Norm-attaining weighted composition operators on weighted Banach spaces of analytic functions

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Abstract

We investigate weighted composition operators that attain their norm on weighted Banach spaces of holomorphic functions on the unit disc of type H^∞. Applications for composition operators on weighted Bloch spaces are given.

1 Introduction and Notation

Let ϕ and ψ be analytic maps on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that $\phi(\mathbb{D}) \subset \mathbb{D}$ and ψ does not coincide with the function zero. These maps induce on the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$, via composition and multiplication, a linear *weighted composition operator* $C_{\phi,\psi}$ defined by $(C_{\phi,\psi})(f) = \psi(f \circ \phi)$. Operators of this type have been studied on various spaces of analytic functions. For a discussion of composition operators on classical spaces of analytic functions we refer the reader to the excellent monographs [8] and [16].

Recall that a bounded linear operator $T : X \to Y$ between Banach spaces X and Y is said to attain its norm on X if there exists $f \in X$ with norm 1 such that $\|T\| = \|Tf\|$. We say that a function f with these properties is an *extremal function* for the norm of T. James, see e.g. [9], proved that a Banach space X is reflexive if and only if every compact operator on X is norm-attaining.

Starting with the paper of Hammond [10] in the setting of the Hardy space H^2, norm-attaining composition operators have been studied on various spaces and by several authors. In particular, Martín [14] characterized norm-attaining composition operators acting on the classical Bloch space $\mathcal{B}$ as well as on the little Bloch space $\mathcal{B}_0$. In this note, inspired by her work, we extend her results to the setting of weighted composition operators acting on weighted Banach spaces of holomorphic functions defined below.

A *weight* v on $\mathbb{D}$ is a strictly positive continuous function on $\mathbb{D}$ which is radial, i.e. $v(z) = v(|z|), z \in \mathbb{D}$, $v(r)$ is decreasing on $[0, 1[$ and satisfies $\lim_{r\to 1} v(r) = 0$. We associate with a weight v the weighted Banach spaces of holomorphic functions

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 := \{f \in H_v^\infty;\ \lim_{|z|\to 1-} v(z)|f(z)| = 0\},$$

both endowed with norm $\|.\|_v$. The spaces H_v^∞ and H_v^0 are not reflexive; see [6, 13]. Spaces of this type and composition operators defined on them have been studied thoroughly. See e.g. [1, 3, 4, 5, 7, 13, 15] and the references therein.

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Our main results are Theorems 2.3 and 3.1. Theorem 2.3 asserts that every bounded weighted composition operator on H_v^∞ is norm attaining, whereas Theorem 3.1 characterizes the bounded weighted composition operators on H_v^0 which are norm attaining. Examples are provided. Our last section includes consequences for norm attaining composition operators on weighted Bloch spaces.

Here are examples of weights $v(z)$ on $\mathbb{D}$:

(1) The polynomial weights $v(z) = (1 - |z|^2)^\alpha, \ \alpha > 0$.

(2) The exponential weights $v(z) = \exp(-\frac{1}{(1-|z|)^\alpha}), \ \alpha > 0$.

(3) The logarithmic weights $v(z) = (\log \frac{e}{1-r})^{-\alpha}, \ \alpha > 0$.

The so called *associated weight* (see [2]) is an important tool to study operators on weighted Banach spaces of analytic functions. For a weight v, the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := (\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\})^{-1} = (\|\delta_z\|_v)^{-1},\ z \in \mathbb{D},$$

where δ_z denotes the point evaluation of z. By [2, Properties 1.2] we know that the associated weight is continuous, radial, $\tilde{v} \geq v > 0$ and that for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $||f_z||_v = 1$, such that $|f_z(z)|\tilde{v}(z) = 1$. It is also shown in [2, Observation 1.12] that $H_{\tilde{v}}^\infty$ coincides isometrically with H_v^∞. In particular, $|| \cdot ||_v = || \cdot ||_{\tilde{v}}$. Under the present assumptions on the weights, it is also true that $H_{\tilde{v}}^0$ coincides isometrically with H_v^0; see [4].

The norm of a bounded weighted composition operator $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$, is given by

$$\|C_{\phi,\psi}\|_{H_v^\infty \to H_w^\infty} = \sup_{z \in D} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))}.$$

See [4, 7, 15]. Moreover, by [7, 15], the essential norm (i.e. the distance to the compact operators) of a bounded composition operator $C_{\phi,\psi} : H_v^0 \to H_w^0$ is given by

$$\|C_{\phi,\psi}\|_{e,H_v^0 \to H_w^0} = \limsup_{|z| \to 1-} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))}.$$

2 Norm attaining weighted composition operators on H_v^∞

The following useful Lemma is due to Horokawa, Izuchi and Zheng [11].

Lemma 2.1 *([11]) Let $(z_m)_m \subset \mathbb{D}$ be a sequence such that $|z_m| \to 1$, when $m \to \infty$. Then there is a subsequence $(z_n)_n$ of $(z_m)_m$ and there is a sequence $(g_k)_k$ in the disc algebra $A(\mathbb{D})$ such that*

$$(i) \quad \sup_{z \in \overline{\mathbb{D}}} \sum_{k=1}^{\infty} |g_k(z)| \leq 1,$$

and

$$(ii) \quad |g_n(z_n)| > 1 - \left(\frac{1}{2}\right)^n \quad \text{for all } n \in \mathbb{N}.$$

Proof. The statement follows from the proof of Theorem 3.1 in [11]. In the notation of [11], $g_k := c_k f^{m_k} g_{n_k}$ and the subsequences are selected by (3.3)-(3.6) in that paper. □

Observe that in the notation of Lemma 2.1, we have $\sum_{k=1, k\neq n}^\infty |g_k(z_n)| \leq \frac{1}{2^n}$ for each $n \in \mathbb{N}$.

Lemma 2.2 *Let v be a weight on $\mathbb{D}$. If $(z_m)_m \subset \mathbb{D}$ be a sequence such that $|z_m| \to 1$, when $m \to \infty$, then there is a subsequence $(z_n)_n$ of $(z_m)_m$ and there is $g \in H_v^\infty$, $||g||_v \leq 1$, such that $|g(z_n)|\tilde{v}(z_n) \to 1$, when $n \to \infty$.*

Proof. We apply Lemma 2.1 to construct functions g_k in the disc algebra $A(\mathbb{D})$ satisfying conditions (i) and (ii). Now, for every k, find $f_k \in H_v^\infty$, $||f_k||_v = 1$, such that $f_k(z_k)\tilde{v}(z_k) = 1$ and put $g(z) := \sum_{k=1}^{\infty} g_k(z) f_k(z)$. It is easy to see that $g \in H(\mathbb{D})$ and $|g(z)|v(z) \leq 1$ for all $z \in \mathbb{D}$, so $g \in H_v^\infty$ and $||g||_v \leq 1$. Moreover, for all n,

$$|g(z_n)|\tilde{v}(z_n) = |\sum_{k=1}^{\infty} g_k(z_n) f_k(z_n)\tilde{v}(z_n)| = |g_n(z_n) f_n(z_n)\tilde{v}(z_n) + \sum_{k=1, k\neq n}^{\infty} g_k(z_n) f_k(z_n)\tilde{v}(z_n)|$$

$$\geq |g_n(z_n)| - \sum_{k=1, k\neq n}^{\infty} |g_k(z_n)| \geq 1 - \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n.$$

Hence $1 - \frac{1}{2^{n-1}} \leq |g(z_n)|\tilde{v}(z_n) \leq 1$ for each $n \in \mathbb{N}$. This implies the conclusion. □

Theorem 2.3 *(a) Every bounded composition operator $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is norm attaining.*

(b) A function $f \in H_v^\infty$ is extremal for $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ if and only if there is a sequence $(z_n)_n$ in $\mathbb{D}$ such that $\lim_{n\to\infty} \tilde{v}(\phi(z_n))|f(\phi(z_n))| = 1$ and $\lim_{n\to\infty} \frac{|\psi(z_n)|w(z_n)}{\tilde{v}(\phi(z_n))} = ||C_{\phi,\psi}||$.

Proof. (a) Since the norm of a bounded weighted composition operator $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$, is given by $\|C_{\phi,\psi}\| = \sup_{z\in D} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))}$, we can find a sequence $(z_n) \subset \mathbb{D}$ and $b \in \overline{\mathbb{D}}$ with $\phi(z_m) \to b$ as $m \to \infty$ such that

$$\lim_m \frac{|\psi(z_m)|w(z_m)}{\tilde{v}(\phi(z_m))} = ||C_{\phi,\psi}||.$$

We distinguish two cases.

Case 1: $|b| = 1$. We apply Lemma 2.2 to $(\phi(z_m))_m$ to find a subsequence $(z_n)_n$ of $(z_m)_m$ and $g \in H_v^\infty$, $||g||_v \leq 1$, such that $\lim_{n\to\infty} g(\phi(z_n))\tilde{v}(\phi(z_n)) = 1$.

Case 2: $|b| < 1$. There exists a $g \in H_v^\infty$, $||g||_v = 1$, with $g(b)\tilde{v}(b) = 1$. Therefore

$$\lim_{n\to\infty} g(\phi(z_n))\tilde{v}(\phi(z_n)) = g(b)\tilde{v}(b) = 1.$$

We have, in both cases,

$$||C_{\phi,\psi} g|| = \sup_{z\in\mathbb{D}} |\psi(z)||g(\phi(z))|w(z) \geq \lim_n \frac{|\psi(z_n)||g(\phi(z_n))|\tilde{v}(\phi(z_n))w(z_n)}{\tilde{v}(\phi(z_n))} =$$

$$= \lim_n \frac{|\psi(z_n)|w(z_n)}{\tilde{v}(\phi(z_n))} = ||C_{\phi,\psi}||.$$

This implies that $g \in H_v^\infty$, $||g||_v \leq 1$, is an extremal function for $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$.

(b) The proof of part (a) shows that if $f \in H_v^\infty$, $||f||_v \leq 1$, satisfies that there is a sequence $(z_n)_n$ in $\mathbb{D}$ such that $\lim_{n\to\infty} \tilde{v}(\phi(z_n))|f(\phi(z_n))| = 1$ and $\lim_n \frac{|\psi(z_n)|w(z_n)}{\tilde{v}(\phi(z_n))} = ||C_{\phi,\psi}||$, then f is an extremal function for $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$.

Conversely, assume that $f \in H_v^\infty$, $||f||_v \leq 1$, satisfies

$$||C_{\phi,\psi}|| = ||C_{\phi,\psi} f||_w = \sup_{z\in\mathbb{D}} w(z)|\psi(z)||f(\phi(z))|.$$

Select a sequence $(z_n)_n$ in $\mathbb{D}$ such that, for each $n \in \mathbb{N}$,

$$\left(1 - \frac{1}{n}\right) ||C_{\phi,\psi}|| < w(z_n)|\psi(z_n)||f(\phi(z_n))| \leq ||C_{\phi,\psi}||.$$

Hence, for each $n \in \mathbb{N}$,

$$\left(1 - \frac{1}{n}\right) \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))} < w(z_n)|\psi(z_n)||f(\phi(z_n))|.$$

This implies $1 - \frac{1}{n} < \tilde{v}(\phi(z_n))|f(\phi(z_n))| \leq 1$; the last inequality because $||f||_v = 1$. We get $\lim_{n\to\infty} \tilde{v}(\phi(z_n))|f(\phi(z_n))| = 1$. By the inequality above for $||C_{\phi,\psi}||$, we conclude

$$\lim_n \frac{|\psi(z_n)|w(z_n)}{\tilde{v}(\phi(z_n))} = ||C_{\phi,\psi}||.$$

□

3 Norm attaining weighted composition operators on H_v^0

Theorem 3.1 *A bounded composition operator $C_{\phi,\psi} : H_v^0 \to H_w^0$ is norm attaining if and only if there are $b \in \mathbb{D}$ and $(z_n)_n$ in $\mathbb{D}$ with $\lim_{n\to\infty} \phi(z_n) = b$ such that*

$$||C_{\phi,\psi}|| = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))},$$

and there is $f \in H_v^0, ||f||_v = 1$, (which is an extremal function) satisfying $\tilde{v}(b)f(b) = 1$.

Proof. Assume first that $C_{\phi,\psi} : H_v^0 \to H_w^0$ is norm attaining. There is $f \in H_v^0, ||f||_v = 1$, such that $||C_{\phi,\psi}|| = ||C_{\phi,\psi}f||_w$. Since $f \in H_v^0 = H_{\tilde{v}}^0$, there is $R \in]0,1[$ such that $\tilde{v}(\zeta)|f(\zeta)| < 1/2$ for each $\zeta \in \mathbb{D}, |\zeta| > R$. If $z \in \mathbb{D}$ satisfies $|\phi(z)| > R$, then

$$w(z)|\psi(z)||f(\phi(z))| \leq \frac{1}{2}\frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} \leq \frac{1}{2}||C_{\phi,\psi}||.$$

Set $A := \{z \in \mathbb{D} \mid |\phi(z)| \leq R\}$. We have

$$||C_{\phi,\psi}|| = ||C_{\phi,\psi}f||_w = \sup_{z\in A} w(z)|\psi(z)||f(\phi(z))|.$$

For each $n \in \mathbb{N}$ we select $z_n \in A$ with

$$\left(1 - \frac{1}{n}\right)||C_{\phi,\psi}|| < w(z_n)|\psi(z_n)||f(\phi(z_n))| \leq ||C_{\phi,\psi}f||_w = ||C_{\phi,\psi}||.$$

Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \phi(z_n) = b$, with $|b| \leq R < 1$, hence $b \in \mathbb{D}$. Moreover, for each $n \in \mathbb{N}$,

$$\left(1 - \frac{1}{n}\right)\frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))} < w(z_n)|\psi(z_n)||f(\phi(z_n))|.$$

Thus, for each $n \in \mathbb{N}$,

$$1 - \frac{1}{n} < \tilde{v}(\phi(z_n))|f(\phi(z_n))| \leq ||f||_v = 1.$$

This implies $\tilde{v}(b)f(b) = 1 = \lim_{n\to\infty} \tilde{v}(\phi(z_n))|f(\phi(z_n))|$. From where it follows

$$||C_{\phi,\psi}|| = \lim_{n\to\infty} w(z_n)|\psi(z_n)||f(\phi(z_n))| =$$

$$= \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))}\tilde{v}(\phi(z_n))|f(\phi(z_n))| = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))}.$$

The proof of this implication is complete.

To prove the other implication, select $b \in \mathbb{D}$, $(z_n)_n$ and $f \in H_v^0, ||f||_v = 1$, as in the assumption. We have

$$||C_{\phi,\psi}|| = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))} = \lim_{n\to\infty} w(z_n)|\psi(z_n)||f(\phi(z_n))|\frac{\tilde{v}(b)}{\tilde{v}(\phi(z_n))}\frac{|f(b)|}{f(\phi(z_n))|} =$$

$$\lim_{n\to\infty} w(z_n)|\psi(z_n)||f(\phi(z_n))| \leq ||C_{\phi,\psi}f||_w.$$

This implies that $f \in H_v^0$ is an extremal function for the operator $C_{\phi,\psi} : H_v^0 \to H_w^0$. □

Corollary 3.2 *Let v be a weight such that for each $b \in \mathbb{D}$ there is $f \in H_v^0, ||f||_v = 1$, such that $\tilde{v}(b)f(b) = 1$. If $C_{\phi,\psi} : H_v^0 \to H_w^0$ is a bounded operator such that $||C_{\phi,\psi}||_e < ||C_{\phi,\psi}||$ (for example if $C_{\phi,\psi}$ is compact and non-zero), then $C_{\phi,\psi} : H_v^0 \to H_w^0$ is norm attaining.*

Proof. According to [15, Theorem 2.2], as $||C_{\phi,\psi}||_e < ||C_{\phi,\psi}||$,

$$\limsup_{|z|\to 1-} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} < \sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))}.$$

Select $(z_m)_m$ in $\mathbb{D}$ such that $||C_{\phi,\psi}|| = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))}$. By the inequality above, there is a subsequence $(z_n)_n$ of $(z_m)_m$ such that $z_n \to a \in \mathbb{D}$ as $n \to \infty$. Put $b = \phi(a)$. By assumption, there is $f \in H_v^0, ||f||_v = 1$, such that $\tilde{v}(b)f(b) = 1$. We can apply Theorem 3.1 to conclude that $f \in H_v^0$ is an extremal function for $C_{\phi,\psi} : H_v^0 \to H_w^0$. □

We discuss now examples of weights satisfying the assumption of Corollary 3.2:

$$(**) \quad \forall b \in \mathbb{D} \;\; \exists f \in H_v^0, \; ||f||_v = 1, \;\; \text{such that} \;\; \tilde{v}(b)f(b) = 1.$$

First recall the following notation: For $a \in \mathbb{D}$, we denote by $\sigma_a(z) := \frac{a-z}{1-\bar{a}z}$. We have $\sigma_a'(z) = \frac{|a|^2-1}{(1-\bar{a}z)^2}$ and $|\sigma_a'(z)| = \frac{1-|\sigma_a(z)|^2}{1-|z|^2}$ for each $z, a \in \mathbb{D}$. See e.g. [17].

Lemma 3.3 *The polynomial weights $v(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, satisfy the condition $(**)$.*

Proof. It is known, see [2], that the polynomial weights satisfy $v = \tilde{v}$. Given $b \in \mathbb{D}$, consider the function

$$f_b(z) := (\sigma_b'(z))^\alpha = \left(\frac{|b|^2-1}{(1-\bar{b}z)^2}\right)^\alpha.$$

Since $f_b \in H^\infty$, we have $f_b \in H_v^0$. Moreover

$$||f_b||_v = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\sigma_b'(z)|^\alpha = \sup_{z\in\mathbb{D}} (1-|\sigma_a(z)|^2)^\alpha = 1$$

Clearly $v(b)|f_b(b)| = 1$. □

Corollary 3.4 *Let v be a polynomial weight $v(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, and let w be an arbitrary weight. A bounded composition operator $C_{\phi,\psi} : H_v^0 \to H_w^0$ is norm attaining if and only if there are $b \in \mathbb{D}$ and $(z_n)_n$ in $\mathbb{D}$ with $lim_{n\to\infty}\phi(z_n) = b$ such that*

$$||C_{\phi,\psi}|| = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))}.$$

Proof. This is a direct consequence of Theorem 3.1 and Lemma 3.3. □

Our next proposition permits us to exhibit more examples of weights satisfying condition $(**)$.

Proposition 3.5 *Let v be a weight such that $v = \tilde{v}$ satisfying condition $(**)$. If w is another weight such that $w = \tilde{w}$, then $u := vw$ satisfies also the condition.*

Proof. First of all it is easy to conclude from [2, Properties 1.2 (iv)] that $\tilde{u}(z) \le \tilde{v}(z)\tilde{w}(z)$ for each $z \in \mathbb{D}$. This implies $u \le \tilde{u} \le \tilde{v}\tilde{w} = vw = u$. Thus $u = \tilde{u} = \tilde{v}\tilde{w}$. Now, to check condition $(**)$ for u, fix $b \in \mathbb{D}$. We find $g \in H_w^\infty, ||g||_w = 1$ with $g(b) = 1/\tilde{w}(b) = 1/w(b)$. Since v satisfies condition $(**)$, there is $f \in H_v^0, ||f||_v = 1$ such that $f(b) = 1/\tilde{v}(b) = 1/v(b)$. Setting $h = fg$, we have $||h||_u \le 1$, $\tilde{u}(b)h(b) = u(b)h(b) = 1$ and $h \in H_u^0$, as $\lim_{|z|\to 1-} u(z)|h(z)| \le \lim_{|z|\to 1-} v(z)|f(z)| = 0$. □

As a consequence of Lemma 3.3 and Proposition 3.5, the following weights satisfy condition $(**)$:

(1) $u(z) = (1-|z|^2)^\alpha \exp(-\frac{1}{(1-|z|)^\beta}),\ \alpha, b > 0.$

(2) $u(z) = (1-|z|^2)^\alpha (\log \frac{e}{1-r})^{-\beta},\ \alpha, b > 0.$

(3) $u(z) = 1-|z|$. Just take $v(z) = 1-|z|^2$, $w(z) = 1/(1+|z|)$ and apply [2, Corollary 1.6].

Example 3.6 Let v be a polynomial weight $v(z) = (1-|z|^2)^\alpha,\ \alpha > 0$. Consider the holomorphic self map $\phi(z) = \sigma_a(z)$ on $\mathbb{D}$ for some $a \in \mathbb{D}$. The composition operator $C_\phi : H_v^0 \to H_v^0$ is not norm attaining. Indeed, for each $z \in \mathbb{D}$, we have

$$\frac{v(z)}{v(\phi(z))} = \frac{(1-|z|^2)^\alpha}{(1-|\sigma_a(z)|^2)^\alpha} = \frac{1}{|(\sigma_a'(z))^\alpha|} = \frac{|1-\overline{a}z|^{2\alpha}}{(1-|a|^2)^\alpha}.$$

Accordingly, a sequence $(z_n)_n$ in $\mathbb{D}$ with $||C_\phi|| = \sup_{z\in\mathbb{D}} \frac{v(z)}{v(\phi(z))} = \lim_{n\to\infty} \frac{v(z_n)}{v(\phi(z_n))}$ must satisfy $|z_n| \to 1$ as $n \to \infty$. We can apply Corollary 3.4 for $\psi(z) = 1$ to conclude that $C_\phi : H_v^0 \to H_v^0$ is not norm attaining.

Examples 3 and 4 in Martín [14] (see also [12]) have the following consequences in our setting that are relevant in connection with Corollary 3.2.

Examples 3.7 (1) Let v be a polynomial weight $v(z) = (1-|z|^2)^\alpha,\ \alpha > 0$. Consider the holomorphic maps $\phi(z) = (z+1)/2$ and $\psi(z) = 1/2$. The weighted composition operator $C_{\phi,\psi} : H_v^0 \to H_v^0$ is not norm attaining by Corollary 3.4 and $||C_{\phi,\psi}||_e = ||C_{\phi,\psi}||$.

(2) Let v be the weight $v(z) = 1-|z|^2$. Consider the lens map $\phi(z) = (\sigma(z)^\alpha - 1)/(\sigma(z)^\alpha + 1), 0 < \alpha < 1$, with $\sigma(z) = (1+z)/(1-z)$, and $\psi(z) = \phi'(z)$. The weighted composition operator $C_{\phi,\psi} : H_v^0 \to H_v^0$ is norm attaining by Corollary 3.4, but $||C_{\phi,\psi}||_e = ||C_{\phi,\psi}||$. Thus the converse of Corollary 3.2 does not hold in general.

4 Consequences for composition operators on weighted Bloch spaces

Let v be a weight. The weighted Bloch space is defined by

$$\mathcal{B}_v = \{f \in H(\mathbb{D}) : f(0) = 0,\ \|f\|_{\mathcal{B}_v} = \sup_{z\in D} v(z)|f'(z)| < \infty\},$$

and the little Bloch space

$$\mathcal{B}_{v,0} = \{f \in \mathcal{B} : \lim_{|z|\to 1-} v(z)|f'(z)| = 0\}.$$

They are Banach spaces endowed with the norm $||\cdot||_{\mathcal{B}_v}$.

The classical Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$ correspond to the weight $v(z) := 1-|z|^2$. Among the many references on these spaces, we mention Zhu [17], for example.

Define the bounded operators $S : \mathcal{B}_v \to H_v^\infty,\ S(h) = h'$ and $S^{-1} : H_v^\infty \to \mathcal{B}_v,\ (S^{-1}h)(z) = \int_0^z h(\xi)d\xi$. Then $SS^{-1} = id_{H_v^\infty}$, $S^{-1}S = id_{\mathcal{B}_v}$ and S, S^{-1} are isometric onto maps. These operators induce isometries between H_v^0 and $\mathcal{B}_{v,0}$.

Consider a composition operator $C_\phi : \mathcal{B}_v \to \mathcal{B}_v$. We can use Theorem 2.3 to find an extremal function $g \in H_v^\infty$ for the norm of $C_{\phi,\phi'} : H_v^\infty \to H_v^\infty$, so that $||C_\phi|| = ||C_{\phi,\phi'}|| = ||SC_\phi S^{-1}|| = ||(SC_\phi S^{-1})g||_v$. Now using that S is an isometry, it follows that $h := S^{-1}(g) \in \mathcal{B}_v$ is an extremal function for the norm of $C_\phi : \mathcal{B}_v \to \mathcal{B}_v$. Accordingly we get the following extension of Martín [14, Theorem 6].

Corollary 4.1 *Every composition operator $C_\phi : \mathcal{B}_v \to \mathcal{B}_v$ is norm-attaining.*

Proceeding similarly for $C_\phi : \mathcal{B}_{v,0} \to \mathcal{B}_{v,0}$ we obtain the following consequence of Corollary 3.4 that is an extension of Martín [14, Theorem 1].

Corollary 4.2 *Let v be a polynomial weight $v(z) = (1-|z|^2)^\alpha$, $\alpha > 0$. A composition operator $C_\phi : \mathcal{B}_{v,0} \to \mathcal{B}_{v,0}$ is norm attaining if and only if there are $b \in \mathbb{D}$ and $(z_n)_n$ in $\mathbb{D}$ with $lim_{n\to\infty}\phi(z_n) = b$ such that*

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))} = \lim_{n\to\infty} \frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))}.$$

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