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Fixed points for cyclic orbital generalized contractions on complete metric spaces

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Abstract

We prove a fixed point theorem for cyclic orbital generalized contractions on complete metric spaces from which we deduce, among other results, generalized cyclic versions of the celebrated Boyd and Wong fixed point theorem, and Matkowski fixed point theorem. This is done by adapting to the cyclic framework a condition of Meir-Keeler type discussed by J. Jachymski [Equivalent conditions and the Meir-Keeler type theorems, J. Math. Anal. Appl. 194 (1995), 293-303]. Our results generalize some theorems of Kirk, Srinavasan and Veeramani, and of Karpagam and Agrawal.

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1 Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of non-negative real numbers and the set of positive integer numbers, respectively.

Let A and B be non-empty subsets of a (non-empty) set X. A map $T : A \cup B \to A \cup B$ is said to be a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

In the last years several authors have studied the existence of fixed points and best proximity points for cyclic maps on metric spaces and Banach spaces (see e.g. [1]-[12]).

One of the remarkable generalizations of the Banach Contraction Principle was given by Kirk, Srinavasan and Veeramani proved in [1] in the frame of cyclic mappings as follows.

Theorem 1.1. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic map. If there exists $k \in (0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x \in A$ and $y \in B$, then T has a unique fixed point z. Moreover $z \in A \cap B$.*

Recently, Karpagam and Agrawal introduced in [2] the notion of a cyclic orbital contraction.

Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic orbital contraction if for some $x \in A$ there exists a $k_x \in (0, 1)$ such that

$$d(T^{2n}x, Ty) \le k_x d(T^{2n-1}x, y),$$

for all $n \in \mathbb{N}$ and $y \in A$.

Then, they gave the following generalization of Theorem 1.1.

Theorem 1.2 ([2]). *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then T has a unique fixed point z. Moreover $z \in A \cap B$.*

On the other hand, and previously, Meir and Keeler proved in [13] their well-known fixed point theorem.

Theorem 1.3 ([13]). *Let (X, d) be a complete metric space and $T : X \to X$ a map. Suppose that the following condition is satisfied:*

(MK) *For each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in X$,*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) < \varepsilon.$$

Then T has a unique fixed point $z \in X$, and for all $x \in X$, $T^n x \to z$.

Later on, Jachymski presented in [14] several modifications of condition (MK), obtaining in this way an interesting variant of Theorem 1.3, which allows to subsume simultaneously the celebrated, and very general (see [15]), Boyd and Wong fixed point theorem, and Matkowski fixed point theorem.

Let us recall some of the conditions introduced in [14].

Given a metric space (X,d) and a map $T : X \to X$, put

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(Tx,y)]\},$$

for all $x, y \in X$.

Now consider the following conditions:

(J1) *For any $x, y \in X$,*

$$m\,(x,y) > 0 \quad \textit{implies} \quad d(Tx,Ty) < m(x,y).$$

(J2) *For each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in X$,*

$$\varepsilon < m(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) \leq \varepsilon.$$

(MK_m) *For each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in X$,*

$$\varepsilon \leq m(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) < \varepsilon.$$

Then, Jachymski proved the following.

Theorem 1.4 ([14]). *Let (X,d) be a complete metric space and $T : X \to X$ a continuous map. If conditions* (J1) *and* (J2) *are satisfied, then T has a unique fixed point $z \in X$ and for all $x \in X$, $T^n x \to z$.*

Remark 1.5. Condition (MK_m) implies (J1) and (J2), but the converse does not hold ([14, Proposition 1 and Example 1]). On the other hand, Boyd and Wong's fixed point theorem and Matkowski's fixed point theorem are consequences of Theorem 1.4 (see [14, Corollary of Theorem 2 and Remark 1]). Moreover, continuity of f cannot be dropped in Theorem 1.4: In fact, Theorems 1.3 and 1.4 are independent (see [14, Example 1]).

In this paper we study cyclic (orbital) contractions on complete metric spaces with a similar approach to the one given by Jachymski. Thus we obtain, among other results, generalized cyclic (orbital) versions of Boyd and Wong's fixed point theorem and Matkowski's fixed point theorem, respectively. Some illustrative examples are also presented. Our results extend Theorem 1.4 to the cyclic framework and generalize Theorems 1.1 and 1.2. They also improve [3, Theorem 2.1] for the case of two non-empty closed subsets.

2 The results

If A and B are two non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic map, we define

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(Tx, y)]\},$$

for all $x, y \in A \cup B$.

Remark 2.1. Note that $M(x, y) = M(y, x)$ for all $x, y \in A \cup B$.

Now consider the following conditions:

($\mathrm{J1}_C$) *For any* $x \in A$, $y \in B$,

$$M(x, y) > 0 \quad \textit{implies} \quad d(Tx, Ty) < M(x, y).$$

($\mathrm{J2}_C$) *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, for any* $x \in A, y \in B$,

$$\varepsilon < M(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) \leq \varepsilon.$$

(MK_C) *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, for any* $x \in A, y \in B$,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon.$$

Remark 2.2. It is clear that condition (MK_C) implies ($\mathrm{J1}_C$) and ($\mathrm{J2}_C$). Example 1 of [14] shows that the converse does not hold in general.

The following result is the key of our study. It is formulated in a cyclic orbital sense, and thus conditions ($\mathrm{J1}_C$) and ($\mathrm{J2}_C$) are suitably generalized to this context.

Lemma 2.3. *Let A and B be non-empty subsets of a metric space (X,d) and $T : A\cup B \to A\cup B$ be a cyclic map. Suppose that there is $x_0 \in A$ such that $T^n x_0 \neq T^{n+1}x_0$ for all $n \in \mathbb{N}\cup\{0\}$, and for which the following condition is satisfied:*

($\mathrm{J1}_{CO}$) *For any $y \in A$ and $n \in \mathbb{N}$,*

$$M\left(T^{2n-1}x_0, y\right) > 0 \quad \textit{implies} \quad d(T^{2n}x_0, Ty) < M(T^{2n-1}x_0, y).$$

Then

$$M(T^n x_0, T^{n+1}x_0) = d(T^n x_0, T^{n+1}x_0),$$

for all $n \in \mathbb{N}$, and hence $\{d(T^n x_0, T^{n+1}x_0)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $\mathbb{R}^+$.

Moreover, if the following condition is satisfied:

($\mathrm{J2}_{CO}$) *For each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $y \in A$ and $n \in \mathbb{N}$,*

$$\varepsilon < M(T^{2n-1}x_0, y) < \varepsilon + \delta \quad \textit{implies} \quad d(T^{2n}x_0, Ty) \leq \varepsilon,$$

then $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).

Proof. Let $x_0 \in A$ with $T^n x_0 \neq T^{n+1}x_0$ *for all* $n \in \mathbb{N}\cup\{0\}$. Define $x_n = T^n x_0$ for each $n \in \mathbb{N}\cup\{0\}$. Then $x_{2n} \in A$, $x_{2n-1} \in B$, and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

We have

$$\begin{aligned}
M(x_{2n-1}, x_{2n}) &= M(x_{2n}, x_{2n-1}) \\
&= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \frac{1}{2}d(x_{2n-1}, x_{2n+1})\} \\
&\leq \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \\
&\quad \frac{1}{2}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]\},
\end{aligned}$$

for all $n \in \mathbb{N}$.

Since $M(x_{2n-1}, x_{2n}) > 0$, it follows from ($\mathrm{J1}_{CO}$) that $d(x_{2n}, x_{2n+1}) < M(x_{2n-1}, x_{2n})$, so that

$$M(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n}),$$

and thus $d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}$.

Similarly, we show that

$$M(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1}) \quad \text{and} \quad d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}),$$

for all $n \in \mathbb{N}$.

Therefore, the sequence $\{(d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $\mathbb{R}^+$, so there exists $a \in \mathbb{R}^+$ such that $a = \lim_{n\to\infty} d(x_n, x_{n+1})$.

Now assume that condition (J2$_{CO}$) is also satisfied. In order to show that then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d), we first prove that $a = 0$.

Indeed, if $a > 0$ then there is $k \in \mathbb{N}$ such that $a < d(x_n, x_{n+1}) < a + \delta$ for all $n \geq k$, where $\delta > 0$ is the constant depending on a for which condition (J2$_{CO}$) is satisfied. Without loss of generality we may assume that k is odd. Since $M(x_k, x_{k+1}) = d(x_k, x_{k+1})$, it follows from (J2$_{CO}$) that $d(x_{k+1}, x_{k+2}) \leq a$, a contradiction. We conclude that $a = 0$.

Finally, choose an arbitrary $\varepsilon > 0$. Let $\delta > 0$ be the constant depending on ε for which condition (J2$_{CO}$) is satisfied, and assume, without loss of generality, that $\delta < \varepsilon$. Then, there is $k_0 \in \mathbb{N}$ such that $d(x_k, x_{k+1}) < \delta/2$ and $d(x_k, x_{k+2}) < \delta/2$ for all $k \geq k_0$.

Let $k > k_0$ with k even. Then $k = 2n$ for some $n \in \mathbb{N}$. We shall show, by induction, that $d(x_{2n}, x_{2n+2j-1}) \leq \varepsilon$ for all $j \in \mathbb{N}$.

Indeed, first note that $d(x_{2n}, x_{2n+1}) = d(x_k, x_{k+1}) < \delta/2 < \varepsilon$.

Now assume that $d(x_{2n}, x_{2n+2j-1}) \leq \varepsilon$ for some $j \in \mathbb{N}$. Observe that

$$M(x_{2n-1}, x_{2n+2j}) < \varepsilon + \delta,$$

because

$$\begin{aligned} d(x_{2n-1}, x_{2n+2j}) &\leq d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+2j-1}) + d(x_{2n+2j-1}, x_{2n+2j}) \\ &< \frac{\delta}{2} + \varepsilon + \frac{\delta}{2}, \end{aligned}$$

$d(x_{2n-1}, x_{2n}) < \delta/2$, $d(x_{2n+2j}, x_{2n+2j+1}) < \delta/2$, and

$$\begin{aligned} & \frac{1}{2}[d(x_{2n-1}, x_{2n+2j+1}) + d(x_{2n}, x_{2n+2j}] \\ \leq\ & \frac{1}{2}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+2j-1}) + d(x_{2n+2j-1}, x_{2n+2j+1}) \\ & + d(x_{2n}, x_{2n+2j-1}) + d(x_{2n+2j-1}, x_{2n+2j})] \\ <\ & \frac{1}{2}[\frac{\delta}{2} + \varepsilon + \frac{\delta}{2} + \varepsilon + \frac{\delta}{2}]. \end{aligned}$$

Hence, if $M(x_{2n-1}, x_{2n+2j}) > \varepsilon$, it follows from condition (J2$_{CO}$) that $d(x_{2n}, x_{2n+2j+1}) \leq \varepsilon$, and if $M(x_{2n-1}, x_{2n+2j}) \leq \varepsilon$, then the inequality $d(x_{2n}, x_{2n+2j+1}) \leq \varepsilon$ follows directly from condition (J1$_{CO}$).

If $k > k_0$ with k odd, then $k = 2n - 1$ for some $n \in \mathbb{N}$ and, thus, a similar argument to the one given for the case that k is even shows that $d(x_{2n-1}, x_{2n+2j}) \leq \varepsilon$ for all $j \in \mathbb{N}$.

From the above facts we immediately deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). $\square$

Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ a cyclic map. If there exist $x_0 \in A$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, and

$$d(T^{2n}x_0, Ty) \leq \varphi(M(T^{2n-1}x_0, y)),$$

for all $y \in A$ and $n \in \mathbb{N}$, then T is called a cyclic orbital generalized contraction for x_0 and φ (a COG-contraction for x_0 and φ, in the sequel).

We point out that the inclusion of condition "$\varphi(t) < t$ for all $t > 0$" in the definition of a COG-contraction was accurately suggested by the referee. In fact, if such a condition is omitted then for A and B bounded sets, any map $T : A \cup B \to A \cup B$ satisfies $d(T^{2n}x_0, Ty) \leq \varphi(M(T^{2n-1}x_0, y))$, when $\varphi(t) := \mathrm{diam}(A \cup B)$ for all $t \in \mathbb{R}^+$.

Theorem 2.4. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a COG-contraction for an $x_0 \in A$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ satisfies the following condition:*

(φ_ε) *For each $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\varepsilon < t < \varepsilon + \delta \quad \textit{implies} \quad \varphi(t) \leq \varepsilon,$$

then T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$.

Proof. Suppose that $T^k x_0 = T^{k+1}x_0$ for some $k \in \mathbb{N} \cup \{0\}$. Then $T^k x_0$ is a fixed point of T, and thus $T^k x_0 \in A \cap B$ because T is cyclic. Obviously $T^n x_0 \to T^k x_0$.

If $T^n x_0 \neq T^{n+1}x_0$ for all $n \in \mathbb{N} \cup \{0\}$, it immediately follows from GOC-contractivity of T for x_0 and φ, that condition (J1$_{CO}$) is satisfied for the sequence $(T^n x_0)_{n\in\mathbb{N}}$. Moreover, ($\varphi_\varepsilon$) clearly implies condition (J2$_{CO}$), so by Lemma 2.3, $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since $A \cup B$ is closed in the complete metric space (X, d), it follows that $(A \cup B, d\mid_{A\cup B})$ is a complete metric space, so there is $z \in A \cup B$ such that $T^n x_0 \to z$. Since T is cyclic we deduce that $z \in A \cap B$. It remains to show that $z = Tz$. Assume the contrary. Then, it easily follows that $M(T^{2n-1}x_0, z) = d(z, Tz)$ eventually. Hence

$$\begin{aligned} d(z,Tz) &\leq d(z,T^{2n}x_0) + d(T^{2n}x_0, Tz) \leq d(z,T^{2n}x_0) + \varphi(M(T^{2n-1}x_0, z)) \\ &= d(z,T^{2n}x_0) + \varphi(d(z,Tz)), \end{aligned}$$

eventually. Taking limit as $n \to \infty$, we deduce that $d(z,Tz) \leq \varphi(d(z,Tz))$, a contradiction. We conclude that z is a fixed point of T. $\square$

Corollary 2.5. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a COG-contraction for an $x_0 \in A$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ is upper semicontinuous from the right on $\mathbb{R}^+$, then T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$.*

Proof. It suffices to note (see, for instance, [14, Remark 1]) that φ satisfies condition (φ_ε). Theorem 2.4 concludes the proof. $\square$

Corollary 2.6. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a COG-contraction for an $x_0 \in A$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ satisfies the following condition:*

(Ma) *φ is non-decreasing on $\mathbb{R}^+$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$,*

then T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$.

Proof. Jachymski showed in [14, Remark 1] that every function φ satisfying condition (Ma) also satisfies condition (φ_ε). Theorem 2.4 concludes the proof. $\square$

The following examples show that under the conditions of Corollary 2.5 or Corollary 2.6 (and hence under the conditions of Theorem 2.4), the fixed point of the cyclic (orbital) map T is not necessarily unique.

Example 2.7. Let $A = \{0, 2\} \cup \{2^{-2n} : n \in \mathbb{N} \cup \{0\}\}$, $B = \{0, 2\} \cup \{2^{-(2n+1)} : n \in \mathbb{N} \cup \{0\}\}$, and $X = A \cup B$. Define $T : A \cup B \to A \cup B$ by $T0 = 0$, $T2 = 2$ and $T2^{-n} = 2^{-(n+1)}$ for all $n \in \mathbb{N} \cup \{0\}$. Then T is a cyclic map on $A \cup B$.

Now construct a function $d : X \times X \to \mathbb{R}^+$ as follows:

$d(x,x) = 0$ for all $x \in X$, $d(x,y) = \max\{x,y\}$ for all $x,y \in X \backslash \{2\}$ with $x \neq y$,

$d(2,0) = d(0,2) = 2$,

$d(2, 2^{-2n}) = d(2^{-2n}, 2) = 2 - 2^{-2n}$ for all $n \in \mathbb{N} \cup \{0\}$, and

$d(2, 2^{-(2n-1)}) = d(2^{-(2n-1)}, 2) = 2 - 2^{-(2n+1)}$ for all $n \in \mathbb{N}$.

It is easy to check that d is a complete metric on X (observe, in particular, that every non-eventually constant Cauchy sequence in (X,d) converges to 0) and that A and B are closed subsets of X.

Now let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\varphi(t) = t/2$ for $t \in [0, 2 - 2^{-3})$, $\varphi(t) = 2 - 2^{-2n}$ for $t \in [2 - 2^{-(2n+1)}, 2 - 2^{-(2n+3)})$, $n \in \mathbb{N}$, and $\varphi(t) = 1$ for $t \geq 2$.

Note that φ is (upper semi)continuous from the right on $\mathbb{R}^+$.

Next we show that T is COG-contraction for $x_0 = 1$ and the above $\dot{\varphi}$.

Indeed, it is clear that $\varphi(t) < t$ for all $t > 0$. Moreover, for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(T^{2n}1, T0) &= d(2^{-2n}, 0) = 2^{-2n} = \varphi(2^{-(2n-1)}) \\ &= \varphi(d(T^{2n-1}1, 0)) = \varphi(M(T^{2n-1}1, 0)), \end{aligned}$$

and

$$\begin{aligned} d(T^{2n}1, T2) &= d(2^{-2n}, 2) = 2 - 2^{-2n} = \varphi(2 - 2^{-(2n+1)}) \\ &= \varphi(d(T^{2n-1}1, 2)) = \varphi(M(T^{2n-1}1, 2)), \end{aligned}$$

whereas that for each $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned} d(T^{2n}1, T2^{-2k}) &= \max\{2^{-2n}, 2^{-(2k+1)}\} \\ &= \frac{1}{2}\max\{2^{-(2n-1)}, 2^{-2k}\} = \varphi(\max\{2^{-(2n-1)}, 2^{-2k}\}) \\ &= \varphi(d(T^{2n-1}1, 2^{-2k})) = \varphi(M(T^{2n-1}1, 2^{-2k})). \end{aligned}$$

Consequently, the conditions of Corollary 2.5 are satisfied. However T has two fixed points.

Example 2.8. Consider the sets A, B and the map $T : A \cup B \to A \cup B$ of the preceding example.

Construct a function $d : X \times X \to \mathbb{R}^+$ as follows:

$d(x,x) = 0$ for all $x \in X$, $d(x,y) = \max\{x,y\}$ for all $x,y \in X \backslash \{2\}$ with $x \neq y$,

$d(2,0) = d(0,2) = d(0, 2^{-2n}) = d(2^{-2n}, 0) = 2,\quad$ and

$d(2, 2^{-(2n+1)}) = d(2^{-(2n+1)}, 2) = 2 + 2^{-(2n+1)}$ for all $n \in \mathbb{N} \cup \{0\}$.

As in the preceding example, d is a complete metric on X with A and B closed subsets of X.

We show that T is COG-contraction for $x_0 = 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\varphi(t) = t/2$ for $t \in [0,2]$ and $\varphi(t) = 2$ for $t > 2$ (note that, in fact, φ is non-decreasing and satisfies $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}^+$.)

Indeed, for each $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, we obtain, as in Example 2.7,

$$d(T^{2n}1, T0) = \varphi(d(T^{2n-1}1, 0)),$$

and

$$d(T^{2n}1, T2^{-2k}) = \varphi(d(T^{2n-1}1, 2^{-2k})).$$

Moreover

$$d(T^{2n}1, T2) = d(2^{-2n}, 2) = 2 = \varphi(2 + 2^{-(2n-1)}) = \varphi(d(T^{2n-1}1, 2)),$$

for all $n \in \mathbb{N}$.

Hence, the conditions of Corollary 2.6 are satisfied. However T has two fixed points.

Observe that the function φ of Example 2.7 is not non-decreasing (in fact, $\varphi(2) < \varphi(t)$ whenever $t \in [2 - 2^{-3}, 2)$), whereas the function φ of Example 2.8 is not upper semicontinuous from the right at $t = 2$. These facts are not casual as the following result shows.

Corollary 2.9. *Let A and B be non-empty closed subsets of a complete metric space (X,d) and $T : A \cup B \to A \cup B$ be a COG-contraction for an $x_0 \in A$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ is non-decreasing and upper semicontinuous from the right on $\mathbb{R}^+$, then T has a unique fixed point z. Moreover $z \in A \cap B$ and $T^n x_0 \to z$.*

Proof. By Corollary 2.5, T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$. Suppose that there exists $u \in A \cup B$ with $u = Tu$, and $u \neq z$. Since T is cyclic, $u \in A \cap B$.

Now, from the facts that

$\lim_{n\to\infty} d(T^n x_0, u) = d(z,u)\quad$ and $\quad\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0$,

it immediately follows that

$\lim_{n\to\infty} M(T^{2n-1}x_0, u) = d(z, u)$.

We consider two cases.

Case 1. There is a subsequence $(T^{2n_k-1}x_0)_{k\in\mathbb{N}}$ of $(T^{2n-1}x_0)_{n\in\mathbb{N}}$ such that $M(T^{2n_k-1}x_0, u) \leq d(z, u)$ for all $k \in \mathbb{N}$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $d(z, u) < M(T^{2n-1}x_0, u)$ for all $n \geq n_0$.

In Case 1, we obtain

$$d(T^{2n_k}x_0, Tu) \leq \varphi(M(T^{2n_k-1}x_0, u)) \leq \varphi(d(z, u)),$$

for all $k \in \mathbb{N}$. So taking limits when $k \to \infty$, we deduce that $d(z, u) \leq \varphi(d(z, u))$, which contradicts that $u \neq z$.

In Case 2, from the upper semicontinuity from the right of φ it follows that

$$\lim_{n\to\infty} \sup \varphi(M(T^{2n-1}x_0, u)) \leq \varphi(d(z, u)).$$

Since $d(T^{2n}x_0, Tu) \leq \varphi(M(T^{2n-1}x_0, u))$, we deduce, taking limit when $n \to \infty$, that $d(z, u) \leq \varphi(d(z, u))$, which contradicts that $u \neq z$.

We conclude that $u = z$, and thus z is the unique fixed point of T. $\square$

As an immediate consequence of Corollary 2.9 we deduce the following improvement of Theorem 1.2.

Corollary 2.10. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a cyclic map. If there exist $x_0 \in A$ and $k_{x_0} \in [0, 1)$ such that*

$$d(T^{2n}x_0, Ty) \leq k_{x_0} M(T^{2n-1}x_0, y),$$

for all $n \in \mathbb{N}$ and $y \in A$, then T has a unique fixed point z. Moreover $z \in A \cap B$ and $T^n x_0 \to z$.

Next we give an easy example where we cannot apply Theorem 1.2, but the conditions of Corollary 2.10 are satisfied.

Example 2.11. Let $X = \{0, 1, 2\}$ and let d be the complete metric on X given by $d(x, x) = 0$ for all $x \in X$, $d(1, 2) = d(2, 1) = 2$, and $d(x, y) = 1$ otherwise. Put $A = \{0, 1\}$ and $B = \{1, 2\}$. It is obvious that A and B are closed subsets of (X, d). Moreover $X = A \cup B$.

Now define $T : A \cup B \to A \cup B$ by $T0 = 0$, $T1 = 2$ and $T2 = 1$. Then $T(A) = B$ and $T(B) = A$, so T is a cyclic map on $A \cup B$.

Since $T^n 0 = 0$, $T^{2n-1}1 = 2$, $T^{2n}1 = 1$, $T^{2n-1}2$ and $T^{2n}2 = 2$, we immediately deduce that, for each $n \in \mathbb{N}$,

$d(T^{2n}0, T1) = d(T^{2n-1}0, 1) = 1, \quad d(T^{2n}1, T0) = d(T^{2n-1}1, 0) = 1,$

and similarly,

$d(T^{2n}0, T2) = d(T^{2n-1}0, 2) = 1, \quad d(T^{2n}2, T0) = d(T^{2n-1}2, 0) = 1.$

Therefore T is not a cyclic orbital contraction (for A and B), and hence we cannot apply Theorem 1.2 to this example.

However, we have $\quad d(T^{2n}0, T0) = 0, \quad$ and

$$d(T^{2n}0, T1) = 1 = \frac{1}{2}d(1,2) = \frac{1}{2}d(1, T1) = \frac{1}{2}M(T^{2n-1}0, 1),$$

for all $n \in \mathbb{N}$. Hence, the conditions of Corollary 2.10 are satisfied for $x_0 = 0 \in A$ and $k_{x_0} = 1/2$.

We conclude the paper by applying the above results to obtain fixed point theorems for cyclic generalized contractions.

Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ a cyclic map. If there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, and

$$d(Tx, Ty) \leq \varphi(M(x, y)),$$

for all $x \in A$, $y \in B$, then T is called a cyclic generalized contraction for φ (a CG-contraction for φ, in the sequel)

Theorem 2.12. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a CG-contraction for a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ satisfies condition (φ_ε) of Theorem 2.4, then T has a unique fixed point z. Moreover $z \in A \cap B$ and $T^n x_0 \to z$ for all $x_0 \in A \cup B$.*

Proof. Let, for instance, $x_0 \in A$. By Theorem 2.4, T has a fixed point $z \in A \cap B$ such that $T^n x_0 \to z$. Suppose that there is $u \in A \cup B$ such that $u = Tu$. Then

$$d(z, u) = d(Tz, Tu) \leq \varphi(M(u, z)) = \varphi(d(u, z)),$$

so $u = z$. This concludes the proof. $\square$

Corollary 2.13. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a CG-contraction for a* *function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ is upper semicontinuous from the right on $\mathbb{R}^+$, then T has a unique fixed point z. Moreover $z \in A \cap B$ and $T^n x_0 \to z$ for all $x_0 \in A \cup B$.*

Corollary 2.14. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ a CG-contraction for a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. If φ satisfies condition (Ma) of Corollary 2.6, then T has a unique fixed point z. Moreover $z \in A \cap B$ and $T^n x_0 \to z$ for all $x_0 \in A \cup B$.*

Note that Corollaries 2.13 and 2.14 provide generalized cyclic counterparts of the Boyd and Wong fixed point theorem [16, Theorem 1], and the Matkowski fixed point theorem [17, Theorem 1.2], respectively.

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References

[1] W.A. Kirk, P.S. Srinavasan and P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory 4 (2003), 79-89.

[2] S. Karpagam and S. Agrawal, Best proximity points theorems for cyclic Meir-Keeler contraction maps, Nonlinear Anal. 74 (2011), 1040-1046.

[3] M. Păcurar and I.A. Rus, Fixed point theory for cyclic φ-contractions, Nonlinear Anal. 72 (2010), 1181-1187.

[4] G. Petruşel, Cyclic representations and periodic points, Studia Univ. Babeş-Bolyai Math. 50 (2005), 107–112.

[5] I.A. Rus, Cyclic representations and fixed points, Ann. T. Popoviciu. Sem. Funct. Eq. Approx. Convexity 3 (2005), 171–178.

[6] A.A. Eldred and P. Veeramani, Convergence and existence for best proximity points, J. Math. Anal. Appl. 323 (2006), 1001–1006.

[7] C. Di Bari, T. Suzuki and C. Vetro, Best proximity points for cyclic Meir-Keeler contractions, Nonlinear Anal. 69 (2008), 3790–3794.

[8] J. Anuradha and P. Veeramani, Proximal pointwise contraction, Topology Appl. 156 (2009), 2942–2948.

[9] M.A. Al-Thafai and N. Shahzad, Convergence and existence for best proximity points, Nonlinear Anal. 70 (2009), 3665-3671.

[10] Sh. Rezapour, M. Derafshpour and N. Shahzad, Best proximity point of cyclic φ-contractions in ordered metric spaces, Topol. Methods Nonlinear Anal. 37 (2011) 193–202.

[11] E. Karapinar, Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett. 24 (2011) 822-825.

[12] G.S. R. Kosuru and P. Veeramani, Cyclic contractions and best proximity pair theorems, arXiv:1012.1434v2 [math.FA] 29 May 2011, 14 pages.

[13] A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326–329.

[14] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, J. Math. Anal. Appl. 194 (1995), 293-303.

[15] J. Jachymski, Equivalence of some contractivity properties over metrical structures, Proc. Amer. Math. Soc. 125 (1997), 2327–2335.

[16] D.W. Boyd and J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458-464.

[17] J. Matkowski, Integrable solutions of functional equations, Dissertationes Math. (Rozprawy Mat.) 127 (1975), 1-68.