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Global regularity in ultradifferentiable classes

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Abstract We study ω-regularity of the solutions of certain operators that are globally C^∞-hypoelliptic in the N-dimensional torus. We also apply these results to prove the global ω-regularity for some classes of sublaplacians. In this way we extend previous work in the setting of analytic and Gevrey classes. Different examples on local and global ω-hypoellipticity are also given.

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1 Introduction

A linear partial differential operator P defined on an open set Ω of $\mathbb{R}^N$ with coefficients in $C^\infty(\Omega)$ (resp., in $\mathcal{A}(\Omega)$) is said to be locally hypoelliptic (resp., locally analytic hypoelliptic) in Ω if for every $f \in C^\infty(U)$ (resp., $f \in \mathcal{A}(U)$), with $U \subset \Omega$ any open set, all the solutions $u \in \mathcal{D}'(U)$ of $Pu = f$ belong to $C^\infty(U)$ (resp., to $\mathcal{A}(U)$). If P is defined on the torus $\mathbb{T}^N$, then P is said to be globally hypoelliptic (resp., globally analytic hypoelliptic) in $\mathbb{T}^N$ if all the solutions $u \in \mathcal{E}'(\mathbb{T}^N)$ of $Pu = f$ belong to $C^\infty(\mathbb{T}^N)$ (resp., to $\mathcal{A}(\mathbb{T}^N)$). We observe that the local hypoellipticity (resp., local analytic hypoellipticity) implies the global hypoellipticity (resp., global analytic hypoellipticity). By the celebrated sum of squares theorem of Hormander [35] the finite type condition is sufficient for the local hypoellipticity of P. But this condition is not sufficient for the local analytic hypoellipticity of P as it was first observed by Baouendi and Goulaouic [7]. Other classes of locally hypoelliptic operators which fail to be locally analytic hypoelliptic have been found and there are important results on analytic regularity (see, f.i., [26, 41, 24, 25, 12, 13] and the references therein). All such operators also *fail* to be locally hypoelliptic in the setting of ultradifferentiable function spaces (see, Propositions 4.1 and 4.2 for example). Cordaro and Himonas [15] proved that the finite type condition is *sufficient* for the global analytic hypoellipticity of some classes of operators in the form of a sum of squares of vector fields with real valued

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and real analytic coefficients. This result was further extended in Refs. [16,32, 33,43]. In particular, Himonas and Petronilho [32,43] showed the global analytic hypoellipticity (and also the global Gevrey hypoellipticity) of certain operators of the type $P = P(t, D_t, D_x)$ defined on the torus $\mathbb{T}^{m+n}$ with real valued coefficients in $\mathcal{A}(\mathbb{T}^m)$ and globally hypoelliptic in $\mathbb{T}^{m+n}$ (see [15,16,33]).

During the last years, many papers have concerned with the study of global solvability and hypoellipticity of linear partial differential operators on compact manifolds, e.g., torus, in large scales of functional spaces (see, e.g., [4–6,8,11,15–17, 19,21,22,28,30–34,43,46]). The theory of global properties of differential operators is not well developed in comparison with the theory of local properties. In particular, global properties are open problems except for some classes of operators. Several works treat the ultradifferentiable setting, and treat especially the Gevrey and analytic cases (see, for instance, [4–6,15,16,28,31–33,46]).

Motivated by the recent work developed in [15,16,32,33,43] and in [2,3], we investigate the global hypoellipticity of linear partial differential operators defined on the torus $\mathbb{T}^N$ in a bigger scale of spaces, namely, in the setting of ultradifferentiable classes as introduced in [10]. Actually, we prove the ω-regularity of solutions of operators of type $P = P(t, D_t, D_x)$ defined on the torus $\mathbb{T}^{m+n}$ with real valued coefficients in $\mathcal{E}_*(\mathbb{T}^m)$ and which are globally hypoelliptic in $\mathbb{T}^{m+n}$. Therefore, we extend the previous work for Gevrey classes of Himonas and Petronilho [32,43] (see, Theorem 3.1). As a consequence, we obtain some applications to sublaplacians that may satisfy the finite type condition or may be of infinite type at most points, see §4. We also characterize the global ω-hypoellipticity of linear partial differential operators with constant coefficients in $\mathbb{T}^N$ in terms of the symbol, see Proposition 3.1 (compare with [23,22,32]).

2 Notation and preliminaries

In this section we recall the definition of ultradifferentiable classes and ultradistributions of Beurling and Roumieu type, as well as the definition of wave front set in this setting and some needed results.

Throughout this article $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 2.1 A *weight* function is an increasing continuous function $\omega : [0,\infty[\to [0,\infty[$ with the following properties:

(α) there exists $L \geq 0$ such that $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$,

(β) $\omega(t) = O(t)$ as t tends to ∞,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\varphi : t \to \omega(e^t)$ is convex.

A weight function ω is called *quasi–analytic* if

$$\int_1^\infty \frac{\omega(t)}{t^2} = \infty.$$

If the integral is finite, then ω is called a *non quasi–analytic* weight function.

A weight function ω is equivalent to a sub-additive weight if, and only if, the following property holds:

$$(\alpha_0) \quad \exists D > 0 \;\; \exists t_0 > 0 \;\; \forall \lambda \geq 1 \;\; \forall t \geq t_0 : \quad \omega(\lambda t) \leq \lambda D \omega(t).$$

For a weight function ω we define $\tilde{\omega} : \mathbb{C} \to [0, \infty[$ by $\tilde{\omega}(z) := \omega(|z|)$ and again denote this function by ω. The *Young conjugate* $\varphi^* : [0, \infty[\to \mathbb{R}$ of φ is given by

$$\varphi^*(s) := \sup\{st - \varphi(t), t \geq 0\}.$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values, it is convex and $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$ and $\varphi^{**} = \varphi$.

Example 2.1 The following are examples of weight functions (eventually after a change on the interval $[0, \delta]$ for a suitable $\delta > 0$):

(1) $\omega(t) = t^\alpha$, $0 < \alpha < 1$;
(2) $\omega(t) = (\log(1 + t))^\beta$, $\beta > 1$;
(3) $\omega(t) = t\,(\log(e + t))^{-\beta}$, $\beta > 0$;
(4) $\omega(t) = t$.

The weight function in (3) is quasi–analytic for $\beta \in]0, 1]$ and non quasi–analytic for $\beta > 1$. The weight function in (4) is also quasi–analytic. Moreover, all the weight functions above satisfy property (α_0). For further examples of quasi–analytic weight functions we refer to [9].

Definition 2.2 Let ω be a weight function. For an open set $\Omega \subset \mathbb{R}^N$ we let

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : \| f \|_{K,\lambda} < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0\},$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \text{for every } K \subset\subset \Omega \text{ there exists } \lambda > 0 \text{ such that } \| f \|_{K,\lambda} < \infty\},$$

where

$$\| f \|_{K,\lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right).$$

The spaces $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$ are endowed with their natural topologies. Then $\mathcal{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space, while $\mathcal{E}_{\{\omega\}}(\Omega)$ is a countable projective limit of (DFN)-spaces, which is reflexive and complete. The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (respectively, $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ω-ultradifferentiable functions of Beurling type (respectively, of Roumieu type) in Ω. By $\mathcal{E}'_{(\omega)}(\Omega)$ and $\mathcal{E}'_{\{\omega\}}(\Omega)$ we denote the duals of $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$. As usual, we also denote by $\mathcal{E}'(\Omega)$ the dual space of $C^\infty(\Omega)$. When ω is quasianalytic the elements of $\mathcal{E}'_{(\omega)}(\Omega)$ (respectively, $\mathcal{E}'_{\{\omega\}}(\Omega)$) are called quasianalytic functionals of Beurling (respectively, Roumieu) type. We observe that in the case $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, the corresponding Roumieu class is the Gevrey class with exponent $s = 1/\alpha$. In particular, $\mathcal{E}_{\{t\}}(\Omega)$ coincides with the space $\mathcal{A}(\Omega)$ of all real analytic functions on Ω.

We will write $*$ to denote (ω) or $\{\omega\}$ when it is not necessary to distinguish between both cases.

If ω is quasi–analytic, the elements with compact support in $\mathcal{E}_{\{\omega\}}(\Omega)$ or in $\mathcal{E}_{(\omega)}(\Omega)$ are trivial. While, if ω is non quasi–analytic, the space $\mathcal{D}_*(K) := \mathcal{E}_*(\Omega) \cap \mathcal{D}(K) \neq \{0\}$, being $K \subset \Omega$ a compact set. Then $\mathcal{D}_*(\Omega) := \mathrm{ind}_n \mathcal{D}_*(K_n)$, where (K_n) is any compact exhaustion of Ω. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ (respectively, $\mathcal{D}'_{\{\omega\}}(\Omega)$) are called ω-ultradistributions of Beurling (respectively, Roumieu) type.

Remark 2.1 We observe that:

(a) If $\sigma(t) = o(\omega(t))$ as t tends to infinity, then

$$\mathcal{E}_{\{\omega\}}(\Omega) \subset \mathcal{E}_{(\sigma)}(\Omega)$$

with continuous inclusion.

(b) If $\omega(t) = o(t)$ as t tends to infinity, then for each constant $l \in \mathbb{N}$, there is a constant $C_l > 0$ such that

$$y \log y \leq y + l\varphi^*\big(\frac{y}{l}\big) + C_l, \quad y > 0.$$

We also recall the notion of wave front sets in the setting of ultradifferentiable classes (see [2]):

Definition 2.3 Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in \mathcal{D}'(\Omega)$. Let ω be a weight function.

1. If $\omega(t) = o(t)$ as t tends to infinity, we define the (ω)-wave front set $WF_{(\omega)}(u)$ of u to be the complement in $\Omega \times (\mathbb{R}^N \setminus \{0\})$ of the set of points (x_0, ξ_0) such that there exist an open neighborhood U of x_0 in Ω, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_L \in \mathcal{E}'(\Omega)$ equal to u in U and such that for every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ satisfying
$$|\xi|^L\, |\widehat{u}_L(\xi)| \leq C_k e^{k\varphi^*(L/k)}, \quad L = 1, 2, \ldots, \quad \xi \in \Gamma. \tag{2.1}$$
2. The $\{\omega\}$-wave front set $WF_{\{\omega\}}(u)$ of u is the complement in $\Omega \times (\mathbb{R}^N \setminus \{0\})$ of the set of points (x_0, ξ_0) such that there exist an open neighborhood U of x_0 in Ω, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_L \in \mathcal{E}'(\Omega)$ equal to u in U which satisfies, for some $C > 0$ and $k \in \mathbb{N}$, the estimates
$$|\xi|^L\, |\widehat{u}_L(\xi)| \leq C e^{\frac{1}{k}\varphi^*(Lk)}, \quad L = 1, 2, \ldots, \quad \xi \in \Gamma. \tag{2.2}$$

Next, denote by $\mathbb{T}^N = \frac{\mathbb{R}^N}{2\pi\mathbb{Z}^N}$ the N-dimensional torus. For a weight function ω let $\mathcal{E}_*(\mathbb{T}^N)$ be the space of all $\mathcal{E}_*$-functions on $\mathbb{T}^N$, which are identified with the $\mathcal{E}_*$-functions on $\mathbb{R}^N$ that are 2π-periodic in each variable. Clearly, $\mathcal{E}_*(\mathbb{T}^N)$ is a closed subspace of $\mathcal{E}_*(\mathbb{R}^N)$. Then, if we set $K_\pi = [0, 2\pi]^N$, $\mathcal{E}_{(\omega)}(\mathbb{T}^N)$ is a Fréchet space whose topology is generated by the sequence $\{\|\ \|_{K_\pi,k}\}_{k\in\mathbb{N}}$ of norms and $\mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ is a dual Fréchet-nuclear space, i.e., $\mathcal{E}_{\{\omega\}}(\mathbb{T}^N) = \mathrm{ind}\lim_k \mathcal{E}_{\{\omega\},k}(\mathbb{T}^N)$ with $\mathcal{E}_{\{\omega\},k}(\mathbb{T}^N) = \{f \in C^\infty(K_\pi) : \|\, f\, \|_{K_\pi,1/k} < \infty\}$ for $k \in \mathbb{N}$.

If either $u \in \mathcal{E}'_*(\mathbb{T}^N)$ or $u \in \mathcal{E}'(\mathbb{T}^N)$, we can define $\hat{u}(\xi) := u((2\pi)^{-N} e^{-ix\cdot\xi})$ for $\xi \in \mathbb{Z}^N$ and show the following result, that may be known for specialists.

Proposition 2.1 *Let ω be a weight function and $u \in \mathcal{E}'(\mathbb{T}^N)$. Then the following holds.*

(i) *Suppose* $\omega(t) = o(t)$ *as* $t \to \infty$. *Then* $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$ *if, and only if, for each* $k \in \mathbb{N}$ *there is* $C_k > 0$ *such that* $|\hat{u}(\xi)| \le C_k e^{-k\omega(\xi)}$ *for all* $\xi \in \mathbb{Z}^N$.

(ii) $u \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ *if and only if there exist* $\varepsilon > 0$, $C > 0$ *such that* $|\hat{u}(\xi)| \le Ce^{-\varepsilon\omega(\xi)}$ *for all* $\xi \in \mathbb{Z}^N$.

In the following, as usual, we denote $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$ $(\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N})$ for $\alpha \in \mathbb{N}_0^N$, where $D_j = -i\partial_{x_j}$ for $j \in \{1, \ldots, N\}$.

Proof (i) Let $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$. Since, for each $\alpha \in \mathbb{N}_0^N$, $\hat{u}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{-ix\cdot\xi} u(x)\, dx$ for $\xi \in \mathbb{Z}^N$, we have

$$\xi^\alpha \hat{u}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{-ix\cdot\xi} D^\alpha u(x)\, dx.$$

As $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$, for each $k \in \mathbb{N}$ there exists C_k such that

$$|\xi^\alpha \hat{u}(\xi)| \le \max_{x \in K_\pi} |\partial^\alpha u(x)| \le C_k e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}, \quad \alpha \in \mathbb{N}_0^N,\ \xi \in \mathbb{Z}^N. \tag{2.3}$$

Now, let $L \in \mathbb{N}_0$ and $\xi \in \mathbb{Z}^N$ and choose $i \in \{1, \ldots, N\}$ such that

$$|\xi_i| = \max_{1 \le j \le N} |\xi_j|.$$

If we set $\alpha = Le_i$ with e_i being the i–th vector of the standard basis of $\mathbb{R}^N$, we have

$$|\xi|^L \le N^{L/2} \max_{1 \le j \le N} |\xi_j|^L = N^{L/2} |\xi_i|^L = N^{L/2} |\xi^\alpha|.$$

Combining this inequality with (2.3) we obtain that, for each $k \in \mathbb{N}$ there exists C_k such that

$$|\xi|^L |\hat{u}(\xi)| \le N^{L/2} C_k e^{k\varphi^*\left(\frac{L}{k}\right)}, \quad L \in \mathbb{N}_0,\ \xi \in \mathbb{Z}^N.$$

By [2, Lemma 3.2] this implies the thesis.

Conversely, let $u \in \mathcal{E}'_{(\omega)}(\mathbb{T}^N)$ satisfying the condition in (i). By [2, Lemma 3.2], for each $k \in \mathbb{N}$ there exists $C_k > 0$ such that $|\xi|^L |\hat{u}(\xi)| \le C_k e^{k\varphi^*\left(\frac{L}{k}\right)}$ for all $\xi \in \mathbb{Z}^N$ and $L \in \mathbb{N}_0$. So, for each $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$, we obtain that the series

$$D^\alpha u(x) = \sum_{\xi \in \mathbb{Z}^N} \xi^\alpha \hat{u}(\xi) e^{ix\cdot\xi}$$

is absolutely convergent. On the other hand, there exist $C > 0$ and $M \in \mathbb{N}$ such that $|\hat{u}(\xi)| \le C(1 + |\xi|)^M$ for all $\xi \in \mathbb{Z}^N$ as u is a distribution on $\mathbb{R}^N$ with compact support.

Fix $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^N$, let $L = |\alpha| + N + M$ and observe that

$$|D^\alpha u(x)| \le \sum_{|\xi| \le e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} |\xi|^{|\alpha|} |\hat{u}(\xi)| + \sum_{|\xi| > e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} |\xi|^{|\alpha|} |\hat{u}(\xi)| =: S_1 + S_2, \quad x \in \mathbb{R}^N, \tag{2.4}$$

where

$$\begin{aligned} S_1 &\le \sum_{|\xi|\le e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} e^{\frac{2k|\alpha|}{L}\varphi^*\left(\frac{L}{2k}\right)} C(1+|\xi|)^M \\ &\le C2^N e^{\frac{2kN}{L}\varphi^*\left(\frac{L}{2k}\right)} e^{\frac{2k|\alpha|}{L}\varphi^*\left(\frac{L}{2k}\right)} \left(1+e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}\right)^M \\ &\le 2^{N+M} C \left(e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}\right)^{N+|\alpha|+M} \le 2^{N+M} C e^{2k\varphi^*\left(\frac{N+|\alpha|+M}{2k}\right)} \end{aligned}$$

and

$$S_2 = \sum_{|\xi|> e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} |\xi|^{L-N-M}|\hat{u}(\xi)| \le C_k e^{2k\varphi^*\left(\frac{L}{2k}\right)} \sum_{|\xi|> e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} |\xi|^{-N-M},$$

as $L=|\alpha|+N+M$ and $\sum_{|\xi|\le e^{\frac{2k}{L}\varphi^*\left(\frac{L}{2k}\right)}} \le 2^N e^{\frac{2kN}{L}\varphi^*\left(\frac{L}{2k}\right)}$. Therefore, by (2.4) we deduce

$$|D^\alpha u(x)| \le 2^{N+M}Ce^{2k\varphi^*\left(\frac{N+|\alpha|+M}{2k}\right)} + C_k c e^{2k\varphi^*\left(\frac{L}{2k}\right)} \le C_k' e^{2k\varphi^*\left(\frac{N+|\alpha|+M}{2k}\right)}, \quad x\in\mathbb{R}^N, \tag{2.5}$$

where $C_k' = 2^{N+M}C + C_k c > 0$ depends only on k, on the dimension N and on u (here, $c=\sum_{|\xi|\neq 0}|\xi|^{-N-M} < \infty$). Moreover, the convexity of φ^* implies that

$$2k\varphi^*\left(\frac{N+|\alpha|+M}{2k}\right) \le k\varphi^*(|\alpha|/k) + k\varphi^*((N+M)/k). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain, for every $k\in\mathbb{N}$, that there exists a constant

$$C_k'' = C_k' e^{k\varphi^*((N+M)/k)} > 0$$

such that

$$|D^\alpha u(x)| \le C_k'' e^{k\varphi^*(|\alpha|/k)}, \quad \alpha\in\mathbb{N}_0^N,\ x\in\mathbb{R}^N.$$

This means that $u\in\mathcal{E}_{(\omega)}(\mathbb{T}^N)$.

(ii) The proof follows proceeding as in case (i) and using [2, Lemma 3.1]. □

As a consequence of Proposition 2.1 we easily obtain that the Fréchet space $\mathcal{E}_{(\omega)}(\mathbb{T}^N)$ is isomorphic to the power series space of infinite type

$$\lambda_\omega := \left\{x\in\mathbb{C}^{\mathbb{Z}^N} : \sum_{\nu\in\mathbb{Z}^N} |x_\nu| e^{k\omega(\nu)} < \infty,\ \forall k\in\mathbb{N}\right\},$$

and that the dual Fréchet nuclear space $\mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ is isomorphic to the sequence space

$$\kappa_\omega := \left\{x\in\mathbb{C}^{\mathbb{Z}^N} : \sum_{\nu\in\mathbb{Z}^N} |x_\nu| e^{\omega(\nu)/k} < \infty,\ \text{for some } k\in\mathbb{N}\right\}$$

(compare with [44,37,38,9]).

Definition 2.4 Let $P = P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be a linear partial differential operator with coefficients $\{a_\alpha\}_{|\alpha|\leq m} \subset \mathcal{E}_*(\mathbb{T}^N)$ ($\{a_\alpha\}_{|\alpha|\leq m} \subset \mathcal{E}(\mathbb{T}^N)$ resp.). The operator P is said to be globally $*$-hypoelliptic in $\mathbb{T}^N$ (globally C^∞-hypoelliptic in $\mathbb{T}^N$ resp.) if the conditions $u \in \mathcal{E}'(\mathbb{T}^N)$ and $Pu \in \mathcal{E}_*(\mathbb{T}^N)$ ($u \in \mathcal{E}'(\mathbb{T}^N)$ and $Pu \in \mathcal{E}(\mathbb{T}^N)$ resp.) imply that $u \in \mathcal{E}_*(\mathbb{T}^N)$ ($u \in \mathcal{E}(\mathbb{T}^N)$ resp.). In case $\omega(t) = t^{1/s}$ with $s \geq 1$ and the operator P is globally $\{t^{1/s}\}$-hypoelliptic in $\mathbb{T}^N$, we say simply that P is globally s-hypoelliptic in $\mathbb{T}^N$ for $s > 1$, and globally analytic hypoelliptic in $\mathbb{T}^N$ for $s = 1$.

3 The Results

If $P = P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ is a linear partial differential operator with constant coefficients on $\mathbb{T}^N$. By [22], the operator P is globally C^∞-hypoelliptic in $\mathbb{T}^N$ if, and only if, there exist $L,\ M,\ C > 0$ such that

$$|P(\xi)| \geq L|\xi|^{-M}, \quad |\xi| \geq C, \tag{3.7}$$

where $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$, $\xi \in \mathbb{Z}^N$, is the symbol of P. Also, by [23, Theorem 2.2] (see also [32]) the operator P is globally s-hypoelliptic in $\mathbb{T}^N$ if, and only if, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|P(\xi)| \geq e^{-\varepsilon|\xi|^{1/s}}, \quad |\xi| \geq C_\varepsilon. \tag{3.8}$$

Since condition (3.7) implies (3.8), [23, Corollary 2.2], if P is globally C^∞-hypoelliptic in $\mathbb{T}^N$ then it is also globally s-hypoelliptic in $\mathbb{T}^N$. On the other hand, there exist examples of linear partial differential operators with constant coefficients that are globally analytic hypoelliptic but not globally C^∞-hypoelliptic. For example, in [23, Theorem 4.1] it is shown that there is $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that the vector field $V = \partial_{x_1} - \alpha\partial_{x_2}$ is globally analytic hypoelliptic but not globally C^∞-hypoelliptic in $\mathbb{T}^2$.

Now, it is easy to prove that

Proposition 3.1 *Let ω be a weight function and $P = P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients on $\mathbb{T}^N$. Then the following holds.*

(i) *Suppose that $\omega(t) = o(t)$ as $t \to \infty$. The operator P is globally (ω)–hypoelliptic in $\mathbb{T}^N$ if, and only if, there exist $L,\ m,\ C > 0$ such that*

$$|P(\xi)| \geq Le^{-m\omega(\xi)}, \quad \xi \in \mathbb{Z}^N, |\xi| \geq C. \tag{3.9}$$

(ii) *The operator P is globally $\{\omega\}$–hypoelliptic in $\mathbb{T}^N$ if, and only if, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|P(\xi)| \geq e^{-\varepsilon\omega(\xi)}, \quad \xi \in \mathbb{Z}^N, |\xi| \geq C_\varepsilon. \tag{3.10}$$

Proof We first recall that if $u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(\xi) e^{i\xi \cdot x}$ and $f = \sum_{\xi \in \mathbb{Z}^N} \hat{f}(\xi) e^{i\xi \cdot x}$, then $Pu = f$ if, and only if, $P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$ for all $\xi \in \mathbb{Z}^N$.

(i) Suppose that condition (3.9) holds and that $Pu = f$ with $u \in \mathcal{E}'_{(\omega)}(\mathbb{T}^N)$ and $f \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$. Since $P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$ for all $\xi \in \mathbb{Z}^N$, by Proposition 2.1(i) we have, for every $k \in \mathbb{N}$, that there exists $C_k > 0$ such that

$$|P(\xi)\hat{u}(\xi)| \le C_k e^{-k\omega(\xi)}, \quad \xi \in \mathbb{Z}^N.$$

By (3.9) it follows that, for every $k > m$,

$$|\hat{u}(\xi)| \le C_k \frac{e^{-k\omega(\xi)}}{|P(\xi)|} \le \frac{C_k}{L} e^{-(k-m)\omega(\xi)}, \quad |\xi| \ge C.$$

Again by Proposition 2.1(i) we have $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$.

Suppose that (3.9) does not hold. Then we can find a sequence $\{\xi_m\}_m \subset \mathbb{Z}^N$ such that $|\xi_m| \to \infty$ and

$$|P(\xi_m)| \le e^{-m\omega(\xi_m)}, \quad m \in \mathbb{N}. \tag{3.11}$$

Let $u \in \mathcal{E}'_{(\omega)}(\mathbb{T}^N)$ defined by $\hat{u}(\xi_m) = 1$ and $\hat{u}(\xi) = 0$ otherwise. Then $u \notin \mathcal{E}_{(\omega)}(\mathbb{T}^N)$ by Proposition 2.1. On the other hand, if we set $f = Pu$, then $\hat{f}(\xi_m) = P(\xi_m)$ for every $m \in \mathbb{N}$ and $\hat{f}(\xi) = 0$ otherwise. So, (3.11) ensures that for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$|\hat{f}(\xi)| \le C_k e^{-k\omega(\xi)}, \quad \xi \in \mathbb{Z}^N.$$

Again by Proposition 2.1 we can conclude that $f \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$.

(ii) follows proceeding as in the previous case. □

Since $\log(t) = o(\omega(t))$ as $t \to \infty$, condition (3.7) implies both conditions (3.9) and (3.10). Then, *if $P = P(D)$ is globally C^∞-hypoelliptic in $\mathbb{T}^N$, then it is also globally (ω) and $\{\omega\}$-hypoelliptic in $\mathbb{T}^N$*. Moreover, it is clear that condition (3.10) implies condition (3.9). This means that if $P = P(D)$ is globally $\{\omega\}$-hypoelliptic in $\mathbb{T}^N$, then it is globally (ω) -hypoelliptic in $\mathbb{T}^N$. But, in case ω and σ are two weight functions such that $\omega = o(\sigma(t))$ as $t \to \infty$ and hence $m\omega(t) - \log(L) = o(\sigma(t))$ for every m, $L > 0$ as $t \to \infty$, the converse holds, i.e., if $P = P(D)$ is globally (ω)-hypoelliptic in $\mathbb{T}^N$, then it is globally $\{\sigma\}$-hypoelliptic in $\mathbb{T}^N$. Consequently, for $\omega(t) = o(t)$ as $t \to \infty$ (non-quasianalytic weight functions always satisfy this condition), if $P = P(D)$ is globally (ω)-hypoelliptic in $\mathbb{T}^N$, then it is globally analytic-hypoelliptic in $\mathbb{T}^N$. Also, in case ω and σ are two weight functions such that $\omega(t) = O(\sigma(t))$ as $t \to \infty$, if $P = P(D)$ is globally (ω)-hypoelliptic in $\mathbb{T}^N$ (globally $\{\omega\}$-hypoelliptic in $\mathbb{T}^N$ resp.), then it is also (σ)-hypoelliptic in $\mathbb{T}^N$ (globally $\{\sigma\}$-hypoelliptic in $\mathbb{T}^N$ resp.).

From [20] it follows that for a fixed $\sigma \ge 1$ there exists $\alpha = \alpha(\sigma) \in \mathbb{R} \setminus \mathbb{Q}$ such that the vector field $V = \partial_{x_1} - \alpha\partial_{x_2}$ (already considered above) is globally s-hypoelliptic in $\mathbb{T}^2$ if $1 \le s \le \sigma$, but V is not globally s-hypoelliptic in $\mathbb{T}^2$ if $s > \sigma$ and hence, it is not globally C^∞-hypoelliptic in $\mathbb{T}^2$. So, by the preceding comments the vector field $V = \partial_{x_1} - \alpha\partial_{x_2}$ is also globally $\{\omega\}$-hypoelliptic and, hence, globally (ω)-hypoelliptic in $\mathbb{T}^2$ for every weight function ω satisfying the condition $t^{1/s} = O(\omega(t))$ as $t \to \infty$ for some $1 < s < \sigma$ with $\sigma > 1$ fixed, but not globally C^∞-hypoelliptic in $\mathbb{T}^2$.

Now, we show that the global C^∞-hypoellipticity in $\mathbb{T}^N$ of a linear partial differential operator $P = P(x, D)$ *with variable coefficients* also implies the global $*$-hypoellipticity of P in $\mathbb{T}^N$ under mild conditions. We first collect some preliminary results.

For $\varphi \in C^\infty(\mathbb{T}^N)$ and $j \in \mathbb{N}_0$ let

$$\| \varphi \|_j = \sum_{|\alpha| \le j} \| \partial^\alpha \varphi \|_\infty = \sum_{|\alpha| \le j} \max_{x \in \mathbb{T}^N} |\partial^\alpha \varphi(x)|.$$

Then $\{\| \, \|_j\}_{j \in \mathbb{N}_0}$ is a sequence of norms on $C^\infty(\mathbb{T}^N)$ generating its Fréchet topology. Next, for $s \in \mathbb{R}_+$ let

$$H^s(\mathbb{T}^N) = \{u \in L^2(\mathbb{T}^N) : \ |u|_s^2 = \sum_{\xi \in \mathbb{Z}^N} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 < \infty\},$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$ for $\xi \in \mathbb{Z}^N$. Then $H^s(\mathbb{T}^N)$ is a Hilbert space with respect to the inner product and norm defined by

$$(u, v)_s = \sum_{\xi \in \mathbb{Z}^N} \langle \xi \rangle^{2s} \hat{u}(\xi) \bar{\hat{v}}(\xi), \quad |u|_s = \sqrt{(u, v)_s}, \quad u, \ v \in H^s(\mathbb{T}^N).$$

So, the following continuous embeddings hold: $C^\infty(\mathbb{T}^N) \hookrightarrow H^s(\mathbb{T}^N)$ for $s \in \mathbb{R}_+$, $H^s(\mathbb{T}^N) \hookrightarrow H^t(\mathbb{T}^N)$ for $s > t \ge 0$, $H^s(\mathbb{T}^N) \hookrightarrow L^2(\mathbb{T}^N)$ for $s \in \mathbb{R}_+$. Such embedding maps have dense range.

We also let, for $s > 0$,

$$H^{-s}(\mathbb{T}^N) = \{u \in \mathcal{E}'(\mathbb{T}^N) : \ |u|_{-s}^2 = \sum_{\xi \in \mathbb{Z}^N} \langle \xi \rangle^{-2s} |\hat{u}(\xi)|^2 < \infty\}.$$

Then $H^{-s}(\mathbb{T}^N)$ is also a Hilbert space with respect to the inner product and norm defined as above. In particular, $H^s(\mathbb{T}^N)$ and $H^{-s}(\mathbb{T}^N)$, for $s > 0$, identify with each other's dual space by duality. Moreover, for $s > 0$, the inclusions $L^2(\mathbb{T}^N) \hookrightarrow H^{-s}(\mathbb{T}^N)$ and $H^{-s}(\mathbb{T}^N) \hookrightarrow \mathcal{E}'(\mathbb{T}^N)$ are continuous with dense range. Also, we have

$$C^\infty(\mathbb{T}^N) = \cap_{s \in \mathbb{R}} H^s(\mathbb{T}^N), \quad \mathcal{E}'(\mathbb{T}^N) = \cup_{s \in \mathbb{R}} H^s(\mathbb{T}^N).$$

Finally, we remark that if $\{s_n\}_{n \in \mathbb{N}}$ is an increasing sequence of positive numbers such that $s_n \to \infty$, then $\{\| \, \|_{s_n}\}_{n \in \mathbb{N}}$ is a sequence of norms also generating the Fréchet topology of $C^\infty(\mathbb{T}^N)$.

We need the following lemma.

Lemma 3.1 *Let $P = P(x, D)$ be a linear partial differential operator with coefficients in $C^\infty(\mathbb{T}^N)$. If P is globally C^∞-hypoelliptic, then for every $j \in \mathbb{N}_0$ there exist $c_j > 0$ and $h \in \mathbb{N}_0$ with $h \ge j$ such that*

$$\| \varphi \|_j \le c_j (\| P\varphi \|_h + |\varphi|_{-1}), \quad \varphi \in C^\infty(\mathbb{T}^N). \tag{3.12}$$

Proof Observe that P is a well defined continuous linear operator from $\mathcal{E}'(\mathbb{T}^N)$ into itself and that $H^{-1}(\mathbb{T}^N)$ is continuously included in $\mathcal{E}'(\mathbb{T}^N)$. Then, the operator P_r defined by

$$P_r \colon D(P_r) = \{\varphi \in H^{-1}(\mathbb{T}^N) : \ P\varphi \in C^\infty(\mathbb{T}^N)\} \to C^\infty(\mathbb{T}^N), \ \varphi \to P\varphi,$$

is closed when $D(P_r)$, that is a subspace of $H^{-1}(\mathbb{T}^N)$, is endowed with the topology induced by the Hilbert space $H^{-1}(\mathbb{T}^N)$. On the other hand, by the global C^∞–hypoellipticity of P we have $D(P_r) = C^\infty(\mathbb{T}^N)$. Thus, the operator

$$J \colon \mathcal{G}_{P_r} \to C^\infty(\mathbb{T}^N), \quad (\varphi, P\varphi) \to \varphi,$$

where $\mathcal{G}_{P_r} = \{(\varphi, P\varphi) : \ \varphi \in C^\infty(\mathbb{T}^N)\} \subset C^\infty(\mathbb{T}^N)\times C^\infty(\mathbb{T}^N)$ is the graph set of P_r, is well-defined, linear and closed when $\mathcal{G}_{P_r}$ is endowed with the Fréchet topology induced by $H^{-1}(\mathbb{T}^N) \times C^\infty(\mathbb{T}^N)$. Since $\mathcal{G}_{P_r}$ is a closed subspace of $H^{-1}(\mathbb{T}^N) \times C^\infty(\mathbb{T}^N)$ and hence a Fréchet space with respect to the induced topology, we can apply the closed graph theorem to conclude that J is continuous. Now, as the sequence $\{\| \ \|_j + | \ |_{-1}\}_{j\in\mathbb{N}_0}$ of norms generates the Fréchet topology of $H^{-1}(\mathbb{T}^N) \times C^\infty(\mathbb{T}^N)$, it follows that for each $j \in \mathbb{N}_0$ there exist $c_j > 0$ and $h \in \mathbb{N}_0$ with $h \geq j$ such that

$$\| \varphi \|_j = \| J(\varphi, P\varphi) \|_j \leq c_j(\| P\varphi \|_h + |\varphi|_{-1})$$

for all $\varphi \in C^\infty(\mathbb{T}^N)$. The proof is complete. □

Theorem 3.1 *Let ω be a weight function satisfying property (α_0) and write, as usual, $*$ for $\{\omega\}$ or (ω). Let $\mathbb{T}^N = \mathbb{T}^{m+n}$ and write $(t, x) \in \mathbb{T}^{m+n}$ for $t \in \mathbb{T}^m$ and $x \in \mathbb{T}^n$. Moreover, if $* = (\omega)$, we assume that $\omega(t) = o(t)$ as $t \to \infty$. Let $P = P(t, D_t, D_x)$ be a linear partial differential operator with coefficients in $\mathcal{E}_*(\mathbb{T}^N)$ and suppose that P is globally C^∞-hypoelliptic in $\mathbb{T}^N$. If $u \in \mathcal{E}'(\mathbb{T}^N)$, $Pu \in \mathcal{E}_*(\mathbb{T}^N)$ and $(t, x, \tau, 0) \notin WF_*(u)$, for any $(t, x) \in \mathbb{T}^{m+n}$ and $\tau \in \mathbb{R}^m \setminus \{0\}$, then $u \in \mathcal{E}_*(\mathbb{T}^N)$.*

Proof Since $Pu =: f \in \mathcal{E}_*(\mathbb{T}^N) \subset C^\infty(\mathbb{T}^N)$ and P is globally C^∞-hypoelliptic in $\mathbb{T}^N$, $u \in C^\infty(\mathbb{T}^N)$. Moreover, by Lemma 3.1 there exist $c_0 > 0$ and $h \in \mathbb{N}_0$ such that

$$\| \varphi \|_0 \leq c_0(\| P\varphi \|_h + |\varphi|_{-1})$$

for every $\varphi \in C^\infty(\mathbb{T}^N)$. As $\partial_x^\alpha \varphi \in C^\infty(\mathbb{T}^N)$ for every $\varphi \in C^\infty(\mathbb{T}^N)$ and $\alpha \in \mathbb{N}_0^n$, it follows that

$$\| \partial_x^\alpha \varphi \|_0 \leq c_0(\| P(\partial_x^\alpha \varphi) \|_h + |\partial_x^\alpha \varphi|_{-1}) \tag{3.13}$$

for every $\varphi \in C^\infty(\mathbb{T}^N)$ and $\alpha \in \mathbb{N}_0^n$.

Since $u \in C^\infty(\mathbb{T}^N)$, $|\partial_x^\alpha \varphi|_{-1} \leq |\partial_x^{\alpha - e_j} \varphi|_0$ where e_j is an element of the standard basis of $\mathbb{R}^n$ such that the corresponding $\alpha_j \geq 1$, $[\partial_x^\alpha, P] = 0$ for every $\alpha \in \mathbb{N}_0^n$ as the coefficients of P depend only on t and $Pu = f$, we obtain via (3.13) that

$$\| \partial_x^\alpha u \|_0 \leq c_0(\| \partial_x^\alpha f \|_h + |\partial_x^{\alpha - e_j} u|_0) \tag{3.14}$$

To conclude the proof we need distinguish two cases: (B) Beurling case; (R) Roumieu case.

(B) Beurling case. Since $f \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$, there exists $A > 0$ such that for each $k \in \mathbb{N}$ there exists $C_k > 0$ for which

$$\| \partial_x^\alpha f \|_0 \leq C_k A^{|\alpha|+1} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.15}$$

Consequently, the following holds.

Lemma 3.2 *There exists $E > 0$ such that for every $k \in \mathbb{N}$ there exists $E_k > 0$ for which*

$$\| \partial_x^\alpha u \|_0 \leq E_k E^{|\alpha|+1} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.16}$$

Proof Fixed any $k \in \mathbb{N}$, the proof is given by induction on $|\alpha|$.

For $\alpha = 0$ we have $\| u \|_0 = E_0 \leq E_0 e^{k\varphi^*\left(\frac{0}{k}\right)}$ as $\varphi^*(0) = 0$

Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$ and suppose that the result holds for all $\beta \in \mathbb{N}_0^n$ with $|\beta| < |\alpha|$. By (3.14), since φ^* is a non negative convex function and $\varphi^*(t)/t$ is an increasing function, we have

$$\begin{aligned}
\| \partial_x^\alpha u \|_0 &\leq c_0 \left(\sum_{|\beta| \leq h} \| \partial_x^{\alpha+\beta} f \|_0 + |\partial_x^{\alpha - e_j} u|_0 \right) \\
&\leq c_0 \left(\sum_{|\beta| \leq h} C_k A^{|\alpha+\beta|+1} e^{2k\varphi^*\left(\frac{|\alpha+\beta|}{2k}\right)} + E_k E^{|\alpha|} e^{k\varphi^*\left(\frac{|\alpha|-1}{k}\right)} \right) \\
&\leq c_0 \left(C_k A^{|\alpha|+1} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)} \sum_{|\beta| \leq h} A^{|\beta|} e^{k\varphi^*\left(\frac{|\beta|}{k}\right)} + E_k E^{|\alpha|} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)} \right) \\
&\leq c_0 \left(C_k D_k A^{|\alpha|+1} + E_k E^{|\alpha|} \right) e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}.
\end{aligned}$$

Here, $D_k = \sum_{|\beta| \leq h} A^{|\beta|} e^{k\varphi^*\left(\frac{|\beta|}{k}\right)} > 0$ depends only on h, f and k.

Let $E = AM$ with $M > 1$ such that $E \geq E_0$ and let $E_k = \max\{C_k D_k, 1\}$. Then the inequality above gives that the result follows if

$$c_0 (E_k A^{|\alpha|+1} + E_k A^{|\alpha|} M^{|\alpha|}) \leq E_k A^{|\alpha|+1} M^{|\alpha|+1},$$

i.e., if

$$c_0 \left(\frac{1}{M^{|\alpha|+1}} + \frac{1}{AM} \right) \leq 1.$$

Therefore, if we take M big enough in order that $c_0 \left(\frac{1}{M} + \frac{1}{AM} \right) \leq 1$, the inductive step is proved and the proof of the lemma is complete.

End of the proof for the Beurling case. By Lemma 3.2 there exists $E > 0$ such that for each $k \in \mathbb{N}$ there exist $E_k > 0$ such that

$$\| \partial_x^\alpha u \|_0 \leq E_k E^{|\alpha|+1} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.17}$$

Since

$$\begin{aligned}
|\xi^\alpha \hat{u}(t, \xi)| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ix\cdot\xi} \partial_x^\alpha u(t, x)\, dx \right| \\
&\leq \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |\partial_x^\alpha u(t, x)|\, dx \\
&\leq \| \partial_x^\alpha u \|_0, \quad t \in \mathbb{T}^m,\ \xi \in \mathbb{Z}^n,\ \alpha \in \mathbb{N}_0^n,
\end{aligned}$$

by (3.17) we have, for each $k \in \mathbb{N}$,

$$|\xi^\alpha \hat{u}(t, \xi)| \leq E_k E^{|\alpha|+1} e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}, \quad t \in \mathbb{T}^m,\ \xi \in \mathbb{Z}^n,\ \alpha \in \mathbb{N}_0^n. \tag{3.18}$$

Next, let $L \in \mathbb{N}_0$ and $\xi \in \mathbb{Z}^n$. Then we select $i \in \{1,\dots,n\}$ so that $|\xi_i| = \max_{1\le j\le n}|\xi_j|$. If we set $\alpha = Le_i$, where e_i is the i-th vector of the standard basis of $\mathbb{R}^n$, we have

$$|\xi|^L \le n^{L/2}\max_{1\le j\le n}|\xi_j|^L = n^{L/2}|\xi_i|^L = n^{L/2}|\xi^\alpha|. \tag{3.19}$$

By (3.18) it follows, for each $k \in \mathbb{N}$, that there exists $F_k > 0$ such that

$$|\xi|^L|\hat{u}(t,\xi)| \le F_k E^{L+1} e^{k\varphi^*\left(\frac{L}{k}\right)}, \quad t \in \mathbb{T}^m,\ \xi \in \mathbb{Z}^n,\ L \in \mathbb{N}_0.$$

Now, by Lemma 3.2 of [2] this means that, for each $k \in \mathbb{N}$ there is $G_k > 0$ such that

$$|\hat{u}(t,\xi)| \le G_k e^{-k\omega(\xi)}, \quad t \in \mathbb{T}^m,\ \xi \in \mathbb{Z}^n.$$

This implies that, for each $k \in \mathbb{N}$,

$$|\hat{u}(\tau,\xi)| = \left|\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} e^{-it\cdot\tau}\hat{u}(t,\xi)\,dt\right| \le G_k e^{-k\omega(\xi)}, \quad (\tau,\xi) \in \mathbb{Z}^{m+n}. \tag{3.20}$$

Let $(\tau_0,\xi_0) \in \mathbb{R}^{m+n}$ with $\xi_0 \neq 0$ and define $\Gamma := \{(\tau,\xi) \in \mathbb{R}^{m+n}:\ |\tau| < c|\xi|\}$ with $c > 1$ such that $(\tau_0,\xi_0) \in \Gamma$. Therefore, $(0,0) \notin \Gamma$, $(\tau_0,0) \notin \Gamma$ and if $(\tau,\xi) \in \Gamma$ then $\xi \neq 0$. Moreover, for every $(\tau,\xi) \in \Gamma \cap \mathbb{Z}^{m+n}$ we have

$$|\xi| = \frac{1}{2}|\xi| + \frac{1}{2}|\xi| \ge \frac{1}{2c}|\tau| + \frac{1}{2}|\xi| \ge \frac{1}{2c}|(\tau,\xi)|$$

and hence, as ω is an increasing function, this yields that

$$\omega(\xi) \ge \omega\Big(\frac{1}{2c}(\tau,\xi)\Big).$$

Now, since ω also satisfies property (α_0), there exist $D > 0$ and $t_0 > 0$ such that

$$\omega\Big(\frac{1}{2c}(\tau,\xi)\Big) \ge \frac{1}{2cD}\omega(\tau,\xi)$$

for $|(\tau,\xi)| \ge 2ct_0$. Therefore, if for each $k \in \mathbb{N}$ we set

$$D_k = \max\Big\{\max_{|(\tau,\xi)|\le 2ct_0}|\hat{u}(\tau,\xi)|e^{k\frac{\omega(\tau,\xi)}{2cD}}, G_k\Big\} < \infty,$$

it follows from (3.20) that

$$|\hat{u}(\tau,\xi)| \le D_k e^{-\frac{k}{2cD}\omega(\tau,\xi)}, \quad (\tau,\xi) \in \Gamma \cap \mathbb{Z}^{m+n}. \tag{3.21}$$

Set $k_0 = [2cD]$. Then (3.21) implies that

$$|\hat{u}(\tau,\xi)| \le D_{h(k_0+1)} e^{-h\omega(\tau,\xi)}, \quad h \in \mathbb{N},\ (\tau,\xi) \in \Gamma \cap \mathbb{Z}^{m+n}. \tag{3.22}$$

Next, fix $\tau_0 \in \mathbb{R}^m\setminus\{0\}$. By hypothesis $(t,x,\tau_0,0) \notin WF_{(\omega)}(u)$ for any $(t,x) \in \mathbb{T}^{m+n}$. Hence, by [2, Definition 3.4 and Lemma 3.2] (see [3, Definition 2.4(i)] in case ω is a non quasi–analytic weight function) there exists a cone Γ_1 containing $(\tau_0,0)$ and for every $h \in \mathbb{N}$ there is $D'_h > 0$ such that

$$|\hat{u}(\tau,\xi)| \le D'_h e^{-h\omega(\tau,\xi)}, \quad (\tau,\xi) \in \Gamma_1 \cap \mathbb{Z}^{m+n}. \tag{3.23}$$

Now, (3.22) and (3.23) imply that $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^{m+n})$ via [2, Proposition 3.3(b) and Lemma 3.2] (see [3, Definition 2.4(i)] in case ω is a non quasi-analytic weight function); see also Proposition 2.1. This completes the proof in the Beurling case. $\square$

The proof of Theorem 3.1 for the Roumieu case is a consequence of Theorem 3.1 for the Beurling case and of [2, Proposition 4.5 and Corollary 4.6]. So, we first recall that, by [2, Proposition 4.5 and Corollary 4.6],

$$WF_{\{\omega\}}(u) = \overline{\cup_{\sigma \in S} WF_{(\sigma)}(u)}, \quad u \in \mathcal{E}'(\mathbb{T}^N), \tag{3.24}$$

and hence

$$\mathcal{E}_{\{\omega\}}(\mathbb{T}^N) = \cap_{\sigma \in S} \mathcal{E}_{(\sigma)}(\mathbb{T}^N), \tag{3.25}$$

where $S = \{\sigma \text{ weight function}: \ \sigma_0 \le \sigma = o(\omega)\}$ with σ_0 and ω two weight functions such that $\sigma_0(t) = o(\omega(t))$ as $t \to \infty$.

(R) Roumieu case. Since $Pu =: f \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ and $(t, x, \tau, 0) \notin WF_{\{\omega\}}(u)$ for any $(t, x) \in \mathbb{T}^{m+n}$ and $\tau \in \mathbb{R}^m \setminus \{0\}$, it follows from (3.24) and (3.25) that, for every $\sigma \in S$, $Pu \in \mathcal{E}_{(\sigma)}(\mathbb{T}^N)$ and $(t, x, \tau, 0) \notin WF_{(\sigma)}(u)$ for any $(t, x) \in \mathbb{T}^{m+n}$ and $\tau \in \mathbb{R}^m \setminus \{0\}$. Since the coefficients of P belong to $\mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ and hence also belong to the space $\mathcal{E}_{(\sigma)}(\mathbb{T}^N)$ for $\sigma \in S$, we can apply the already proved Beurling case to conclude that $u \in \mathcal{E}_{(\sigma)}(\mathbb{T}^N)$ for every $\sigma \in S$. Therefore, (3.25) implies that $u \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$. $\square$

4 Applications

Now, we study the global $*$-hypoellipticity for some classes of sublaplacians, whose global G^s-hypoellipticity for $s \ge 1$ was treated in [15,16,32,33,43]. For the notation see, for example, [15,16].

Theorem 4.1 *Let ω and σ be two weight functions such that σ satisfies property (α_0), and $\omega(t) = o(\sigma(t))$ and $\omega(t) = o(t)$ as $t \to \infty$. Let $\mathbb{T}^N = \mathbb{T}^{m+n}$ and*

$$X_j = \sum_{h=1}^{m} a_{jh}(t)\partial_{t_h} + \sum_{k=1}^{n} b_{jk}(t)\partial_{x_k}, \quad j = 0, \ldots, l,$$

with a_{jh}, $b_{jk} \in \mathcal{E}_{\{\sigma\}}(\mathbb{T}^m)$ and real valued. Let $c \in \mathcal{E}_{\{\sigma\}}(\mathbb{T}^m)$. Moreover, suppose that the following conditions are satisfied:

(i) *Every point in $\mathbb{T}^N$ is of finite type for $X_1, \ldots, X_l$.*
(ii) *The vector fields $\{\sum_{h=1}^m a_{jh}(t)\partial_{t_h}\}_{j=1}^l$ span $T(\mathbb{T}^m)$ for every $t \in \mathbb{T}^m$.*

Then $P = \sum_{j=1}^l X_j^2 + X_0 + c(t)$ is globally (ω)-hypoelliptic in $\mathbb{T}^N$.

We recall that a point in $\mathbb{T}^N$ is of finite type for $X_1, \ldots, X_l$ if the Lie algebra generated by the vector fields $X_1, \ldots, X_l$ spans the tangent space of $\mathbb{T}^N$ there.

Proof By condition (i), we can apply Hormander's Theorem [35] to conclude that P is locally C^∞-hypoelliptic and, hence, globally C^∞-hypoelliptic in $\mathbb{T}^N$. On the other hand, let $u \in \mathcal{E}'(\mathbb{T}^N)$ be such that $Pu \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$. Since, by condition (ii), P is elliptic in t, we have $(t, x, \tau, 0) \notin WF_{(\omega)}(u)$ for any $(t, x) \in \mathbb{T}^n$ and $\tau \in \mathbb{R}^m \setminus \{0\}$ (see [2, Theorem 4.1] or, also, [3, Theorem 3.15] for non quasi-analytic weight functions). So, we can apply Theorem 3.1 to conclude that $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$. □

The Roumieu version of Theorem 4.1 continues to hold for c depending also on both variables t and x as in [33]. Indeed, we have

Theorem 4.2 *Let ω be a weight function satisfying property (α_0). Let $\mathbb{T}^N = \mathbb{T}^{m+n}$ and*

$$X_j = \sum_{h=1}^{m} a_{jh}(t)\partial_{t_h} + \sum_{k=1}^{n} b_{jk}(t)\partial_{x_k}, \quad j = 0, \ldots, l,$$

with a_{jh}, $b_{jk} \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^m)$ and real valued. Let $c \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$. Moreover, suppose that the following conditions are satisfied:

(i) *Every point in $\mathbb{T}^N$ is of finite type for $X_1, \ldots, X_l$.*
(ii) *The vector fields $\{\sum_{h=1}^m a_{jh}(t)\partial_{t_h}\}_{j=1}^l$ span $T(\mathbb{T}^m)$ for every $t \in \mathbb{T}^m$.*

Then $P = \sum_{j=1}^l X_j^2 + X_0 + c(t, x)$ is globally $\{\omega\}$-hypoelliptic in $\mathbb{T}^N$.

Proof In case $c(t, x) = c(t)$ for all $(t, x) \in \mathbb{T}^{m+n}$ the result follows from Theorem 3.1 as in Theorem 4.1. In case the function c depends on both variables t and x, the proof is not a consequence of Theorem 3.1. In fact, here the commutator $[\partial_x^\alpha, P] \neq 0$ and we need a different subelliptic estimate from (3.12).

We recall from [16, Lemma 2.1] that there exist $c_0 > 0$ such that

$$|\partial_x^\alpha \varphi|_0 \leq c_0(|P(\partial_x^\alpha \varphi)|_0 + |\partial_x^\alpha \varphi|_{-1}) \tag{4.26}$$

for every $\varphi \in C^\infty(\mathbb{T}^N)$ and $\alpha \in \mathbb{N}_0^n$ (here, $|\ |_0$ denotes the L^2–norm, i.e., $|f|_0 = (1/(2\pi)^{n+m} \int_{\mathbb{T}^{n+m}} |f(x)|^2 dx)^{1/2}$). Moreover, if $Pu =: f \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$ for some $u \in \mathcal{E}'(\mathbb{T}^N)$, then $u \in C^\infty(\mathbb{T}^N)$. Now, as in the proof of Theorem 3.1, $|\partial_x^\alpha \varphi|_{-1} \leq |\partial_x^{\alpha - e_j} \varphi|_0$ where e_j is an element of the standard basis of $\mathbb{R}^n$ such that the corresponding $\alpha_j \geq 1$. Since the vectors fields $X_1, X_2, \ldots, X_l$ depend only on the variable t, and hence commute with ∂_x^α, we obtain via (4.26) that

$$|\partial_x^\alpha u|_0 \leq c_0(|\partial_x^\alpha f|_0 + |[\partial_x^\alpha, P]u|_0 + |\partial_x^{\alpha - e_j} u|_0). \tag{4.27}$$

Since f, $c \in \mathcal{E}_{\{\omega\}}(\mathbb{T}^N)$, there exist $A > 0$ and $k \in \mathbb{N}$ so that

$$\max\{|\partial_x^\alpha f|_0, |\partial_x^\alpha c|_0\} \leq \max\{\| \partial_x^\alpha f \|_0, \| \partial_x^\alpha c \|_0\} \leq A^{|\alpha|+1} e^{\frac{1}{k}\varphi^*(k|\alpha|)}, \quad \alpha \in \mathbb{N}_0^n. \tag{4.28}$$

Therefore, the following lemma holds (compare with [33, Lemma 3.1]).

Lemma 4.1 *There exist $B > 0$ and $l \in \mathbb{N}$ such that*

$$|\partial_x^\alpha u|_0 \leq B^{|\alpha|+1} e^{\frac{1}{l}\varphi^*(l|\alpha|)}, \quad \alpha \in \mathbb{N}_0^n. \tag{4.29}$$

Proof The proof is given by induction on $|\alpha|$. Since $\varphi^*(0) = 0$ we have

$$|u|_0 = B_0 = B_0 e^{\frac{1}{l}\varphi^*(0)},$$

for any $l \in \mathbb{N}_0$ with $l \geq k$.

Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$ and suppose that (4.29) holds for all $\beta \in \mathbb{N}_0^n$ with $|\beta| < |\alpha|$. The dependence only on the variable t of the vectors fields $X_1, X_2, \dots, X_l$ gives

$$[\partial_x^\alpha, P]u = c\partial_x^\alpha u - \partial_x^\alpha(cu) = -\sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} c \; \partial_x^\beta u.$$

As in [2], since ω satisfies (α_0) we can assume that it is equivalent to a sub-additive weight function and then

$$\frac{e^{\frac{1}{l}\varphi^*(l|\alpha-\beta|)}}{(\alpha-\beta)!} \frac{e^{\frac{1}{l}\varphi^*(l|\beta|)}}{\beta!} \leq \frac{e^{\frac{1}{l}\varphi^*(l|\alpha|)}}{\alpha!}.$$

By the inductive hypothesis and by (4.28) we obtain that

$$\begin{aligned}
|[P, \partial_x^\alpha]u|_0 &\leq \sum_{\beta<\alpha} \binom{\alpha}{\beta} A^{|\alpha-\beta|+1} e^{\frac{1}{k}\varphi^*(k|\alpha-\beta|)} B^{|\beta|+1} e^{\frac{1}{l}\varphi^*(|\beta|l)} \\
&\leq \alpha! \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} \frac{e^{\frac{1}{l}\varphi^*(l|\alpha-\beta|)}}{(\alpha-\beta)!} B^{|\beta|+1} \frac{e^{\frac{1}{l}\varphi^*(l|\beta|)}}{\beta!} \\
&\leq e^{\frac{1}{l}\varphi^*(l|\alpha|)} \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} B^{|\beta|+1}.
\end{aligned}$$

Then, by (4.27) we obtain

$$\begin{aligned}
|\partial_x^\alpha u|_0 &\leq c_0 \left(|\partial_x^\alpha f|_0 + |[P, \partial_x^\alpha]u|_0 + |\partial_x^{\alpha-e_j} u|_0 \right) \\
&\leq c_0 \left(A^{|\alpha|+1} e^{\frac{1}{l}\varphi^*(l|\alpha|)} + e^{\frac{1}{l}\varphi^*(l|\alpha|)} \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} B^{|\beta|+1} + B^{|\alpha|} e^{\frac{1}{l}\varphi^*(l|\alpha|-l)} \right) \\
&\leq c_0 \; e^{\frac{1}{l}\varphi^*(l|\alpha|)} \left(A^{|\alpha|+1} + \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} B^{|\beta|+1} + B^{|\alpha|} \right).
\end{aligned}$$

We look for B of the form $B = MA$, for some $M > 1$. Then, it suffices to choose M such that

$$c_0 \left(A^{|\alpha|+1} + \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} M^{|\beta|+1} A^{|\beta|+1} + M^{|\alpha|} A^{|\alpha|} \right) \leq M^{|\alpha|+1} A^{|\alpha|+1}.$$

The conclusion follows as in the proof of [33, Lemma 3.1]. □

End of the proof of Theorem 4.2. By Lemma 4.1 there exist $B > 0$ and $l \in \mathbb{N}$ such that

$$|\partial_x^\alpha u|_0 \leq B^{|\alpha|+1} e^{\frac{1}{l}\varphi^*(l|\alpha|)}, \quad \alpha \in \mathbb{N}_0^n. \tag{4.30}$$

It follows from (4.30) that

$$\begin{aligned}|\xi^\alpha \hat{u}(t,\xi)| &= \left|\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} e^{-ix\cdot\xi}\partial_x^\alpha u(t,x)\,dx\right| \\ &\le \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} |\partial_x^\alpha u(t,x)|\,dx \le c|\partial_x^\alpha u|_0 \\ &\le cB^{|\alpha|+1}e^{\frac{1}{l}\varphi^*(l|\alpha|)}, \quad t\in\mathbb{T}^m,\ \xi\in\mathbb{Z}^n,\ \alpha\in\mathbb{N}_0^n.\end{aligned} \tag{4.31}$$

The same argument used in the proof of Theorem 3.1 for the Beurling case gives (see formula (3.19)), by (4.31),

$$|\xi|^L|\hat{u}(t,\xi)| \le cn^{L/2}B^{L+1}e^{\frac{1}{l}\varphi^*(lL)}, \quad t\in\mathbb{T}^m,\ \xi\in\mathbb{Z}^n,\ L\in\mathbb{N}_0.$$

Since L and ξ are arbitrary, we can apply Lemma 3.1 of [2] to conclude that there exist $\varepsilon>0$ and $C>0$ such that

$$|\hat{u}(t,\xi)| \le Ce^{-\varepsilon\omega(\xi)}, \quad t\in\mathbb{T}^m, \xi\in\mathbb{Z}^n.$$

This implies that

$$|\hat{u}(\tau,\xi)| = \left|\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} e^{-it\cdot\tau}\hat{u}(t,\xi)\,dt\right| \le Ce^{-\varepsilon\omega(\xi)}, \quad (\tau,\xi)\in\mathbb{Z}^{m+n}. \tag{4.32}$$

Let $(\tau_0,\xi_0)\in\mathbb{R}^{m+n}$ with $\xi_0\neq 0$ and define $\Gamma := \{(\tau,\xi)\in\mathbb{R}^{m+n} :\ |\tau| < c|\xi|\}$ with $c>1$ such that $(\tau_0,\xi_0)\in\Gamma$. Then, $(0,0)\notin\Gamma$, $(\tau_0,0)\notin\Gamma$ and if $(\tau,\xi)\in\Gamma$ then $\xi\neq 0$. Since ω is an increasing function and satisfies property (α_0), we can argue again as in the proof of Theorem 3.1 for the Beurling case to show via (4.32) that there exist $\varepsilon'>0$ and $E>0$ such that

$$|\hat{u}(\tau,\xi)| \le Ee^{-\varepsilon'\omega(\tau,\xi)}, \quad (\tau,\xi)\in\Gamma\cap\mathbb{Z}^{m+n}. \tag{4.33}$$

Next, let $\tau_0\in\mathbb{R}^m\setminus\{0\}$. By condition (ii), P is elliptic in t, and consequently $(t,x,\tau_0,0)\notin WF_{\{\omega\}}(u)$ for any $(t,x)\in\mathbb{T}^{m+n}$. Hence, by [2, Definition 3.4 and Lemma 3.1] (see [3, Definition 2.4(ii)] in case ω is a non quasi–analytic weight function) there exist $\varepsilon''>0$, $F>0$ and a cone Γ_1 containing $(\tau_0,0)$ such that

$$|\hat{u}(\tau,\xi)| \le Fe^{-\varepsilon''\omega(\tau,\xi)}, \quad (\tau,\xi)\in\Gamma_1\cap\mathbb{Z}^{m+n}. \tag{4.34}$$

Now, (4.33) and (4.34) imply that $u\in\mathcal{E}_{\{\omega\}}(\mathbb{T}^{m+n})$ via [2, Proposition 3.3(a) and Lemma 3.1] (see also Proposition 2.1). This completes the proof. □

Theorems 4.1 and 4.2 ensure, for example, that the operator in $\mathbb{T}^3$ given by

$$P_1 = \partial_{t_1}^2 + (\partial_{t_2} - a(t_1)\partial_x)^2 \tag{4.35}$$

is globally (ω)- $(\{\omega\}$-$)$ hypoelliptic in $\mathbb{T}^3$ for every weight function ω satisfying property (α_0) if the function a belongs to $\mathcal{A}(\mathbb{T})$, is real valued and not constant on $\mathbb{T}$. We recall that the global analytic and Gevrey hypoellipticity of P_1 was already established by Cordaro and Himonas [15] and by Himonas and Petronilho [32], and that P_1 is not in general locally analytic hypoelliptic. Indeed, Hanges and Himonas [24] proved that if $a(t_1)=t_1^{k-1}$, $k=3,5,7,\ldots$, then P_1 is not analytic hypoelliptic at 0. Christ [12] extended this result for all $k\ge 3$ and improved it in [13] showing that P_1 is not analytic hypoelliptic at 0 for any analytic function $a(t_1)$ with $a(0)=a'(0)=0$. We prove that P_1 is also not locally $*$-hypoelliptic for some weight function ω.

Proposition 4.1 *Let $P_1 = \partial_{t_1}^2 + (\partial_{t_2} - a(t_1)\partial_x)^2$ with a in $\mathcal{A}(\mathbb{T})$, real valued and not constant on $\mathbb{T}$. If $a(0) = a'(0) = 0$, then there exists a weight function ω with $\omega(t) = o(t)$ as $t \to \infty$ such that P_1 is neither $\{\sigma\}$-hypoelliptic nor (σ)-hypoelliptic at 0 for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ as $t \to \infty$.*

Proof Since, by [13], P_1 is not analytic hypoelliptic at 0, there exists $u \in \mathcal{D}'(U) \setminus \mathcal{A}(U)$ for some $U \subset \mathbb{R}^3$ open neighbourhood of 0 such that $P_1 u \in \mathcal{A}(U)$. So, $WF_{\{\omega_1\}}(u) \neq \emptyset$, where $\omega_1(t) = \max(t-1, 0)$. By [2, Proposition 4.5 and Corollary 4.6] (see (3.24)) this implies that there exists a weight function ω with $\omega(t) = o(t)$ as $t \to \infty$ such that $WF_{(\omega)}(u) \neq \emptyset$; hence, $u \notin \mathcal{E}_{(\omega)}(U)$. Since $\mathcal{E}_{(\sigma)}(U) \subset \mathcal{E}_{\{\sigma\}}(U) \subset \mathcal{E}_{(\omega)}(U)$ for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ as $t \to \infty$ (see Remark 2.4 (a)), and $\mathcal{A}(U) \subset \mathcal{E}_{(\sigma)}(U) \subset \mathcal{E}_{\{\sigma\}}(U)$, we can conclude that P_1 is neither $\{\sigma\}$-hypoelliptic nor (σ)-hypoelliptic at 0 for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ as $t \to \infty$. □

Theorems 4.1 and 4.2 also imply that the generalized Baouendi-Goulaouic operator

$$P_2 = \partial_t^2 + a^2(t)\partial_{x_1}^2 + b^2(t)\partial_{x_2}^2 \tag{4.36}$$

is globally (ω)- ($\{\omega\}$-) hypoelliptic in $\mathbb{T}^3$ for every weight function ω satisfying property (α_0) if the functions a and b belong to $\mathcal{A}(\mathbb{T})$, are real valued and not identically 0 on $\mathbb{T}$. It is known that the operator P_2 is globally analytic and Gevrey hypoelliptic on $\mathbb{T}^3$, see [15,32], but P_2 is not in general locally Gevrey hypoelliptic. For example, Christ [14] proved that, if $a(t) = t^{p-1}$ and $b(t) = t^{q-1}$, for some $1 \leq p \leq q \in \mathbb{N}$, then P_2 is locally Gevrey hypoelliptic for every $s \geq q/p$, but it is not locally Gevrey hypoelliptic in any class G^s with $s < q/p$. These operators include the well-known Baouendi-Goulaouic operator $\partial_t^2 + \partial_{x_1}^2 + t^2\partial_{x_2}^2$. Also in this case, we have that P_2 is not locally $*$-hypoelliptic for some weight function ω.

Proposition 4.2 *Let $a(t) = t^{p-1}$, $b(t) = t^{q-1}$, for $1 \leq p \leq q \in \mathbb{N}$, and $t \in \mathbb{T}$, and let $P_2 = \partial_t^2 + a^2(t)\partial_{x_1}^2 + b^2(t)\partial_{x_2}^2$. Then for every $s \in [1, q/p)$, there exists a weight function ω with $\omega(t) = o(t^{1/s})$ as $t \to \infty$ such that P_2 is neither $\{\sigma\}$-hypoelliptic nor (σ)-hypoelliptic at 0 for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ and $\sigma(t) = o(t^{1/s})$ as $t \to \infty$.*

Proof By [14], P_2 is not Gevrey hypoelliptic at 0 in any class G^s with $s < q/p$. We fix then $s < q/p$. There exists $u \in \mathcal{D}'(U) \setminus G^s(U)$ for some $U \subset \mathbb{R}^3$ open neighbourhood of 0 such that $P_2 u \in G^s(U)$. So, $WF_{\{t^{1/s}\}}(u) \neq \emptyset$. By [2, Proposition 4.5 and Corollary 4.6] (see (3.24)) this implies that there exists a weight function ω with $\omega(t) = o(t^{1/s})$ as $t \to \infty$ such that $WF_{(\omega)}(u) \neq \emptyset$; hence, $u \notin \mathcal{E}_{(\omega)}(U)$. Since $\mathcal{E}_{(\sigma)}(U) \subset \mathcal{E}_{\{\sigma\}}(U) \subset \mathcal{E}_{(\omega)}(U)$ and $G^s(U) \subset \mathcal{E}_{(\sigma)}(U)$ for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ and $\sigma(t) = o(t^{1/s})$ as $t \to \infty$, we can conclude that P_2 is neither $\{\sigma\}$-hypoelliptic nor (σ)-hypoelliptic at 0 for all weight function σ satisfying $\omega(t) = o(\sigma(t))$ and $\sigma(t) = o(t^{1/s})$ as $t \to \infty$. □

We end the paper with another application of Theorem 3.1 to a class of operators which may be of infinite type in the setting of non-quasianalytic ultradifferentiable classes.

Theorem 4.3 *Let ω and σ two weight functions such that σ satisfies property (α_0), and $\omega(t) = o(\sigma(t))$ and $\omega(t) = o(t)$ as $t \to \infty$. Let $\mathbb{T}^N = \mathbb{T}^{m+n}$ and let*

$$P = -\Delta_t - \sum_{j=1}^{n} a_j(t)\partial_{x_j}^2, \tag{4.37}$$

where $a_j \in \mathcal{E}_{\{\sigma\}}(\mathbb{T}^m)$ with $a_j \geq 0$. If each a_j is not identically equal to zero on $\mathbb{T}^m$, then P is globally $\{\sigma\}$-hypoelliptic and globally (ω)-hypoelliptic on $\mathbb{T}^N$.

Proof By Himonas [27, Theorem 1.1], the operator P is globally C^∞-hypoelliptic in $\mathbb{T}^N$. On the other hand, if $u \in \mathcal{E}'(\mathbb{T}^N)$ is such that $Pu \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$, then by [2, Theorem 4.1] (see also [3, Theorem 3.15] if ω is a non quasi-analytic weight) we have $(t, x, \tau, 0) \notin WF_{(\omega)}(u)$ for any $(t, x) \in \mathbb{T}^n$ and $\tau \in \mathbb{R}^m \setminus \{0\}$ as P is elliptic at every point $t \in \mathbb{T}^m$. We can apply Theorem 3.1 to conclude that $u \in \mathcal{E}_{(\omega)}(\mathbb{T}^N)$.

In the Roumieu case the result follows in a similar way. □

We remark that the operator P in (4.37) may be of infinite type at most points and is not locally hypoelliptic if the weights are non-quasianalytic, see [27, Remark 1.1].

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References

1. Albanese, A.A., Corli, A., Rodino, L.: Hypoellipticty and local solvability in Gevrey classes. Math. Nachr. 242, 5–16 (2002)
2. Albanese, A.A., Jornet, D., Oliaro, A.: Quasianalytic wave front sets for solutions of linear partial differential operators. Integr. Equ. Oper. Theory 66, 153–181 (2010)
3. Albanese, A.A., Jornet, D., Oliaro, A.: Wave front sets for ultradistribution solutions of linear pertial differential operators with coefficients in non-quasianalytic classes. Math. Nachr. doi:10.1002/mana.201010039 (to appear).
4. Albanese, A.A., Zanghirati, L.: Global hypoellipticity and global solvability in Gevrey classes on the n–dimensional torus. J. Diff. Eq. 199, 256–268 (2004)
5. Albanese, A.A., Popivanov, P.: Global analytic and Gevrey solvability of sublaplacians under Diophantine conditions. Ann. Mat. Pura e Appl. 185, 395–409 (2006)
6. Albanese, A.A., Popivanov, P.: Gevrey hypoellipticity and solvability on the multidimensional torus of some classes of linear partial differential operators. Ann. Univ. Ferrara 52, 65–81 (2006)
7. Baouendi, M.S., Goulaouic, C.: Nonanalytic–hypoellipticity for some degenerate elliptic operators. Bull. Amer. Math. Soc. 78, 483–486 (1972)
8. Bergamasco, A.P.: Remarks about global analytic hypoellipticity. Trans. Amer. Math. Soc. 351, 4113–4126 (1999)
9. Bonet, J., Meise, R., Melikhov, S.N.: A comparison of two different ways to define classes of ultradifferentiable functions. Bull. Belg. Math. Soc. Simon Stevin 14, 425–444 (2007)
10. Braun, R.W., Meise, R., Taylor, B.A.: Ultradifferentiable functions and Fourier analysis. Results Math. 17, 206–237 (1990)
11. Chen, W., Chi, M.Y.; Hypoelliptic vector fields and almost periodic periodic motions on the torus $\mathbb{T}^n$. Comm. Part. Diff. Eq. 25, 337–354 (2000)
12. Christ, M.: Certain sums of squares of vector fields fail to be analytic hypoelliptic. Comm. Part.Diff. Eq. 16, 1695–1707 (1991)
13. Christ, M.: A class of hypoelliptic PDE admitting non-analytic solutions. Contemporary Mathematics, Symposium in Complex Analysis 137 (1992)
14. Christ, M.: Intermediate optimal Gevrey exponents occur. Comm. Part. Diff. Eq. 22, 359–379 (1997)
15. Cordaro, P.D., Himonas, A.A.: Global analytic hypoellipticity for a class of degenerate elliptic operators on the torus. Math. Res. Lett. 1, 501–510 (1994)
16. Cordaro, P.D., Himonas, A.A.: Global analytic regularity for sums of squares of vector fields. Trans. Amer. Math. Soc. 350, 4993–5001 (1998)
17. Dickinson, D., Gramchev, T., Yoshino, M.: Perturbations of vector fields on tori: resonant normal forms and Diophantine phenomena. Proc. Edinb. Math. Soc. 45, 731–759 (2002)
18. Ferńandez, C., Galbis, A., Jornet, D.: Pseudodifferential operators of Beurling type and the wave front set. J. Math. Appl. Math. 340, 1153–1170 (2008)

19. Gramchev, T., Popivanov, P., Yoshino, M.: Some note on Gevrey hypoellipticity and solvability on torus. J. Math. Soc. Japan 43, 501–514 (1991)
20. Gramchev, T., Popivanov, P., Yoshino, M.: Some examples of global Gevrey hypoellipticity and solvability. Proc. Japan Acad. 69, 395–398 (1993)
21. Gramchev, T., Popivanov, P., Yoshino, M.: Global properties in spaces of generalized functions on the torus for second order differential operators with variable coefficients. Rend. Sem. Univ. Pol. Torino 51, 145–172 (1993)
22. Greenfield, S., Wallach, N.: Global hypoellipticity and Liouville numbers. Proc. Amer. Math. Soc. 31, 112–114 (1972)
23. Greenfield, S.: Hypoelliptic vector fields and continued fractions. Proc. Amer. Math. Soc. 31, 115–118 (1972)
24. Hanges, N., Himonas, A.A.: Singular solutions for sums of squares of vector fields. Comm. Part. Diff. Eq. 16, 1503–1511 (1991)
25. Hanges, N., Himonas, A.A.: Analytic hypoellipticity for generalized Baouendi–Goulaouic operators. J. Funct. Anal. 125, 309–325 (1994)
26. Helfer, B.: Conditions nécessaires d'hypoanalyticité puor des opérateurs invariants à gauche homogènes sur un groupe nilpotent gradué. J. Diff. Eq. 44, 460–481 (1982)
27. Himonas, A.A.: On degenerate elliptic operators of infinite type. Math. Z. 220, 449–460 (1996)
28. Himonas, A.A.: Global analytic and Gevrey hypoellipticity of sublaplacians under diophantine conditions. Proc. Amer. Math. Soc. 129, 2001–2007 (2000)
29. Himonas, A.A., Petronilho, G.: On global hypoellipticity of degenerate elliptic operators. Math. Z. 230, 241–257 (1999)
30. Himonas, A.A., Petronilho, G.: Global hypoellipticity and simultaneous approximability. J. Func. Anal. 170, 356–365 (2000)
31. Himonas, A.A., Petronilho, G.: Propagation of regularity and global hypoellipticity. Michigan Math. J. 50, 471–481 (2002)
32. Himonas, A.A., Petronilho, G.: On Gevrey regularity of globally C^∞ hypoelliptic operators. J. Diff. Eq. 207, 267–284 (2004)
33. Himonas, A.A., Petronilho, G.: On C^∞ and Gevrey regularity of sublaplacians. Trans. Amer. Math. Soc. 358, 4809–4820 (2006)
34. Himonas, A.A., Petronilho, G., dos Santos, L.A.C.: Regularity of a class of subLaplacians on the 3–dimensional torus. J. Funct. Anal. 240, 568–591 (2006)
35. Hörmander, L.: Hypoelliptic second order differential equations. Acta Mat. 119, 147–171 (1967)
36. Hörmander, L.: The Analysis of Linear Partial Differential Operators. Berlin Heidelbergh New York: Springer–Verlag, Vol. I, 2nd Edition (1990)
37. Langenbruch, M.: Ultradifferentiable functions on compact intervals, Math. Nachr. 140, 109–126 (1989)
38. Meise, R.: Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals, J. reine angew. Math. 363, 59–95 (1985)
39. Metivier, G.: Une class d'operateurs non hypoelliptiques analytiques. Indiana Univ. Math. J. 29, 823–860 (1980)
40. Omori, H., Kobayashi, T.: Global hypoellipticity of subelliptic operators on closed manifolds. Hokkaido Math. J. 28, 613–633 (1999)
41. Pham The Lai, Robert, D.: Sur un probléme aus valeurs propres non linéaire. Israel J. Math. 36, 169–186 (1980)
42. Petronilho, G.: Simultaneous reduction of a family of commuting real vector fields and global hypoellipticity. Israel J. Math. 155, 81–92 (2006)
43. Petronilho, G.: On Gevrey solvability and regularity. Math. Nachr. 282, 470–481 (2009)
44. Petzsche, H.-J.: Die Nuklearität der Ultradistributionsräume und der Satz von Kern I, Manuscripta Math. 24, 133–171 (1978)
45. Rodino, L.: Linear Partial Differential Operators in Gevrey Classes. Singapore New Jersey London Hong Kong: World Scientific (1993)
46. Tartakoff, D.: Global (and local) analyticity for second order operators constructed from rigid vector fields on product of tori. Trans. Amer. Math. Soc. 348, 2577–2583 (1996)

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