Bonet Solves, JA.; Fernandez, C.; Galbis, A.; Ribera Puchades, JM. (2014). Shrinking and boundedly complete Schauder frames in Fréchet spaces. Journal of Mathematical Analysis and Applications. 410(2):953-966. doi:10.1016/j.jmaa.2013.09.010.

Shrinking and boundedly complete Schauder frames in Fréchet spaces

José Bonet, Carmen Fernández, Antonio Galbis, Juan M. Ribera

Authors' addresses:

José Bonet (corresponding author): Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 València, Spain
email: jbonet@mat.upv.es
telephone number: +34963879497.
fax number: +34963879494.

Carmen Fernández and Antonio Galbis: Departamento de Análisis Matemático, Universitat de València, E-46100 Burjasot (València), Spain.
email: fernand@uv.es and antonio.galbis@uv.es

Juan M. Ribera: Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 València, Spain.
email: juaripuc@mat.upv.es

Abstract

We study Schauder frames in Fréchet spaces and their duals, as well as perturbation results. We define shrinking and boundedly complete Schauder frames on a locally convex space, study the duality of these two concepts and their relation with the reflexivity of the space. We characterize when an unconditional Schauder frame is shrinking or boundedly complete in terms of properties of the space. Several examples of concrete Schauder frames in function spaces are also presented.

Key words and phrases. Atomic decomposition, Schauder basis, Schauder frame, Fréchet spaces, (LB)-spaces, reflexivity, locally convex spaces, shrinking, boundedly complete.

MSC 2010: primary: 46A04, secondary: 42C15, 46A08, 46E10, 46A13, 46A25, 46A35, 46E10.

Introduction

Schauder frames are used to represent an arbitrary element x of a function space E as a series expansion involving a fixed countable set $(x_j)_j$ of elements in that space such that the coefficients of the expansion of x depend in a linear and continuous way on x. Unlike Schauder bases, the expression of an element x in terms of the sequence $(x_j)_j$, i.e. the reproduction formula for x, is not necessarily unique. In the classical literature of function spaces the Schauder frames are usually referred to as atomic decompositions. In abstract theory of Banach spaces the concept of atomic decomposition is often associated with a certain sequence space selected a priori while the notion of Schauder frame makes no reference to any sequence space. However, the two concepts are closely related and some papers in the area ([8], [4], [5]) are written in terms of atomic decompositions whereas others ([7], [1], [24]) are stated in terms of Schauder frames. Atomic decompositions appeared in applications to signal processing and sampling theory among other areas. Feichtinger characterized Gabor atomic decompositions for modulation spaces [10] and the general theory was developed in his joint work with Gröchenig [11] and [12]. In these papers, the authors show that reconstruction through atomic decompositions is not limited to Hilbert spaces. Indeed, they obtain atomic decompositions for a large class of Banach spaces, namely the coorbit spaces. Atomic decompositions or Schauder frames are a less restrictive structure than bases, because a complemented subspace of a Banach space with basis has always a natural Schauder frame, that is obtained from the basis of the superspace. Even when the complemented subspace has a basis, there is not a systematic way to find it. There is a vast literature dedicated to the subject. The related topic of frame expansions in Banach spaces was considered for example in [6] and [8].

Carando and Lasalle [4] and [5] studied atomic decompositions and their relationship with duality and reflexivity of Banach spaces. They extended the concepts of shrinking and boundedly complete Schauder basis to the atomic decomposition framework. They considered when an atomic decomposition for a Banach space generates, by duality, an atomic decomposition for its dual space and characterized the reflexivity of a Banach space in terms of properties of its atomic decompositions. Unconditional atomic decompositions allowed them to prove James-type results characterizing shrinking and boundedly complete unconditional atomic decompositions in terms of the containment in the Banach space of copies of ℓ_1 and c_0 respectively.

Very recently, Pilipovic and Stoeva [32] (see also [31]) studied series expansions in (countable) projective or inductive limits of Banach spaces. In this article we begin a systematic study of Schauder frames in locally convex spaces, but our main interest lies in Fréchet spaces and their duals. The main difference with respect to the concept considered in [32] is that our approach does not depend on a fixed representation of the Fréchet space as a projective limit of Banach spaces. We mention the following preliminary example as a motivation for our work: Leontiev proved that for each bounded convex domain G of the complex plane $\mathbb{C}$ there is a sequence of complex numbers $(\lambda_j)_j$ such that every holomorphic function $f \in H(G)$ can be expanded as a series of the form $f(z) = \sum_{j=1}^{\infty} a_j e^{\lambda_j z}$, converging absolutely and uniformly on the compact subsets of G. It is well-known that this expansion is not unique. We refer the reader e.g. to Korobeinik's survey [20]. A priori it is not clear whether the coefficients a_j in the expansion can be selected depending continuously on the function f. However, Korobeinik and Melikhov [22, Th. 4.3 and remark 4.4(b)] showed that this is the case when the boundary of the open set G is of class C^2; thus obtaining what we call below an uncon-

ditional Schauder frame for the Fréchet space $H(G)$. These are the type of phenomena and reproducing formulas that we try to understand in our paper.

Our main purpose is to investigate the relation between the properties of an existing Schauder frame in a Fréchet space E and the structure of the space, for example if E is reflexive or if it contains copies of c_0 or ℓ_1. For complete barrelled spaces, we show in 1.4 that having a Schauder frame is equivalent to being complemented in a complete locally convex space with a Schauder basis. Perturbation results for Schauder frames are given in Theorem 1.6. We introduce shrinking and boundedly complete Schauder frames on a locally convex space, study the duality of these two concepts and their relation with the reflexivity of the space; see Theorem 2.9. Unconditional Schauder frames are studied in Section 3. We completely characterize, for a given unconditional Schauder frame, when it is shrinking or boundedly complete in terms of properties of the space in Theorems 3.9 and 3.12. As a tool, that could be of independent interest, we show Rosenthal ℓ_1 Theorem for boundedly retractive inductive limits of Fréchet spaces; see Proposition 3.11. Some examples of concrete Schauder frames in function spaces are also included in Section 4. Our Theorem 4.2 shows a remarkable relation between the existence of a continuous linear extension operator for C^∞ functions defined on a compact subset K of $\mathbb{R}^n$ and the existence of an unconditional Schauder frame in $C^\infty(K)$ using exponentials.

1 Schauder frames in locally convex spaces

Throughout this work, E denotes a locally convex Hausdorff linear topological space (briefly, a lcs) with additional hypotheses added as needed and $cs(E)$ is the system of continuous seminorms describing the topology of E. The symbol E' stands for the topological dual of E and $\sigma(E', E)$ for the weak* topology on E'. We set E'_β for the dual E' endowed with the topology $\beta(E', E)$ of uniform convergence on the bounded sets of E. We will refer to E'_β as the strong dual of E. The bidual E'' of E is the dual of E'_β. Basic references for lcs are [18] and [27]. If $T : E \to F$ is a continuous linear operator, its transpose is denoted by $T' : F' \to E'$, and it is defined by $T'(v)(x) := v(T(x)), x \in E, v \in F'$. We recall that a Fréchet space is a complete metrizable lcs. An (LF)-space is a lcs that can be represented as an inductive limit of a sequence $(E_n)_n$ of Fréchet spaces, and in case all the spaces E_n are Banach spaces, we call it an (LB)-space. In most of the results we need the assumption that the lcs is barrelled. The reason is that Banach-Steinhaus theorem holds for barrelled lcs. Every Fréchet space and every (LB)-space is barrelled. We refer the reader to [18] and [30] for more information about barrelled spaces.

Definition 1.1 Let E be a lcs, $\{x_j\}_{j=1}^\infty \subset E$ and $\{x'_j\}_{j=1}^\infty \subset E'$. We say that $\Big(\{x'_j\}, \{x_j\}\Big)$ is an *Schauder frame of* E if

$$x = \sum_{j=1}^{\infty} x'_j(x)\, x_j, \qquad \text{for all } x \in E,$$

the series converging in E.

A lcs E which admits a Schauder frame is separable. Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^\infty \subset E$ and let $\{e'_j\}_{j=1}^\infty \subset E'$ denote the coefficient functionals. Clearly, $\Big(\{e'_j\}, \{e_j\}\Big)$ is

a Schauder frame for E. The main difference with Schauder basis is that, in general, one may have a sequence $\{x_j\}_{j=1}^{\infty} \subset E$ and two different sequences $\{x'_j\}_{j=1}^{\infty} \subset E'$ and $\{y'_j\}_{j=1}^{\infty} \subset E'$ so that both $\left(\{x'_j\},\{x_j\}\right)$ and $\left(\{y'_j\},\{x_j\}\right)$ are Schauder frames. See the comments after Theorem 4.2.

Proposition 1.2 *Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $\left(\{x'_j\},\{x_j\}\right)$ is a Schauder frame for E, then $\left(\{P'(x'_j)\},\{P(x_j)\}\right)$ is a Schauder frame for $P(E)$.*

In particular, if E is isomorphic to a complemented subspace of a lcs with a Schauder basis, then E admits a Schauder frame.

Proof. Since $\langle P'(x'_j), y\rangle = \langle x'_j, P(y)\rangle = \langle x'_j, y\rangle$ for all $y \in P(E)$ and $j \in \mathbb{N}$, we obtain a Schauder frame:

$$y = P(y) = P\left(\sum_{j=1}^{\infty} x'_j(y)\, x_j\right) = \sum_{j=1}^{\infty} \langle P'\left(x'_j\right), y\rangle P(x_j).$$

□

As usual ω denotes the countable product $\mathbb{K}^{\mathbb{N}}$ of copies of the scalar field, endowed by the product topology, and φ stands for the space of sequences with finite support. A sequence space $\bigwedge$ is a lcs which contains φ and is continuously included in ω.

Lemma 1.3 *Let $\{x_j\}_{j=1}^{\infty}$ be a fixed sequence of non-zero elements in a lcs E and let us denote by $\bigwedge$ the vector space*

$$\bigwedge := \{\alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E\}. \tag{1.1}$$

Endowed with the system of seminorms

$$\mathcal{Q} := \left\{ q_p\left((\alpha_j)_j\right) := \sup_n p\left(\sum_{j=1}^{n} \alpha_j x_j\right),\ \text{for all } p \in cs(E) \right\} \tag{1.2}$$

$\bigwedge$ is a sequence space and the canonic unit vectors form a Schauder basis. If E is complete, then $\bigwedge$ is complete. In particular, if E is a Fréchet (resp. Banach) space, so is $\bigwedge$.

Proof. It is routine to check that the unit vectors are a topological basis of $\bigwedge$. Since

$$q\left(\sum_{i=1}^{n} \alpha_i e_i\right) \leq q\left(\sum_{i=1}^{n+m} \alpha_i e_i\right)$$

for every $q \in \mathcal{Q}$ and for all $m, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{n+m} \in \mathbb{K}$ we can apply [18, 14.3.6] to conclude that the unit vectors are also a Schauder basis. □

Theorem 1.4 *Let E be a complete barrelled locally convex space. The following conditions are equivalent:*

(1) *E admits a Schauder frame.*

(2) *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.*

(3) *E is isomorphic to a complemented subspace of a complete locally convex space with a Schauder basis.*

In particular, a Fréchet space E admits a Schauder frame if and only if it is isomorphic to a complemented subspace of a Fréchet space with a Schauder basis.

Proof. (1) $\Rightarrow$ (2) Let $\left(\{x'_j\},\{x_j\}\right)$ be a Schauder frame of E. We may assume that $x_j \neq 0$ for all $j \in \mathbb{N}$. Let $\bigwedge$ be the complete lcs of sequences defined as in Lemma 1.3. We define $F_n : E \longrightarrow E$ as $F_n(x) := \sum_{j=1}^{n} x'_j(x)\, x_j$. Since E is barrelled the sequence $(F_n)_n$ is equicontinuous, that is, for every $p \in cs(E)$ there exists $p' \in cs(E)$ such that $p(F_n(x)) \leq p'(x)$ for every $x \in E$ and for every $n \in \mathbb{N}$. Consequently the map $U : E \longrightarrow \bigwedge$, $U(x) := \left(x'_j(x)\right)_j$, is injective and continuous. Moreover, the map $S : \bigwedge \longrightarrow E$, $S\left((\alpha_j)_j\right) := \sum_{j=1}^{\infty} \alpha_j x_j$, is linear and continuous, since

$$p\left(S\left((\alpha_j)_j\right)\right) = p\left(\sum_{j=1}^{\infty} \alpha_j x_j\right) \leq \sup_n p\left(\sum_{j=1}^{n} \alpha_j x_j\right) = q_p\left((\alpha_j)_j\right).$$

As $S \circ U = I_E$ we conclude that U is an isomorphism into its range $U(E)$ and $U \circ S$ is a projection of $\bigwedge$ onto $U(E)$.

(2) $\Rightarrow$ (3) is trivial, while (3) $\Rightarrow$ (1) is consequence of Proposition 1.2. □

The following Corollary is a consequence of an important result of Pełczyński. A locally convex space is said to satisfy the bounded approximation property if the identity of E is the pointwise limit of an equicontinuous net of finite rank operators. If the locally convex space is separable, then the net can be replaced by a sequence. Pełczyński [29] (see also [25, Theorem 2.11]) proved that a separable Fréchet space has the bounded approximation property if and only if it is isomorphic to a complemented Fréchet space with a Schauder basis.

Corollary 1.5 *A separable Fréchet space E admits a Schauder frame if and only if E has the bounded approximation property.*

Proof. It follows from Theorem 1.4 and the aforementioned result of Pełczyński [29]. □

Taskinen [35] gave examples of a complemented subspace F of a Fréchet Schwartz space E with a Schauder basis, such that F is nuclear and does not have a basis. By Theorem 1.4, F has a Schauder frame. Vogt [36] gave examples of nuclear (hence separable) Fréchet spaces E which do not have the bounded approximation property. These separable Fréchet spaces E do not admit a Schauder frame, although by Komura-Komura's Theorem [27, Theorem 29.8] they are isomorphic to a subspace of the countable product $s^{\mathbb{N}}$ of copies of the space of rapidly decreasing sequence, that has a Schauder basis.

To end this section we discuss perturbation results. The following result, that is needed below, can be found in [16, page 436]: *Let E be a complete lcs and let $T : E \to E$ be an operator*

with the property that there exists $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ such that $p(Tx) \leq C_p p_0(x)$ for all $x \in E$ (that is, T maps a neighborhood into a bounded set) and moreover C_{p_0} can be chosen strictly smaller than 1. Then $I - T$ is invertible (with continuous inverse on E).

Theorem 1.6 *Let $(\{x'_j\}, \{x_j\})$ be a Schauder frame of a complete lcs E.*

(1) *If $(y_j)_j$ is a sequence in E satisfying that there is $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with:*

(i) $\sum_{j=1}^{\infty} |x'_j(x)| p(x_j - y_j) \leq p_0(x) C_p$ for each $x \in E$ and

(ii) C_{p_0} can be chosen strictly smaller than 1,

then, there exists $(y'_j)_j$ a sequence in E' such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.

(2) *If $(y'_j)_j$ is a sequence in E' satisfying that there is $p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with:*

(i) $\sum_{j=1}^{\infty} |(x'_j - y'_j)(x)| p(x_j) \leq p_0(x) C_p$ for each $x \in E$ and

(ii) C_{p_0} can be chosen strictly smaller than 1,

then, there exists $(y_j)_j$ a sequence in E such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.

Proof. In case (1) we consider the operator $T(x) = \sum_{j=1}^{\infty} x'_j(x)(x_j - y_j)$. It is well defined as the series is absolutely convergent in E, hence convergent, and T is continuous as

$$p(Tx) \leq \sum_{j=1}^{\infty} |x'_j(x)| p(x_j - y_j) \leq p_0(x) C_p.$$

Now, $S := I - T$ is invertible, therefore one can take $y'_j = (S^{-1})'(x'_j)$ to conclude.

In case (2) we argue in the same way with the operator $T(x) = \sum_{j=1}^{\infty} (x'_j - y'_j)(x) x_j$, and the sequence $(y_j)_j$ is given by $S^{-1}(x_j)$, $j \in \mathbb{N}$. □

Our next result should be compared with [9, Proposition 2].

Corollary 1.7 *Let $(\{x'_j\}, \{x_j\})$ be a Schauder frame of a complete lcs E. Suppose that there exists $p_0 \in cs(E)$ such that $|x'_j(x)| \leq p_0(x)$ for every $x \in E, j \in \mathbb{N}$. Let $(y_j)_j \subset E$ such that $\sum_{j=1}^{\infty} p(y_j - x_j) < \infty$ for every $p \in cs(E)$ and $\sum_{j=1}^{\infty} p_0(y_j - x_j) < 1$. Then there exists $(y'_j)_j \subset E'$ such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.*

Corollary 1.8 *Let E be a Fréchet space with fundamental system of seminorms $(p_k)_k$ and let $(\{x'_j\}, \{x_j\})$ be a Schauder frame of E. Suppose that $(y'_j)_j \subset E'$ satisfies*

$$p_1^*(x'_j - y'_j) < \frac{1}{1 + j^2 p_j(x_j) + 3^j p_1(x_j)} \text{ where } p_1^*(x') = \sup\{|x'(x)| : p_1(x) \leq 1\}.$$

Then there exists $(y_j)_j \subset E$ such that $(\{y'_j\}, \{y_j\})$ is a Schauder frame for E.

Given a Schauder frame $(\{x'_j\},\{x_j\})$ on a complete lcs E, if $x'_1(x_1)\neq 1$ the map $x\to\sum_{j=2}^{\infty}x'_j(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x\to x'_1(x)x_1$; see [19, p. 207]. Hence there exists $(y'_j)_j\subset E'$ such that $(\{y'_j\}_j,\{x_{j+1}\}_j)$ is a Schauder frame and similarly there exists $(y_j)_j\subset E$ such that $(\{x'_{j+1}\}_j,\{y_j\}_j)$ is a Schauder frame. That is, we can remove an element and still obtain Schauder frames. We recall that for a Schauder basis $(x_j)_j$ with functional coefficients $(x'_j)_j$ one has $x'_1(x_1)=1$.

2 Duality of Schauder frames

Given a Schauder frame $\left(\{x'_j\},\{x_j\}\right)$ of E it is rather natural to ask whether $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of E'. This is always the case when E' is endowed with the weak* topology $\sigma(E',E)$.

Lemma 2.1 *If $\left(\{x'_j\},\{x_j\}\right)$ is a Schauder frame of E, then $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of $(E',\sigma(E',E))$.*

Proof. For every $x'\in E'$ and $x\in E$ we have

$$x'(x)=x'\left(\sum_{j=1}^{\infty}x'_j(x)\,x_j\right)=\sum_{j=1}^{\infty}x'_j(x)\,x'(x_j)=\left(\sum_{j=1}^{\infty}x'(x_j)\,x'_j\right)(x),$$

and $x'=\sum_{j=1}^{\infty}x'(x_j)\,x'_j$ with convergence in $(E',\sigma(E',E))$. □

We investigate conditions to ensure that $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of the strong dual $(E',\beta(E',E))$ of E. Moreover, we investigate the relation between the existence of certain Schauder frames and reflexivity. We recall that in the case of bases this questions lead to the concept of shrinking basis and boundedly complete basis; see [18].

Given a Schauder frame $\left(\{x'_j\},\{x_j\}\right)$ of a lcs E we denote, for each $n\in\mathbb{N}$, $T_n(x):=\sum_{j=n+1}^{\infty}x'_j(x)\,x_j$, that is a continuous linear operator on E.

Definition 2.2

1. A Schauder frame $\left(\{x'_j\},\{x_j\}\right)$ of a lcs E is said to be *shrinking* if, for all $x'\in E'$,
$$\lim_{n\to\infty}x'\circ T_n=0$$
uniformly on the bounded subsets of E.
2. A Schauder frame $\left(\{x'_j\},\{x_j\}\right)$ of a lcs E is said to be *boundedly complete* if the series $\sum_{j=1}^{\infty}x'_j(x'')\,x_j$ converges in E for every $x''\in E''$.

Proposition 2.3 *Let E be a lcs and let $\left(\{x'_j\},\{x_j\}\right)$ be a Schauder frame of E. The following are equivalent:*

(1) *$\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame for E'_β.*

(2) *For all* $x' \in E'$, $\sum_{j=1}^{\infty} x'(x_j)\, x'_j$ *is convergent in* E'_β.

(3) $\left(\{x'_j\}, \{x_j\}\right)$ *is shrinking.*

Moreover, if $\left(\{x'_j\}, \{x_j\}\right)$ *is a shrinking Schauder frame of* E*, then* $\left(\{x_j\}, \{x'_j\}\right)$ *is a boundedly complete Schauder frame of* E'_β.

Proof. (1) $\Rightarrow$ (2) is clear by the definition of Schauder frame.

(2) $\Rightarrow$ (3) From the assumption and lemma 2.1, $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ in the topology $\beta(E', E)$. As $x' \circ T_n = \sum_{j=n+1}^{\infty} x'(x_j)\, x'_j$ we conclude.

Finally, we prove (3) $\Rightarrow$ (1). Every $x' \in E'$ can be written as $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ with convergence in the weak* topology $\sigma(E', E)$. Given a bounded set B in E,

$$\sup_{x\in B} \left| \left(x' - \sum_{j=1}^{n} x'(x_j)\, x'_j \right)(x) \right| = \sup_{x\in B} |x' \circ T_n(x)|$$

which tends to zero, hence $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ in the topology $\beta(E', E)$.

Finally, if $\left(\{x'_j\}, \{x_j\}\right)$ is a shrinking Schauder frame of E, then $\left(\{x_j\}, \{x'_j\}\right)$ is a Schauder frame of E'_β. Moreover, given $x''' \in E'''$ set $x' := x'''|_E$ to obtain

$$\sum_{j=1}^{\infty} x'''(x_j)\, x'_j = \sum_{j=1}^{\infty} \left(x'''\big|_E\right)(x_j)\, x'_j = \sum_{j=1}^{\infty} x'(x_j)\, x'_j = x'.$$

□

A space E is called Montel if it is barrelled and every bounded subset of E is relatively compact. Since the pointwise convergence of an equicontinuous sequence of operators implies the uniform convergence on the compact sets, every Schauder frame of a Montel space E is shrinking. Beanland, Freeman and Liu [1] have shown that every infinite dimensional Banach space which admits a Schauder frame has also a Schauder frame which is not shrinking. The main tool in the proof is the existence of weak* null sequences in the unit sphere of E'. This result has inspired the following characterization of Fréchet spaces with a Schauder frame that are Montel. In fact, since a Fréchet space E is Montel if and only if every weak* null sequence in E' is also strongly convergent [3], an adaptation of the proof of [1, Theorem 2.3] gives the following result.

Theorem 2.4 *Let* E *be a separable Fréchet space with the bounded approximation property. Then* E *is Montel if and only if every Schauder frame of* E *is shrinking.*

Recall that a boundedly complete Schauder basis $(e_j)_j$ in a lcs E is a basis such that if $(\alpha_j)_j \in \omega$ and $\left(\sum_{j=1}^{k} \alpha_j e_j\right)_k$ is bounded, then $\sum_{j=1}^{\infty} \alpha_j e_j$ is convergent.

In [4] it is shown that a basis $(e_j)_j$ in a Banach space X is boundedly complete if and only if the Schauder frame $\left(\{e'_j\}, \{e_j\}\right)$ is boundedly complete. This extends to arbitrary barrelled spaces.

Proposition 2.5 *Let E be a barrelled lcs with a Schauder basis $(e_j)_j$. Then the following are equivalent:*

(1) *The basis is boundedly complete.*

(2) *The Schauder frame $\left(\{e'_j\},\{e_j\}\right)$ is boundedly complete.*

Proof. To prove (1) $\Rightarrow$ (2) we fix $x''\in E''$ and we prove that $\sum_{j=1}^{\infty} e'_j\left(x''\right)e_j$ converges in E. For every $x'\in E'$ and $x\in E$ we have

$$\lim_{k\to\infty}\left(\sum_{j=1}^{k} x'\left(e_j\right)e'_j\right)(x)=\lim_{k\to\infty} x'\left(\sum_{j=1}^{k} e'_j\left(x\right)e_j\right)=x'(x).$$

Since E is barrelled we conclude that $\left\{\sum_{j=1}^{k} x'\left(e_j\right)e'_j, k\in\mathbb{N}\right\}$ is $\beta(E',E)$-bounded. Consequently $\left\{\sum_{j=1}^{k} x''\left(e'_j\right)x'(e_j), k\in\mathbb{N}\right\}$ is a bounded set of scalars for every $x'\in E'$, which means that $\left\{\sum_{j=1}^{k} x''\left(e'_j\right)e_j, k\in\mathbb{N}\right\}$ is $\sigma(E,E')$-bounded. As all topologies of the same dual pair have the same bounded sets ([18, 8.3.4]) we finally obtain that $\left\{\sum_{j=1}^{k} x''\left(e'_j\right)e_j, k\in\mathbb{N}\right\}$ is a bounded subset of E and the conclusion follows.

To prove (2) $\Rightarrow$ (1) we fix $\left(\alpha_j\right)_j\subset\mathbb{K}$ such that $\left(\sum_{j=1}^{k}\alpha_j e_j\right)_k$ is bounded and we show that $\sum_{j=1}^{\infty}\alpha_j e_j$ is convergent in E. Since $\left(\sum_{j=1}^{k}\alpha_j e_j\right)_k$ is $\sigma\left(E'',E'\right)$-relatively compact then it has a $\sigma\left(E'',E'\right)$-cluster point $x''\in E''$. By hypothesis, $\sum_{j=1}^{\infty} x''\left(e'_j\right)e_j$ is convergent in E, so to conclude it suffices to check that $x''\left(e'_j\right)=\alpha_j$. To this end we fix $j\in\mathbb{N}$ and $k>j$ and observe that

$$e'_j\left(\sum_{i=1}^{k}\alpha_i e_i\right)=\sum_{i=1}^{k}\alpha_i e'_j\left(e_i\right)=\alpha_j.$$

As $x''(e_j)$ is a cluster point of $\left\{e'_j\left(\sum_{i=1}^{k}\alpha_i e_i\right)\right\}_{k=1}^{\infty}$ we finally deduce $x''\left(e'_j\right)=\alpha_j$. □

Remark 2.6 Let $\left(\{x'_j\},\{x_j\}\right)$ be a Schauder frame of E and let $P:E\to E$ be a continuous linear projection. It is easy to see that if $\left(\{x'_j\},\{x_j\}\right)$ is shrinking (boundedly complete) then $\left(\{P'(x'_j)\},\{P\left(x_j\right)\}\right)$ is a shrinking (boundedly complete) Schauder frame for $P\left(E\right)$.

Lemma 2.7 *Suppose that $\left(\{x'_j\},\{x_j\}\right)$ is a Schauder frame of a barrelled lcs E such that for all $k\in\mathbb{N}$*

$$\lim_{n\to\infty}\left(x'_k-\sum_{j=1}^{n} x'_k\left(x_j\right)x'_j\right)=0 \text{ in } E'_\beta. \tag{2.1}$$

Then $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of the closed linear span $H=\overline{\operatorname{span}\left\{x'_j\right\}}^{E'_\beta}$.

Proof. We fix $x' \in H$ and show that $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ with convergence in E'_β. To this end we fix U a neighborhood of zero in E'_β and consider $F_n(x) = \sum_{j=1}^{n} x'_j(x) x_j$, $n \in \mathbb{N}, x \in E$. Since $(F'_n)_n \subset L(E')$ is equicontinuous, there is another $\beta(E', E)$-neighborhood V, $V \subset U$, such that $F'_n(V) \subset \frac{1}{3}U$ for each $n \in \mathbb{N}$. Find $u = \sum_{k=1}^{s} \alpha_k x'_k$, $\alpha_k \in \mathbb{K}$, $s \in \mathbb{N}$, with $x' - u \in \frac{1}{3}V$. By condition (2.1) we can find $n_0 \in \mathbb{N}$ such that $u - F'_n(u) \in \frac{1}{3}V$ for each $n \geq n_0$. Finally,

$$x' - F'_n(x') = x' - u - F'_n(x' - u) + u - F'_n(u) \in \frac{1}{3}V + \frac{1}{3}U + \frac{1}{3}V \subset U \text{ if } n \geq n_0.$$

Thus $E'_\beta\text{-}\lim_{n\to\infty} F'_n(x') = x'$ and the conclusion follows. □

Remark 2.8 (a) Observe that if $\left(\{x_j\}, \{x'_j\}\right)$ is a Schauder frame of the closed linear span $H = \overline{\operatorname{span}\left\{x'_j\right\}}^{E'_\beta}$ then (2.1) holds since $x'_k \in H$ for each $k \in \mathbb{N}$.

(b) If $\{x_j\}$ is a Schauder basis in E with functional coefficients $\{x'_j\}$ then (2.1) also holds, since $x'_k - \sum_{j=1}^{n} x'_k(x_j)\, x'_j = 0$ for every $n \geq k$.

(c) If E is a Montel space, (2.1) holds since every weakly convergent sequence in a Montel space is also strongly convergent to the same limit, by [18, 11.6.2].

Theorem 2.9 *Let $\left(\{x'_j\}, \{x_j\}\right)$ be a Schauder frame of a lcs E. Then,*

(1) *If $\left(\{x'_j\}, \{x_j\}\right)$ is a boundedly complete Schauder frame, E is a barrelled and complete lcs E with E''_β barrelled, then E is complemented in its bidual E''_β.*

(2) *If E is reflexive and* (2.1) *in Lemma 2.7 holds, then $\left(\{x'_j\}, \{x_j\}\right)$ is shrinking.*

(3) *If $\left(\{x'_j\}, \{x_j\}\right)$ is shrinking and boundedly complete, then E is semi-reflexive. If, in addition, E is barrelled then it is reflexive.*

Proof.

(1) Since $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete the linear map $P : E'' \to E$, $P(x'') := \sum_{j=1}^{\infty} x''\left(x'_j\right) x_j$ is well defined. Since E''_β is barrelled we can apply Banach-Steinhaus theorem to conclude that P is continuous, and it is clearly surjective. As E is barrelled, it can be canonically identified with a topological subspace of its bidual E''_β. Then it is easy to see that P is a projection.

(2) As E is reflexive then it is barrelled and Lemmas 2.1 and 2.7 hold. In particular, for each $x' \in H = \overline{\operatorname{span}\{x'_j\}}^{E'_\beta}$ we have $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ with convergence in E'_β. Since E is semi-reflexive, $\beta(E', E)$ and $\sigma(E', E)$ are topologies of the same dual pair. Hence, by Lemma 2.1 we obtain $H = \overline{\operatorname{span}\{x'_j\}}^{E'_\beta} = \overline{\operatorname{span}\{x'_j\}}^{(E', \sigma(E', E))} = E'$. The result follows by Proposition 2.3.

(3) Fix $x'' \in E''$. Since the Schauder frame is boundedly complete then $\sum_{j=1}^{\infty} x'_j(x'')\, x_j$ converges to an element $x \in E$. We claim that $x'' = x$. In fact, since the Schauder frame is

shrinking, for every $x' \in E'$ we have $x' = \sum_{j=1}^{\infty} x'(x_j)\, x'_j$ with convergence in E'_β. Thus

$$\langle x'', x' \rangle = \langle x'', \sum_{j=1}^{\infty} x'(x_j)\, x'_j \rangle = \sum_{j=1}^{\infty} x'(x_j)\, x''(x'_j) = \left(\sum_{j=1}^{\infty} x''(x'_j)\, x_j \right)(x') = \langle x, x' \rangle.$$

It follows $x'' = x$. □

For a Fréchet space E, the bidual E''_β is again a Fréchet space, therefore barrelled. For LB-spaces, this is not always the case. In fact, if $\lambda_1(A)$ is the Grothendieck example of a non-distinguished Fréchet space, $\lambda_1(A)$ is the strong dual of an LB-space E. The bidual of E, being the strong dual of $\lambda_1(A)$, is not barrelled. See [23, Chapter 31, Sections 6 and 7] and [27, Example 27.19].

3 Unconditional Schauder frames

In this section we assume that E is a complete lcs and we denote by $\mathcal{U}_0(E)$ the set of absolutely convex and closed 0-neighborhoods. We refer the reader to [18] for unconditional convergence of series in locally convex spaces.

Definition 3.1 A Schauder frame $\left(\{x'_j\}, \{x_j\}\right)$ for a lcs E is said to be *unconditional* if for every $x \in E$ we have $x = \sum_{j=1}^{\infty} x'_j(x)\, x_j$ with unconditional convergence.

Remark 3.2 By [26, p.116] a series $\sum_{j=1}^{\infty} x_j$ in a (sequentially) complete lcs converges unconditionally if and only if the limits $\lim_{n\to\infty} \sum_{j=1}^{n} a_j x_j$ exist uniformly for $(a_j)_j$ in the unit ball of ℓ_∞, and the operator $\ell_\infty \to E,\ \{a_j\} \mapsto \sum_{j=1}^{\infty} a_j x_j$, is continuous.

Lemma 3.3 *Let X be a normed space, E a barrelled space and G any lcs. Then every separately continuous bilinear map $B : X \times E \to G$ is continuous.*

Proof. Let $W \in U_0(G)$ and let U_X be the closed unit ball of X. Now we take $T := \{y \in E :\ B(x,y) \in W \text{ for every } x \in U_X\} = \bigcap_{x \in U_X} B_x^{-1}(W)$, where $B_x = B(x, \cdot)$. Note that T is an absolutely convex closed subset since each $B_x : E \to G$ is continuous. Fixing $y \in E$, since $B_y : X \to G$ is continuous then $B_y^{-1}(W) \in \mathcal{U}_0(X)$, what means that there exists $\lambda > 0$ such that $\lambda U_X \subset B_y^{-1}(W)$. Therefore $B(x, \lambda y) \in W$ for every $x \in U_X$ and $\lambda y \in T$, that is, T is absorbent. Since E is barrelled then $T \in U_0(E)$ and from $B(U_X \times T) \subset W$ we conclude that B is continuous. □

Corollary 3.4 *Let $\left(\{x'_j\}, \{x_j\}\right)$ be an unconditional Schauder frame for a complete barrelled lcs E. Then, the bilinear map*

$$B : E \times \ell_\infty \to E,\ B(x,a) := \sum_{j=1}^{\infty} a_j x'_j(x)\, x_j,$$

is continuous.

The property of having an unconditional Schauder frame is also inherited by complemented subspaces.

Proposition 3.5 *Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $\left(\{x'_j\}, \{x_j\}\right)$ is an unconditional Schauder frame for E, then $\left(\{P'(x'_j)\}, \{P(x_j)\}\right)$ is an unconditional Schauder frame for $P(E)$.*

In particular if E is isomorphic to a complemented subspace of a lcs with a unconditional Schauder basis, then E admits an unconditional Schauder frame.

Similarly to Lemma 1.3 we have the following.

Lemma 3.6 *Let $(x_j)_j$ be a fixed sequence of non-zero elements in a lcs E and let us denote by $\widetilde{\bigwedge}$ the space*

$$\widetilde{\bigwedge} := \{\alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is unconditionally convergent in } E\}. \tag{3.1}$$

Endowed with the system of seminorms

$$\widetilde{\mathcal{Q}} := \left\{ \widetilde{q}_p\left((\alpha_j)_j\right) := \sup_{b \in B_{\ell^\infty}} p\left(\sum_{j=1}^{\infty} b_j \alpha_j x_j\right), \text{ for all } p \in cs(E) \right\}, \tag{3.2}$$

$\widetilde{\bigwedge}$ is a complete lcs of sequences and the canonical unit vectors are an unconditional basis.

Theorem 3.7 *Let E be a complete, barrelled lcs. The following conditions are equivalent:*

(1) *E admits an unconditional Schauder frame.*

(2) *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as unconditional Schauder basis.*

(3) *E is isomorphic to a complemented subspace of a complete locally convex space with unconditional Schauder basis.*

Proof. The proof follows the steps of Theorem 1.4 but the continuity of the map

$$U : E \longrightarrow \widetilde{\bigwedge},\ x \to \left(x'_j(x)\right)_j,$$

now follows from Corollary 3.4. □

In our next two results, bipolars are taken in E'' that is $U^{\circ\circ} = \{x'' \in E'' : |x''(x')| \leq 1 \text{ for all } x' \in U^\circ\}$.

Lemma 3.8 *Let E be a lcs and let U be an absolutely convex and closed 0-neighborhood. For every $z \in E''$ such that $p_{U^{\circ\circ}}(z) > 0$ there exists $(x_\alpha) \subset E$ with $p_U(x_\alpha) \leq p_{U^{\circ\circ}}(z)$ and $x_\alpha \to z$ in $\sigma(E'', E')$.*

Proof. First, we observe that $x := \frac{z}{p_{U^{\circ\circ}}(z)} \in U^{\circ\circ}$, a set that coincides with $\overline{U}^{\sigma(E'',E')}$ by the bipolar Theorem ([18, 8.2.2]). Therefore there exists $(y_\alpha)_\alpha \subset U$ such that $y_\alpha \to x$ in $\sigma(E'', E')$. Now, it suffices to take $x_\alpha := p_{U^{\circ\circ}}(z)\, y_\alpha$. □

Theorem 3.9 *Let E be a complete, barrelled lcs which admits an unconditional Schauder frame $\left(\{x'_j\}, \{x_j\}\right)$. Then, $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete if and only if E does not contain a copy of c_0.*

Proof. Suppose that E contains a copy of c_0. Since E is separable, there exists a projection $P : E \to E$ such that $P(E)$ is isomorphic to c_0 ([18, 8.5.9]). If $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete, then $\left(\{P'(x'_j)\}, \{P(x_j)\}\right)$ is a boundedly complete Schauder frame in $P(E) \simeq c_0$. By Proposition 2.9, c_0 is complemented in its bidual, a contradiction.

In order to show the converse, suppose that E does not contain a copy of c_0 and $\left(\{x'_j\}, \{x_j\}\right)$ is not boundedly complete. Then there exists $x'' \in E''$, $x'' \neq 0$, such that $\sum_{j=1}^{\infty} x''\left(x'_j\right) x_j$ is not convergent in E. We can find an absolutely convex 0-neighborhood U_1 and two sequences (p_i), (q_i) of natural numbers such that $p_1 < q_1 < p_2 < q_2 < \ldots$ and $p_{U_1}\left(\sum_{i=p_j}^{q_j} x''(x'_i)\, x_i\right) \geq 1$ for each $j \in \mathbb{N}$. We set $y_j := \sum_{i=p_j}^{q_j} x''(x'_i)\, x_i$ and define $T : \varphi \to E$ by $T\left((a_j)_j\right) := \sum_{j=1}^{\infty} a_j y_j$. We first prove that T is continuous when φ is endowed with the $\|\cdot\|_\infty$- norm. To this end, take U an absolutely convex neighborhood of the origin in E. Since $x'' \neq 0$, $x'' \in E''$, there is an absolutely convex 0-neighborhood U_2 in E such that $p_{U_2^{\circ\circ}}(x'') > 0$. Put $V := U_1 \cap U_2 \cap U$. Clearly $p_{V^{\circ\circ}}(x'') \geq p_{U_2^{\circ\circ}}(x'') > 0$. We can apply Corollary 3.4 to find an absolutely convex closed 0-neighborhood W in E such that $W \subset V$ and

$$p_V\left(\sum_{i=1}^{\infty} d_i x'_i(z)\, x_i\right) \leq p_W(z)\, \|d\|_\infty \tag{3.3}$$

for each $n \in \mathbb{N}$, each $d \in \ell^\infty$ and $z \in E$. For $a = (a_j)_j \in \varphi$, and $s := \max(\operatorname{supp} a)$, the support of a being the set of non-zero coordinates of a, we define $b_i = a_j$ for $p_j \leq i \leq q_j$, and $b_i = 0$ otherwise. We have

$$\sum_{j=1}^{\infty} a_j y_j = \sum_{j=1}^{s} a_j y_j = \sum_{i=p_1}^{q_s} b_i x''\left(x'_i\right) x_i.$$

Given $\varepsilon > 0$, we can apply Lemma 3.8 to find $y \in E$, $p_W(y) \leq p_{W^{\circ\circ}}(x'')$ and

$$\max_{p_1 \leq i \leq q_s} \left|\left(x'' - y\right)\left(x'_i\right)\right| \leq \frac{\varepsilon}{2q_s\, \|a\|_\infty \max\left(p_V(x_i), 1\right)}.$$

This implies

$$\begin{aligned} p_V\left(\sum_{i=p_1}^{q_s} b_i x''\left(x'_i\right) x_i\right) &\leq p_V\left(\sum_{i=p_1}^{q_s} b_i x'_i(y)\, x_i\right) + \sum_{i=p_1}^{q_s} |b_i| \left|\left(x'' - y\right)\left(x'_i\right)\right| p_V(x_i) \leq \\ &\leq p_V\left(\sum_{i=p_1}^{q_s} b_i x'_i(y)\, x_i\right) + \frac{\varepsilon}{2}. \end{aligned}$$

Now, by the estimate (3.3), we obtain

$$p_V\left(\sum_{i=p_1}^{q_s} b_i x_i'(y)\, x_i\right) \leq \left(\max_{p_1\leq i\leq q_s}|b_i|\right) p_W(y) \leq \left(\max_{1\leq j\leq s}|a_j|\right) p_{W^{\circ\circ}}\left(x''\right).$$

Then,

$$p_V\left(\sum_{j=1}^{s} a_j y_j\right) \leq \|a\|_\infty\, p_{W^{\circ\circ}}\left(x''\right) + \frac{\varepsilon}{2}.$$

Since this holds for each $\varepsilon > 0$, we get

$$p_U\left(\sum_{j=1}^{\infty} a_j y_j\right) \leq p_V\left(\sum_{j=1}^{\infty} a_j y_j\right) \leq \|a\|_\infty\, p_{W^{\circ\circ}}\left(x''\right).$$

Thus the operator $T : (\varphi, \|\cdot\|_\infty) \to E$ is continuous. Since E is complete, T admits a unique continuous extension $\widetilde{T} : c_0 \to E$. As by assumption E does not contain c_0, we can apply Theorem 4 in [33, p.208] to conclude that $\left(\widetilde{T}(e_j)\right)_j$ has a convergent subsequence $\left(\widetilde{T}(e_{j_k})\right)_k$. That is, $(y_j)_j$ admits a convergent subsequence $(y_{j_k})_k$. Moreover, since $\widetilde{T} : (c_0, \sigma(c_0, l_1)) \to (E, \sigma(E, E'))$ is also continuous then $\left(\widetilde{T}(e_j)\right)_j = (y_j)_j$ is $\sigma(E, E')$-convergent to 0, hence $(y_{j_k})_k$ must converge to 0 in E. This is a contradiction, since $p_{U_1}(y_j) \geq 1$ for each $j \in \mathbb{N}$. □

Definition 3.10 [30] An (LF)-space $E = \operatorname{ind}_{n\to} E_n$ is called *boundedly retractive* if for every bounded set B in E there exists $m = m(B)$ such that B is contained and bounded in E_m and E_m and E induce the same topology on B.

By [13] an (LF)-space E is boundedly retractive if and only if each bounded subset in E is in fact bounded in some step E_n and for each n there is $m > n$ such that E_m and E induce the same topology on the bounded sets of E_n.

For (LB)-spaces, this is equivalent to the a priori weaker condition that for all $n \in \mathbb{N}$, there exists $m > n$ such that for all $k > m$, E_m and E_k induce the same topology in the unit ball B_n of E_n ([28]). In particular (LB)-spaces with compact linking maps $E_n \hookrightarrow E_{n+1}$ are boundedly retractive. More information about these and related concepts can be seen in [37].

Obviously, each Fréchet space F can be seen as a boundedly retractive (LF)-space, just take $F_n = F$ for all $n \in \mathbb{N}$. In particular 3.12 holds for Fréchet spaces. Every strict (LF)-space is boundedly retractive. In particular, for a open subset Ω in $\mathbb{R}^d$, the space $\mathcal{D}(\Omega)$ is a boundedly retractive (LF)-space. The space $\mathcal{E}'(\Omega)$ and the space HV in Example 1 of Section 4 are boundedly retractive (LB)-spaces.

Rosenthal ℓ_1-theorem was extended to Fréchet spaces by Díaz in [9], showing that every bounded sequence in a Fréchet space has a subsequence that is either weakly Cauchy or equivalent to the unit vectors in ℓ_1.

Proposition 3.11 (*Rosenthal ℓ_1-theorem for (LF)-spaces*). *Let $E = \operatorname{ind}_{n\to} E_n$ be a boundedly retractive (LF)-space. Every bounded sequence in E has a subsequence which is $\sigma(E, E')$-Cauchy or equivalent to the unit vector basis of ℓ_1. In particular, E does not contain a copy of ℓ_1 if and only if every bounded sequence in E has a $\sigma(E, E')$-Cauchy subsequence.*

Proof. Let $(x_j)_j$ be a bounded sequence in E and assume that has no $\sigma(E, E')$-Cauchy subsequence. There is $n_0 \in \mathbb{N}$ such that $(x_j)_j$ is a bounded sequence in E_{n_0}. Now select $m \geq n_0$ such that E_m and E induce the same topology on the bounded sets of E_{n_0}. Since $(x_j)_j$ is bounded in E_m and it has no $\sigma(E_m, E_m')$-Cauchy subsequence, we can apply Rosenthal's ℓ_1-Theorem in the Fréchet space E_m to conclude that there is a subsequence $(x_{j_k})_k$ which is equivalent to the unit vector basis of ℓ_1. That is, there exist c_1 and a continuous seminorm p in E_m such that

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \sup_k p(x_{j_k}) \sum_{k=1}^{\infty} |\alpha_k|,$$

for every $\alpha = (\alpha_k)_k \in \ell_1$.

As the inclusion $E_{n_0} \hookrightarrow E_m$ is continuous, we find a continuous seminorm q in E_{n_0} such that for $x \in E_{n_0}$ one has $p(x) \leq q(x)$. Then, for each $\alpha = (\alpha_k)_k \in \ell_1$,

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq q\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \sup_k q(x_{j_k}) \sum_{k=1}^{\infty} |\alpha_k|.$$

Set $F := \{\sum_{k=1}^{\infty} \alpha_k x_{j_k} : \alpha = (\alpha_k)_k \in \ell_1\} \subset E_{n_0}$. Then p and q restricted to F are equivalent norms, and F endowed with any of them is a Banach space isomorphic to ℓ_1. The spaces E_{n_0} and E_m induce on F the same (Banach) topology. Denote by U_F the closed unit ball of F and by τ_m and τ the topologies of E_m and E, respectively. Then τ and τ_m coincide on U_F, which is an absolutely convex 0-neighbourhood for $\tau_m|_F$. Applying a result of Roelcke [30, 8.1.27] we conclude that τ_m and τ coincide in F; hence, there is a continuous seminorm r on E such that $p(z) \leq r(z)$ for every $z \in F$. This implies, for each $\alpha = (\alpha_k)_k \in \ell_1$,

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq r\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \left(\sup_k r(x_{j_k})\right) \sum_{k=1}^{\infty} |\alpha_k|.$$

Thus, $(x_{j_k})_k$ is equivalent to the unit vectors of ℓ_1 in E and the inclusion $F \hookrightarrow E$ is a topological isomorphism into. Then, E contains an isomorphic copy of ℓ_1. □

We use the notation $\mu(E', E)$ for the topology on E' of uniform convergence on the absolutely convex and $\sigma(E, E')$-compact sets. In the proof of the next result we utilize the fact that a boundedly retractive (LF)-space E does not contain ℓ_1 if and only if every $\mu(E', E)$-null sequence in E' is $\beta(E', E)$-convergent to 0. This was proved by Domański and Drewnowski and by Valdivia independently for Fréchet spaces. The proof can be seen in [2] and the proof for arbitrary boundedly retractive (LF)-spaces follows the same steps as in [2, Theorem 10] but using Proposition 3.11 instead of Rosenthal ℓ_1-theorem for Fréchet spaces.

Theorem 3.12 *Let E be a boundedly retractive (LF)-space. Assume that E admits an unconditional Schauder frame $\left(\{x_j'\}, \{x_j\}\right)$. Then, $\left(\{x_j'\}, \{x_j\}\right)$ is shrinking if and only if E does not contain a copy of ℓ_1.*

Proof. We first assume that $\left(\{x'_j\},\{x_j\}\right)$ is shrinking. Then, by Proposition 2.3, $\left(\{x'_j\},\{x_j\}\right)$ is a Schauder frame for E'_β and, in particular, E'_β is separable. Consequently E contains no subspace isomorphic to ℓ_1.

Conversely, assume that E does not contain a copy of ℓ_1. By Lemma 2.1, $\left(\{x_j\},\{x'_j\}\right)$ is a Schauder frame of $(E', \sigma(E', E))$. We check that, for all $x' \in E'$,

$$\sum_{j=1}^{\infty} x'(x_j)x'_j \tag{3.4}$$

is subseries summable to x' in E'_β. Since for each $x \in E$ the convergence of

$$\sum_{j=1}^{\infty} x'_j(x)x_j \tag{3.5}$$

is unconditional and E is sequentially complete, then (3.5) is subseries summable and we conclude that (3.4) is also $\sigma(E', E)$-subseries summable. We can apply Orlicz-Pettis' Theorem ([18, p. 308]) to obtain that (3.4) is $\mu(E', E)$-unconditionally convergent to x'. Therefore it is $\beta(E', E)$-convergent to x', as E does not contain a copy of ℓ_1. Consequently $\left(\{x'_j\},\{x_j\}\right)$ is shrinking. □

4 Examples

In this section we will present some examples of Schauder frame on locally convex spaces. These Schauder frames are shrinking and boundedly complete since all the spaces involved are Montel spaces.

Example 1. This example was obtained by Taskinen in [34]. Denote by $\mathbb{D}$ the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and for each n let v_n be the weight $v_n(z) := \min\left\{1, |\log(1-|z|)|^{-n}\right\}$. We consider the weighted Banach space of holomorphic functions

$$H^\infty_{v_n} := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|_{v_n} = \sup_{z\in\mathbb{D}} |f(z)|\, v_n(z) < \infty \right\}.$$

Since $v_{n+1} \leq v_n$ then $H^\infty_{v_n} \subset H^\infty_{v_{n+1}}$ continuously and we consider the inductive limit

$$HV = \mathrm{ind}_{n\to\infty} H^\infty_{v_n}.$$

The unit disc $\mathbb{D}$ is decomposed as $\mathbb{D} := \bigcup_j D_j$ with $\overset{\circ}{D}_j \neq \emptyset$ for all $j \in \mathbb{N}$ in such a way that the set of elements of $\mathbb{D}$ belonging to more that one of the D_j's has Lebesgue measure 0. Let us fix, for all $j \in \mathbb{N}$, $\lambda_j \in \overset{\circ}{D}_j$. As proved in [34], we can obtain such a decomposition with the property that

$$S : HV \to HV,\ f \mapsto (Sf)(z) := \sum_{j=1}^{\infty} \frac{m(D_j)\, f(\lambda_j)}{\left(1-\overline{\lambda_j} z\right)^2},$$

is an isomorphism.

Theorem 4.1 [34, Theorem 1] *Under the conditions above, let* $u_j(f) := \left(S^{-1}f\right)(\lambda_j)$ *and* $f_j(z) := \frac{m(D_j)}{(1-\overline{\lambda_j}z)^2}$ *be given. Then* $(\{u_j\},\{f_j\})$ *is a shrinking and boundedly complete Schauder frame for* HV.

Proof. Each $f \in HV$ can be written as

$$f = S\left(S^{-1}(f)\right) = \sum_{j=1}^{\infty} \left(S^{-1}f\right)(\lambda_j) f_j,$$

hence $(\{u_j\},\{f_j\})$ is a Schauder frame in HV. Since HV is a Montel space we can apply Theorem 2.9 to conclude that the Schauder frame is shrinking. □

As pointed in [34, p. 330], the coefficients in the series expansion above are not unique.

Example 2. Let K be a compact subset of $\mathbb{R}^p$ that coincides with the closure of its interior, i.e. $K = \overline{\overset{\circ}{K}}$. Let $C^\infty(K)$ be the space of all complex-valued functions $f \in C^\infty(\overset{\circ}{K})$ uniformly continuous in $\overset{\circ}{K}$ together with all partial derivatives. The Fréchet space topology in $C^\infty(K)$ is defined by the norms:

$$q_n(f) := \sup\left\{\left|f^{(\alpha)}(x)\right| : x \in K,\, |\alpha| \leq n\right\}, n \in \mathbb{N}_0.$$

A continuous and linear extension operator is a continuous and linear operator $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$ such that $T(f)|_K = f$. Not every compact set admits a continuous and linear extension operator but every convex compact set does. Further information can be found in [15].

Theorem 4.2 *Let* $K \subset \mathbb{R}^p$ *be a compact set which is the closure of its interior. The following conditions are equivalent:*

(1) *There exists a continuous and linear extension operator* $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$.

(2) *There are sequences* $(\lambda_j)_j \subset \mathbb{R}^p$ *and* $(u_j)_j \in C^\infty(K)'$ *such that* $\left(\{u_j\},\left\{e^{2\pi i x\cdot\lambda_j}\right\}\right)$ *is an unconditional Schauder frame for* $C^\infty(K)$.

Proof. (1) ⇒ (2). We consider $M > 0$ such that $K \subset [-M,M]^p$ and choose $\phi \in D([-2M,2M]^p)$ such that $\phi(x) = 1$ for all x in a neighborhood of $[-M,M]^p$. For every $f \in C^\infty(K)$ we define $Hf = \phi(T(f)) \in D(]-2M,2M[^p)$. Then $H : C^\infty(K) \to D(]-2M,2M[^p)$ is a continuous and linear map and $Hf|_K = f$. After extending Hf as a periodic C^∞ function in $\mathbb{R}^p$ we get

$$Hf(x) := \sum_{j\in\mathbb{Z}^p} a_j e^{2\pi i x\cdot\lambda_j}, \text{ where } \lambda_j = \frac{1}{4M}(j_1,\ldots,j_p)$$

and $a_k = a_k(Hf)$ are the Fourier coefficients of Hf. By [21], $\sup_{j\in\mathbb{Z}^p}|a_j|\,|j|^m < \infty$ for every m, which implies that the series $f = \sum_{j\in\mathbb{Z}^p} a_j e^{2\pi i x\cdot\lambda_j}$ converges absolutely in $C^\infty(K)$. Each a_k, being a Fourier coefficient of Hf, depends linearly and continuously on f. Then $\left(u_j(\cdot), e^{2\pi i x\cdot\lambda_j}\right)_{j\in\mathbb{Z}^p}$ is a Schauder frame for $C^\infty(K)$, with $u_j \in C^\infty(K)'$ defined by $u_j(f) = a_j(Hf)$.

(2) $\Rightarrow$ (1). For every $f \in C^\infty(K)$ we have

$$f(x) = \sum_{j=1}^{\infty} u_j(f) e^{2\pi i x \cdot \lambda_j} \text{ in } C^\infty(K)$$

and

$$\sum_{j=1}^{\infty} u_j(f) b_j e^{2\pi i x \cdot \lambda_j}$$

converges in $C^\infty(K)$ for every $(b_j) \in \ell_\infty$. After differentiation, we obtain that the series

$$\sum_{j=1}^{\infty} u_j(f) 2\pi b_j \lambda_j^\alpha e^{2\pi i x \cdot \lambda_j}$$

converges in $C^\infty(K)$ for every $\alpha \in \mathbb{N}_0^p$ and $(b_j) \in \ell_\infty$. In particular, this series converges for a fixed x_0 in the interior of K, from where it follows

$$\sum_{j=1}^{\infty} \left| u_j(f) 2\pi \lambda_j^\alpha \right| < +\infty$$

for every $\alpha \in \mathbb{N}_0^p$. Consequently $T(f)(x) := \sum_{j=1}^{\infty} u_j(f) e^{2\pi i x \cdot \lambda_j}$ defines a C^∞ function in $\mathbb{R}^p$ and we obtain that $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$ is a linear extension operator. The continuity of T follows from the Banach-Steinhaus theorem, as $T(f)$ is the pointwise limit of $T_n(f) := \sum_{j=1}^{n} u_j(f) f_j$, $f_j(x) := e^{2\pi i x \cdot \lambda_j}$. □

Assume that condition (1) in the previous theorem holds. Then, for a fixed $j_0 \in \mathbb{Z}^p$ we can choose ϕ such that the j_0-th Fourier coefficient of $\phi T(e^{2\pi i \lambda^{j_0} \cdot})$ is not equal to 1. According to the comment after Corollary 1.8, we may remove one of the exponentials in the Schauder frame above and still obtain a Schauder frame.

Choosing $\psi \neq \phi$ in the proof above, we find a different sequence $(v_j) \in C^\infty(K)'$ such that $\left(\{v_j\}, \left\{e^{2\pi i x \cdot \lambda_j}\right\}\right)$ is an unconditional Schauder frame for $C^\infty(K)$. In fact, according to [21], no system of exponentials can be a basis in $C^\infty([0,1])$.

Example 3. We give a Schauder frame of the Schwartz space $\mathcal{S}(\mathbb{R}^p)$ of rapidly decreasing functions. It is inspired by the work of Pilipovic, Stoeva and Teofanov [31], although their Theorem 4.2 cannot be directly applied to conclude that one gets a Schauder frame. Let $a, b > 0$, and $\Lambda = a\mathbb{Z}^p \times b\mathbb{Z}^p$ be given. For $z = (x, \xi) \in \mathbb{R}^{2p}$ and $f \in L^2(\mathbb{R}^p)$ we put $\pi(z)f(t) = e^{2\pi i \xi t} f(t - x)$. Let us assume that $g \in \mathcal{S}(\mathbb{R}^p)$ and $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a Gabor frame in $L^2(\mathbb{R}^p)$. As proved by Janssen (see [17, Corollary 11.2.6]) the dual window is also a function $h \in \mathcal{S}(\mathbb{R}^p)$ and every $f \in L^2(\mathbb{R}^p)$ can be written as

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \, \pi(\lambda)h. \tag{4.1}$$

For every $\lambda \in \Lambda$ we consider $u_\lambda \in \mathcal{S}'(\mathbb{R}^p)$ defined by $u_\lambda(f) = \langle f, \pi(\lambda)g \rangle$.

Proposition 4.3 *$((u_\lambda)_{\lambda \in \Lambda}, (\pi(\lambda)h)_{\lambda \in \Lambda})$ is an unconditional Schauder frame for $\mathcal{S}(\mathbb{R}^p)$.*

Proof. According to [17, Corollary 11.2.6], the topology of $\mathcal{S}(\mathbb{R}^p)$ can be described by the sequence of seminorms

$$q_n(f) := \sup_{z\in\mathbb{R}^{2p}} |\langle f, \pi(z)g\rangle|\, v_n(z),\ n \in \mathbb{N},$$

where $v_n(z) = (1 + |z|)^n$. So, we only need to check that, for every $n \in \mathbb{N}$,

$$\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|\, q_n\left(\pi(\lambda)h\right) < \infty. \tag{4.2}$$

To this end, we fix $N > n$ large enough. Since

$$|\langle \pi(\lambda)h, \pi(z)g\rangle| \leq |\langle h, \pi(z-\lambda)g\rangle| \leq q_N(h)v_N(z-\lambda)^{-1}$$

and v_n is submultiplicative we obtain that (4.2) is dominated by

$$q_N(h)q_N(f)\sum_{\lambda\in\Lambda} \left(v_N(\lambda)\right)^{-1} v_n(\lambda) < \infty$$

and the proof is finished. □

This example is closely related to the fact that $\{\pi(\lambda)g :\ \lambda \in \Lambda\}$ is a Gabor frame for each modulation space defined in terms of a polynomially moderate weight; see for instance [17, Corollary 12.2.6].

Acknowledgement. This research was partially supported by MEC and FEDER Project MTM2010-15200. The authors are thankful to D. Freeman for pointing out several relevant references to them.

References

[1] K. Beanland, D. Freeman and R. Liu. Upper and lower estimates for Schauder frames and atomic decompositions. arXiv:1202.2492v1

[2] J. Bonet and M. Lindström. Convergent sequences in duals of Fréchet spaces. In *Functional analysis (Essen, 1991)*, volume 150 of *Lecture Notes in Pure and Appl. Math.*, pages 391–404.

[3] J. Bonet, M. Lindström and M. Valdivia. Two theorems of Josefson-Nissenzweig type for Fréchet spaces. *Proc. Amer. Math. Soc.*, 117(2):363–364, 1993.

[4] D. Carando and S. Lassalle. Duality, reflexivity and atomic decompositions in Banach spaces. *Studia Math.*, 191(1):67–80, 2009.

[5] D. Carando, S. Lassalle, and P. Schmidberg. The reconstruction formula for Banach frames and duality. *J. Approx. Theory*, 163(5):640–651, 2011.

[6] P. Casazza, O. Christensen, and D. T. Stoeva. Frame expansions in separable Banach spaces. *J. Math. Anal. Appl.*, 307(2):710–723, 2005.

[7] P.G. Casazza, S.J. Dilworth, E. Odell, Th.Schlumprecht and A. Zsk. Coefficient quantization for frames in Banach spaces. *J. Math. Anal. Appl.*, 348(1):66–86, 2008.

[8] P. G. Casazza, D. Han, and D. R. Larson. Frames for Banach spaces. In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, volume 247 of *Contemp. Math.*, pages 149–182. Amer. Math. Soc., Providence, RI, 1999.

[9] J. C. Díaz. Montel subspaces in the countable projective limits of $L^P(\mu)$-spaces. *Canad. Math. Bull.*, 32(2):169–176, 1989.

[10] H. G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. *Rocky Mountain J. Math.*, 19(1):113–125, 1989. Constructive Function Theory—86 Conference (Edmonton, AB, 1986).

[11] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions I. *J. Funct. Anal.*, 86(2):307–340, 1989.

[12] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions II. *Monatsh. Math.*, 108(2-3):129–148, 1989.

[13] C. Fernández. Regularity conditions on (LF)-spaces. *Arch. Math. (Basel)*, 54(4):380–383, 1990.

[14] D. Freeman, E. Odell, Th. Schlumprecht and A. Zsák. Unconditional structures of translates for $L_p(\mathbb{R}^d)$. arXiv:1209.4619v1.

[15] L. Frerick. Extension operators for spaces of infinite differentiable Whitney jets. *J. Reine Angew. Math.*, 602:123–154, 2007.

[16] H. G. Garnir, M. De Wilde, and J. Schmets. *Analyse fonctionnelle. Théorie constructive des espaces linéaires à semi-normes. Tome I: Théorie générale.* Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 36. Birkhäuser Verlag, Basel, 1968.

[17] Karlheinz Gröchenig. *Foundations of time-frequency analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.

[18] H. Jarchow. *Locally convex spaces.* B. G. Teubner, Stuttgart, 1981. Mathematische Leitfäden. [Mathematical Textbooks].

[19] J. L. Kelley and Isaac Namioka. *Linear topological spaces.* With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, Kennan T. Smith. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J., 1963.

[20] Y. F. Korobeĭnik. Representative systems. *Uspekhi Mat. Nauk*, 36(1(217)):73–126, 248, 1981.

[21] Y. F. Korobeĭnik. On absolutely representing systems in spaces of infinitely differentiable functions. *Studia Math.*, 139(2):175–188, 2000.

[22] Y. F. Korobeĭnik and S. N. Melikhov. A linear continuous right inverse operator for the representation operator, and applications to convolution operators. *Siberian Math. J.*, 34(1):59–72, 1993.

[23] G. Köthe. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.

[24] R. Liu. On shrinking and boundedly complete Schauder frames of Banach spaces. *J. Math. Anal. Appl.*, 365(1):385–398, 2010.

[25] C. Matyszczyk. Approximation of analytic and continuous mappings by polynomials in Fréchet spaces. *Studia Math.*, 60(3):223–238, 1977.

[26] C. W. McArthur and J. R. Retherford. Some applications of an inequality in locally convex spaces. *Trans. Amer. Math. Soc.*, 137:115–123, 1969.

[27] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[28] H. Neus. Über die Regularitätsbegriffe induktiver lokalkonvexer Sequenzen. *Manuscripta Math.*, 25(2):135–145, 1978.

[29] A. Pełczyński. Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis. *Studia Math.*, 40:239–243, 1971.

[30] P. Pérez and J. Bonet. *Barrelled locally convex spaces*, volume 131 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 113.

[31] S. Pilipović, D. Stoeva, and N. Teofanov. Frames for Fréchet spaces. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, (32):69–84, 2007.

[32] S. Pilipović and D. T. Stoeva. Series expansions in Fréchet spaces and their duals, construction of Fréchet frames. *J. Approx. Theory*, 163(11):1729–1747, 2011.

[33] B. Qingying and L. Ronglu. Locally convex spaces containing no copy of c_0. *J. Math. Anal. Appl.*, 172(2):205–211, 1993.

[34] J. Taskinen. Atomic decomposition of a weighted inductive limit. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 97(2):325–337, 2003.

[35] J. Taskinen. A Fréchet-Schwartz space with basis having a complemented subspace without basis. *Proc. Amer. Math. Soc.*, 113(1):151–155, 1991.

[36] D. Vogt. An example of a nuclear Fréchet space without the bounded approximation property. *Math. Z.*, 182(2):265–267, 1983.

[37] J. Wengenroth. *Derived functors in functional analysis*, volume 1810 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.