omega \to \mathbb{R}$ having a density probability function, $f_Y(y)$, such that $E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy < +\infty$, where $E\left[\cdot\right]$ denotes the expectation operator. The space of all 2-r.v.'s defined on $(\Omega, \mathcal{F}, P)$ and endowed with the norm

$$\|Y\| = \left(E\left[Y^2\right]\right)^{1/2},$$

has a Banach space structure, denoted by L_2. Let X^j, $j = 1, ...m$ be 2-r.v.'s, a m-dimensional second order random vector is given by $\mathbf{X}^T = (X^1, ..., X^m)$. The space of all m-dimensional random vectors of second order (2-r.v.v.'s) with the norm

$$\|\mathbf{X}\|_m = \max_{j=1,...,m} \|X^j\| \tag{2.1}$$

is a Banach space and will be called the L_2^m-space. Given an interval $T \subseteq \mathbb{R}$, a stochastic process $\{X(t),\, t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ is called a second order stochastic process (2-s.p.), if for each $t \in T$, $X(t)$ is a 2-r.v. In an analogous way, if for each $t \in T$, $\mathbf{X}(t)$ is a m-dimensional 2-r.v.v., then $\{\mathbf{X}(t),\, t \in T\}$ is a second order m-dimensional vector stochastic process (2-v.s.p.). The covariance matrix function of $\{\mathbf{X}(t),\, t \in T\}$ is defined by

$$\mathbf{\Lambda}_{\mathbf{X}(t)} = E\left[\left(\mathbf{X}(t) - E\left[\mathbf{X}(t)\right]\right)\left(\mathbf{X}(t) - E\left[\mathbf{X}(t)\right]\right)^{\mathsf{T}}\right] = \left(v^{ij}(t)\right)_{m\times m}, \tag{2.2}$$

where $v^{ij}(t) = E[X^i(t)X^j(t)] - E[X^i(t)]\,E[X^j(t)]$, $1 \le i,j \le m$, $t \in T$. In what follows, we shall assume that each r.v., r.v.v. and v.s.p. are of second order unless the contrary is stated.
We say that a sequence of 2-r.v.v.'s $\{\mathbf{X}_n\}_{n\ge 0}$ is mean square (m.s.) convergent to $\mathbf{X} \in L_2^m$, and it will be denoted by $\mathbf{X}_n \xrightarrow[n\to\infty]{m.s.} \mathbf{X}$, if $\lim_{n\to\infty} \|\mathbf{X}_n - \mathbf{X}\|_m = 0$.

We say that a 2-v.s.p. $\{\mathbf{X}(t) : t \in T\}$ in L_2^m is m.s. continuous at $t \in T$, T an interval of the real line, if $\lim_{\tau\to 0} \|\mathbf{X}(t+\tau) - \mathbf{X}(t)\|_m = 0, \quad t, t+\tau \in T$, and it is m.s. differentiable at $t \in T$, if there exists a 2-v.s.p., denoted by $\left\{\dot{\mathbf{X}}(t) : t \in T\right\}$, such that

$$\lim_{\tau\to 0} \left\| \frac{\mathbf{X}(t+\tau) - \mathbf{X}(t)}{\tau} - \dot{\mathbf{X}}(t) \right\|_m = 0, \quad t, t+\tau \in T.$$

Definition 2.1 *Let S be a bounded set in L_2^m, an interval $T \subseteq \mathbb{R}$ and $h > 0$, we say that $F : S \times T \to L_2^m$ is m.s. randomly bounded time uniformly continuous in S if*

$$\lim_{h\to 0} \omega(S,h) = 0,$$

where $\omega(S,h) = \sup_{\mathbf{X}\in S\subset L_2^m} \sup_{|t-t'|\le h} \|F(\mathbf{X},t) - F(\mathbf{X},t')\|_m$.

Example 2.2 *Consider the function $F(\mathbf{X},t) = \mathbf{A}\mathbf{X} + \mathbf{C}(t)$, $0 \le t \le t_e$, where $\mathbf{X}^T = (X^1, X^2)$, and*

$$\mathbf{C}(t) = \begin{pmatrix} 0 \\ \frac{G(t)+\alpha B(t)}{L} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & \frac{-R}{L} \end{pmatrix}, \tag{2.3}$$

where $B(t)$ is the Brownian motion, $G(t)$ is a differentiable deterministic function in the interval $[t_0, t_e]$ and C, L, R, α are positive real constants.

Note that

$$(F(\mathbf{X},t) - F(\mathbf{X},t^*))^T = \left(0, \frac{G(t) - G(t^*) + \alpha(B(t) - B(t^*))}{L}\right). \tag{2.4}$$

Using expression (3.115) of [3, p.63] and (2.4) it follows that, $\|F(\mathbf{X},t) - F(\mathbf{X},t^*)\|_m \le \frac{1}{L}\max\left\{0, |G(t) - G(t^*)| + \alpha|t-t^*|^{\frac{1}{2}}\right\}$. Hence $F(\mathbf{X},t)$ is randomly bounded uniformly continuous.

3 Analysis of the mean square convergence

Let us consider the random initial value problem (1.1) under the following hypotheses on its right-hand side $F : S \times T \to L_2^m$, with $S \subset L_2^m$:

- H1: $F(\mathbf{X},t)$ is m.s. randomly bounded time uniformly continuous.
- H2: $F(\mathbf{X},t)$ satisfies the m.s. Lipschitz condition

$$\|F(\mathbf{X},t)-F(\mathbf{Y},t)\|_m \leq k(t)\,\|\mathbf{X}-\mathbf{Y}\|_m\,, \quad \int_{t_0}^{t_e} k(t)dt < +\infty.$$

Note that under hypotheses H1 and H2, Theorem 5.1.2. of [3, p. 118] guarantees the existence and uniqueness of a m.s. solution $\mathbf{X}(t)$ in $[t_n, t_{n+1}] \subset [t_0, t_e]$. Moreover, conditions H1 and H2 guarantee the m.s. continuity of $F(\mathbf{X},t)$ with respect to both variables.
Now, let us introduce the random improved Euler method for problem (1.1) defined by

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Phi(\mathbf{X}_n, t_n, h), \;\; \mathbf{X}_0 = \mathbf{X}\left(t_0\right), \;\; n \geq 0, \tag{3.1}$$

where $\Phi(\mathbf{X}_n,t_n,h) = \frac{h}{2}\left[F(\mathbf{X}_n,t_n) + F(\mathbf{X}_n + hF(\mathbf{X}_n,t_n),t_{n+1})\right]$, $F(\mathbf{X}_n,t_n)$ and $\mathbf{X}_n$ are 2-r.v.v.'s, $h = t_{n+1}-t_n$, with $t_n = t_0+nh$, for $n = 0,1,2,\ldots$. We wish to prove that under hypotheses H1 and H2, the scheme (3.1) is m.s. convergent in the fixed station sense, i.e., fixed $t \in [t_0, t_e]$ and taking n such that $t = t_n = t_0 + nh$, the m.s. error

$$\mathbf{e}_n = \mathbf{X}_n - \mathbf{X}(t) = \mathbf{X}_n - \mathbf{X}\left(t_n\right), \tag{3.2}$$

tends to zero in L_2^m, as $h \to 0$, $n \to \infty$ with $t - t_0 = nh$.
With the notation introduced previously, let e_n^j, X_n^j, $F^j(\mathbf{X},t)$ be the j-th entry of the random vectors $\mathbf{e}_n$, $\mathbf{X}_n$, $F(\mathbf{X},t)$, respectively. Thus, $e_n^j = X_n^j - X^j(t_n)$, and by the m.s. fundamental theorem of calculus, see [3, p.104], it follows that

$$X^j\left(t_{n+1}\right) = X^j\left(t_n\right) + \int_{t_n}^{t_{n+1}} \dot{X}^j(u)du, \quad n \geq 0. \tag{3.3}$$

From (3.1)-(3.3) and using that $\dot{X}^j(u) = F^j\left(\mathbf{X}(u),u\right)$, it follows that

$$e_{n+1}^j = e_n^j + \int_{t_n}^{t_{n+1}} G^j(u)du, \qquad \text{for all} \;\; n \geq 0, \tag{3.4}$$

where $G^j(u) = \frac{1}{2}F^j\left(\mathbf{X}_n,t_n\right) + \frac{1}{2}F^j(\mathbf{X}_n + hF(\mathbf{X}_n,t_n),t_{n+1}) - F^j\left(\mathbf{X}(u),u\right)$. As $F(\mathbf{X},t)$ is a m.s. continuous with respect to both variables, for j fixed, $G^j(u)$ is also m.s. continuous for each $u \in [t_n, t_{n+1}]$. Taking 2-norms in (3.4) and using property 3 of [3, p.102], it follows that

$$\left\|e_{n+1}^j\right\| \leq \left\|e_n^j\right\| + \int_{t_n}^{t_{n+1}} \left\|G^j(u)\right\| du. \tag{3.5}$$

Let us bound the integrand appearing in (3.5) in the following way

$$\begin{aligned}\left\|G^j(u)\right\| \leq \frac{1}{2}\left\|F^j\left(\mathbf{X}_n,t_n\right) - F^j\left(\mathbf{X}(u),u\right)\right\| \\ + \frac{1}{2}\left\|F^j(\mathbf{X}_n + hF(\mathbf{X}_n,t_n),t_{n+1}) - F^j\left(\mathbf{X}(u),u\right)\right\|.\end{aligned} \tag{3.6}$$

Now, we proceed by bounding the first term of the right-hand side of (3.6) by considering (2.1)

$$\begin{aligned}\left\|F^j\left(\mathbf{X}_n,t_n\right)-F^j\left(\mathbf{X}(u),u\right)\right\| &\leq \left\|F\left(\mathbf{X}_n,t_n\right)-F\left(\mathbf{X}(t_n),t_n\right)\right\|_m \\ &+\left\|F\left(\mathbf{X}(t_n),t_n\right)-F\left(\mathbf{X}(u),t_n\right)\right\|_m \\ &+\left\|F\left(\mathbf{X}(u),t_n\right)-F\left(\mathbf{X}(u),u\right)\right\|_m . \qquad (3.7)\end{aligned}$$

For the two first terms of latter right-hand side, using hypothesis H2 and (3.2), one gets the following bounds

$$\left\|F\left(\mathbf{X}_n,t_n\right)-F\left(\mathbf{X}(t_n),t_n\right)\right\|_m \leq k(t_n)\left\|\mathbf{e}_n\right\|_m , \qquad (3.8)$$

$$\|F(\mathbf{X}(t_n),t_n)-F(\mathbf{X}(u),t_n)\|_m \leq k(t_n)\left\{\max_{j=1,\ldots,m}\left\|X^j(t_n)-X^j(u)\right\|\right\} , \qquad (3.9)$$

with $u\in[t_n,t_{n+1}]$. Note that by applying (3.3) in $[t_n,u]\subset[t_n,t_{n+1}]$ and using again property 3 of [3, p.102], it follows that

$$\left\|X^j(t_n)-X^j(u)\right\| = \left\|\int_{t_n}^{u}\dot{X}^j(v)dv\right\| \leq \int_{t_n}^{u}\left\|\dot{X}^j(v)\right\| dv \leq M_{\dot{\mathbf{X}}}\left(u-t_n\right), \qquad (3.10)$$

where $M_{\dot{\mathbf{X}}} = \max_{j=1,\ldots,m}\sup\left\{\left\|\dot{X}^j(v)\right\| \;;\; t_0\leq v\leq t_e\right\}$. Taking into account that $h=t_{n+1}-t_n>u-t_n$, from (3.9) and (3.10) one gets

$$\left\|F\left(\mathbf{X}(t_n),t_n\right)-F\left(\mathbf{X}(u),t_n\right)\right\|_m \leq k(t_n)hM_{\dot{\mathbf{X}}}. \qquad (3.11)$$

Let $S_{\mathbf{X}}$ be the bounded set in L_2^m defined by the exact theoretical solution of problem (1.1), $S_{\mathbf{X}}=\left\{\mathbf{X}(t)\,,\; t_0\leq t\leq t_e\right\}$. Then by hypothesis H1 and Definition 2.1, we have

$$\left\|F\left(\mathbf{X}(u),t_n\right)-F\left(\mathbf{X}(u),u\right)\right\|_m \leq \omega\left(S_{\mathbf{X}},h\right), \qquad (3.12)$$

and by (3.8), (3.11) and (3.12), it follows that (3.7) is bounding by

$$\left\|F^j\left(\mathbf{X}_n,t_n\right)-F^j\left(\mathbf{X}(u),u\right)\right\| \leq k(t_n)\left\|\mathbf{e}_n\right\|_m + k(t_n)hM_{\dot{\mathbf{X}}}+\omega\left(S_{\mathbf{X}},h\right). \qquad (3.13)$$

The second term of the right-hand size of (3.6) can be bounded in a similar manner by considering that

$$\begin{aligned}&\left\|F^j(\mathbf{X}_n+hF(\mathbf{X}_n,t_n),t_{n+1})-F^j\left(\mathbf{X}(u),u\right)\right\| \\ &\leq \left\|F(\mathbf{X}_n+hF(\mathbf{X}_n,t_n),t_{n+1})-F\left(\mathbf{X}(t_n)+hF(\mathbf{X}(t_n),t_n),t_{n+1}\right)\right\|_m \\ &+\left\|F\left(\mathbf{X}(t_n)+hF(\mathbf{X}(t_n),t_n),t_{n+1}\right)-F\left(\mathbf{X}(u),t_{n+1}\right)\right\|_m \\ &+\left\|F\left(\mathbf{X}(u),t_{n+1}\right)-F\left(\mathbf{X}(u),u\right)\right\|_m ,\end{aligned}$$

then, by applying hypotheses H1, H2, Definition 2.1, property 3 of [3, p.102] and (3.10), one gets the following bound

$$\begin{aligned}&\left\| F^j(\mathbf{X}_n + hF(\mathbf{X}_n, t_n), t_{n+1}) - F^j\left(\mathbf{X}(u), u\right) \right\| \\ &\le k(t_{n+1}) \left\{ \|\mathbf{e}_n\|_m (1 + hk(t_n)) + 2hM_{\dot{\mathbf{X}}} \right\} + \omega\left(S_{\mathbf{X}}, h\right). \end{aligned} \tag{3.14}$$

Hence, by (3.5), (3.6), (3.13) and (3.14) one obtains

$$\|\mathbf{e}_{n+1}\|_m \le \|\mathbf{e}_n\|_m \left\{ \frac{h}{2} k(t_n) + \frac{h}{2} k(t_{n+1}) \left(1 + hk(t_n)\right) + 1 \right\} + B \tag{3.15}$$

where $B = h\left\{\omega(S_{\mathbf{X}}, h) + hM_{\dot{\mathbf{X}}}\left(k(t_{n+1}) + \frac{1}{2}k(t_n)\right)\right\}$. By (3.15) and lemma 1.2 of [7, p.18], one gets, $\|\mathbf{e}_n\|_m \le \exp(n\delta) \|\mathbf{e}_0\|_m + \frac{\exp(n\delta)-1}{\delta} B$, where $\delta = \frac{h}{2}k(t_n) + \frac{h}{2}k(t_{n+1})\left(1 + hk(t_n)\right)$. As $nh = t - t_0$, $t_{n+1} = h + t$, $\|\mathbf{e}_0\|_m = 0$, the last inequality can be written in the form

$$\begin{aligned}\|\mathbf{e}_n\|_m \le &\frac{\exp\left\{\frac{t-t_0}{2}\left\{k(t) + k(t+h)(1 + hk(t))\right\}\right\} - 1}{\frac{1}{2}\left\{k(t) + k(t+h)(1 + hk(t))\right\}} \\ &\times \left\{\omega(S_{\mathbf{X}}, h) + hM_{\dot{\mathbf{X}}}\left\{k(t+h) + \frac{1}{2}k(t)\right\}\right\}. \end{aligned} \tag{3.16}$$

From (3.16), it follows that $\{\mathbf{e}_n\}$ is m.s. convergent to zero as $h \to 0$. Summarizing the following result has been established:

Theorem 3.1 *With the previous notation, under the hypotheses H1 and H2, the random improved Euler method (3.1) is m.s. convergent and the discretization error* $\mathbf{e}_n$ *defined by (3.2) satisfies the inequality (3.16) for* $t = t_0 + nh$, $h > 0$, $t_0 \le t \le t_e$.

Note that, under hypotheses $H1$ and $H2$, the mean square convergence is straightforward adapted to similar schemes such as Euler modified which yields by taking $\Phi(X_n, t_n, h) = hF(\mathbf{X}_n + \frac{1}{2}hF(\mathbf{X}_n, t_n), t_n + \frac{1}{2}h)$, or the classical Runge–Kutta method (see, [3, p.67]), that is, taking $\Phi(\mathbf{X}_n, t_n, h) = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$, where $k_1 = F(\mathbf{X}_n, t_n)$, $k_2 = F(\mathbf{X}_n + \frac{1}{2}hk_1, t_n + \frac{1}{2}h)$, $k_3 = F(\mathbf{X}_n + \frac{1}{2}hk_2, t_n + \frac{1}{2}h)$, $k_4 = F(\mathbf{X}_n + hk_3, t_n + h)$ in (3.1).

4 Numerical results and conclusions

In this section we apply the Euler improved method and Euler method see [5], to a second order differential equation. As a first example, we introduce randomness into the equation by means of the initial conditions and source term. This is because the theoretical solution is available. After that, we consider a second example where the theoretical solution is not available by introducing randomness into the equation through one of its coefficients, initial conditions and source term. To make more complete this last example, we use the Monte Carlo (M.C.) method to compare the numerical results.

Example 4.1 *Let us consider the random differential equation given by*

$$L\ddot{Q}(t) + R\dot{Q}(t) + \frac{1}{C}Q(t) = G(t) + \alpha B(t), \quad Q(0) = Q_0, \quad \dot{Q}(0) = I_0, \quad (4.1)$$

where $Q(t)$ is the charge at time t, L is the inductance, R is the resistance, C is the capacitance and $H(t) = G(t) + \alpha B(t)$ represents the potential source at time t being $B(t)$ a Brownian motion and $G(t) = 24\sin(10t)$. By introducing the vector $\mathbf{X}(t)^T = (X^1(t), X^2(t)) = \left(Q(t), \dot{Q}(t)\right)$, model (4.1) can be written as follows: $\dot{\mathbf{X}}(t) = F(\mathbf{X}(t), t) = \mathbf{A}\mathbf{X}(t) + \mathbf{C}(t)$, where $\mathbf{C}(t)$ and $\mathbf{A}$ are given by (2.3). By example 2.2, H1 holds, and H2 is easy to check. So, by Theorem 3.1, the random improved Euler method is m.s. convergent and in this case it is given by

$$\mathbf{X}_n = \left(R_{\mathbf{A},h}\right)^n \mathbf{X}_0 + \frac{1}{2}h\sum_{i=0}^{n-1}\left[\left(R_{\mathbf{A},h}\right)^{n-1-i}\right]\left[(I + \mathbf{A}h)\mathbf{C}(t_i) + \mathbf{C}(t_{i+1})\right], \; n \geq 1, \quad (4.2)$$

where $R_{\mathbf{A},h} := I + \mathbf{A}h + \frac{h^2}{2}\mathbf{A}^2$, being I the identity matrix of size 2. Henceforth we will assume that Q_0 and I_0 are 2-r.v.'s independent of $B(t)$ for each $t \geq 0$ such that $\mathrm{E}[Q_0] = 0.1$, $\mathrm{E}\left[(Q_0)^2\right] = 0.5$, $\mathrm{E}[I_0] = 0$, $\mathrm{E}\left[(I_0)^2\right] = 0.05$. Taking into account these data, (2.2) and (4.2), one obtains the approximations for the mean and variance of the charge $Q(t)$ by means of random improved Euler method. We compare our results with respect to the Euler approximations, see [5], and the theoretical ones, see [3, p. 154]. Tables 1 and 2 show our results by taking $\alpha = 0.5$, $C = 0.02$, $R = 6$, $L = 0.5$. Clearly the improved Euler method provides better approximations as h goes to zero. Now, suppose that the resistance R is a r.v. following a uniform distribution on $[6, 7]$ and the rest of the data as were stated before. Therefore, the matrix A becomes random and the computations of the mean and variance from (4.2) are more laborious than the last example. Because R is bounded, condition $H2$ holds true. Condition $H1$ follows from example 2.2. From [5], we see that conditions $H1$ and $H2$ are sufficient for the m.s. convergence of Euler method. Hence the m.s. convergence of the Euler method as well as Euler improved method are guaranteed in this last case. As a consequence of this fact and theorem 4.3.1 of [3], the mean and variance obtained from Euler and Euler improved methods are also convergent. In tables 3 and 4 we show the numerical results. Column $\mu_X^{50000}(t)$ include M.C. approximations, taking 5×10^4 silumations. Simulations for the Brownian motion were made by using the function

$$SB(r, L) = \sqrt{2}\sum_{n=0}^{L}\frac{\sin\left(n + \frac{1}{2}\right)\pi t}{\left(n + \frac{1}{2}\right)\pi}z.$$

points	Euler	improved Euler	Euler	improved Euler	theoretical value
$t_n = nh$	$h = 1/40$	$h = 1/40$	$h = 1/80$	$h = 1/80$	
0.2	0.244167	0.276526	0.261959	0.275337	0.274961
0.4	0.288691	0.210613	0.248255	0.213224	0.214190
0.6	-0.460131	-0.393853	-0.424472	-0.393106	-0.39308
0.8	0.022128	0.069489	0.445390	0.064280	0.062646
1.0	0.452318	0.331677	0.388030	0.335071	0.336368

Table 1
Approximations for the mean of Example 4.1

points	Euler	improved Euler	Euler	improved Euler	theoretical value
$t_n = nh$	$h = 1/40$	$h = 1/40$	$h = 1/80$	$h = 1/80$	
0.2	0.011265	0.022419	0.017010	0.023005	0.023149
0.4	0.009948	0.003969	0.004293	0.004403	0.004403
0.6	0.000090	0.000192	0.000074	0.000203	0.000206
0.8	0.000172	0.000097	0.000141	0.000103	0.000105
1.0	0.000094	0.000087	0.000087	0.000087	0.000087

Table 2
Approximations of the variance of Example 4.1

points	Euler	improved Euler	Euler	improved Euler	$\mu_X^{50000}(t)$
$t_n = nh$	$h = 1/40$	$h = 1/40$	$h = 1/80$	$h = 1/80$	
0.2	0.240867	0.270931	0.257344	0.26980	0.26863
0.4	0.277701	0.206378	0.240992	0.208869	0.209859
0.6	-0.423197	-0.365072	-0.39205	-0.364483	-0.364567
0.8	0.017576	0.062072	0.038597	0.057478	0.056053
1.0	0.411907	0.307001	0.357109	0.310004	0.311212

Table 3
Approximations for the mean of Example 4.1

points	Euler	improved Euler	Euler	improved Euler	$\mu_X^{50000}(t)$
$t_n = nh$	$h = 1/40$	$h = 1/40$	$h = 1/80$	$h = 1/80$	
0.2	0.016869	0.029248	0.023341	0.029812	0.030141
0.4	0.006256	0.002112	0.004004	0.002316	0.002405
0.6	0.000480	0.000476	0.000452	0.000486	0.000496
0.8	0.000124	0.000090	0.000107	0.000090	0.000089
1.0	0.000573	0.000269	0.000387	0.000275	0.000277

Table 4
Approximations of the variance of Example 4.1

Where z is a standard normal r.v. and L is the truncation order of the trigonometric series. In this calculations $L = 100$. We can observe the convergence of the mean and variance by using Euler and Euler Improved methods as h goes to zero. Also the results of the mean and variance obtained with M.C. method are alike.

In this paper, we have presented the proof of the mean square convergence of an improved Euler random numerical scheme for systems of random differential equations. The more remarkable contribution of the present work is that the results here established do not depend on the sample or trajectory behavior of the data processes which allow us to apply the techniques to further random differential equations. In addition, we take advantage of the improved Euler random numerical scheme for computing directly the main statistic properties such as the expectation and variance of the mean square approximations. An important feature of this approach is that the mean square calculus guarantees that these approximations converge (in the mean square sense) to the exact ones. On the other hand, the example shows not only the theoretical aspects treated throughout the paper but also that the scheme here developed improves the corresponding approximations obtained from the Euler random numerical scheme as well as they agree with the theoretical results.

Acknowledgement
This work has been partially supported by the Spanish M.C.Y.T. grants MTM2009-08587, DPI2010–20891–C02–01, Universidad Politécnica de Valencia grant PAID06-09-2588 and Mexican Conacyt.

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