Beltrán Meneu, MJ.; Bonet Solves, JA.; Fernández Rosell, MC. (2013). Classical operators on weighted Banach spaces of entire functions. Proceedings of the American Mathematical Society. 141(12):4293-4303. doi:S0002-9939-2013-11685-0.

CLASSICAL OPERATORS ON WEIGHTED BANACH SPACES OF ENTIRE FUNCTIONS

MARÍA J. BELTRÁN, JOSÉ BONET AND CARMEN FERNÁNDEZ

(Communicated by)

Abstract. We study the operators of differentiation and of integration and the Hardy operator on weighted Banach spaces of entire functions. We estimate the norm of the operators, study the spectrum, and analyze when they are surjective, power bounded, hypercyclic and (uniformly) mean ergodic.

1. Introduction and notation

The aim of this paper is to study the following three operators on weighted spaces of entire functions defined by means of sup norms: the differentiation operator $Df(z) = f'(z)$, the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ and the Hardy operator $Hf(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta$, $z \in \mathbb{C}$. The continuity of the differentiation and the integration operators between weighted Banach spaces of holomorphic functions has been studied by Harutyunyan and Lusky [13]. The continuity of these two operators on weighted Banach spaces of entire functions associated to a weight v is determined by the growth or decline of $v(r)e^{\alpha r}$ for some $\alpha > 0$ in an interval $[r_0, \infty[$. The surjectivity and the spectrum of the differentiation operator on weighted Banach spaces of entire functions were studied by Atzmon and Brive [2]. Although there is a huge literature on the Hardy operator on different function spaces (see e.g. [1]), it seems that it has not yet been studied in the context considered in our paper. Bonet [8] (see also [9]) investigated when the operator of differentiation is hypercyclic or chaotic on weighted Banach spaces of entire functions. It is our purpose to continue this work by analyzing other operators as well as other properties related to the dynamical behaviour of the operator, like being power bounded or mean ergodic; thus complementing also work by Bonet and Ricker [10] about mean ergodic multiplication operators.

A continuous and linear operator T from a Banach space E into itself is called *power bounded* if the sequence of its iterates $(T^n)_{n\in\mathbb{N}}$ is equicontinuous. By the uniform boundedness principle this happens if and only if the orbit $(x, Tx, T^2x, \dots)$ is bounded for every $x \in E$. The operator $T \in L(E)$ is called *hypercyclic* if there is $x \in E$ with a dense orbit. We refer the reader to the recent books by Bayart and Matheron [3] and by Grosse-Erdmann and Peris [12] for linear dynamics. The

---

2010 *Mathematics Subject Classification.* Primary: 47B38, Secondary: 47A16; 46E15.

The authors were partially supported by MEC and FEDER Project MTM2010-15200, by GV Project Prometeo/2008/101, by grant F.P.U. AP2008-00604, and Conselleria d'Educació de la GVA, Project GV/2010/040.

operator T is said to be *mean ergodic* if the limits

$$Px := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n x, \; x \in E$$

exist in E. A power bounded operator T is mean ergodic precisely when $E = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Im}(I-T)}$. Here I stands for the identity on E. If the convergence is in the operator norm, the operator is called *uniformly mean ergodic.* Clearly, if T is mean ergodic, then $\lim_{n\to\infty} ||T^n x||/n = 0$ for each $x \in E$, and if it is uniformly mean ergodic, $\lim_{n\to\infty} ||T^n||/n = 0$. If this condition is satisfied, the operator T is uniformly mean ergodic if and only if $\mathrm{Im}(I-T)$ is closed [16]. An operator T is said to be quasi-compact if T^n is compact for some $n \in N$. Quasi-compact operators share some properties of compact operators, in particular the spectrum $\sigma(T)$ of a quasi-compact operator T reduces to its eigenvalues and $\{0\}$. Our notation for functional analysis and operator theory is standard. We refer the reader e.g. to [21] and [23]. For ergodic theory of operators on Banach spaces, see [15].

In what follows $\mathcal{H}(\mathbb{C})$ and $\mathcal{P}$ will denote the spaces of entire functions and of polynomials, respectively. The space $\mathcal{H}(\mathbb{C})$ will be endowed with the compact open topology τ_{co}. It is easy to see that the three operators, D, J and H are continuous on $\mathcal{H}(\mathbb{C})$.

Throughout the paper, a *weight* v is a continuous function $v : [0, +\infty[\to]0, +\infty[$ which is non-increasing on $[0,\infty[$ and satisfies $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. For such a weight, the *weighted Banach spaces of entire functions* are defined by

$$H_v^\infty(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) \mid ||f||_v := \sup_{z\in\mathbb{C}} v(|z|)|f(z)| < +\infty\},$$
$$H_v^0(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) \mid \lim_{|z|\to\infty} v(|z|)|f(z)| = 0\},$$

endowed with the sup norm $\|\cdot\|_v$. Clearly $H_v^0(\mathbb{C})$ is a closed subspace of $H_v^\infty(\mathbb{C})$ which contains the polynomials. Both are Banach spaces and the closed unit ball of $H_v^\infty(\mathbb{C})$ is τ_{co}-compact. The polynomials are contained and dense in $H_v^0(\mathbb{C})$ but the monomials are not in general a Schauder basis [19]. The Cesàro means of the Taylor polynomials satisfy $||C_n f||_v \le ||f||_v$ for each $f \in H_v^\infty(\mathbb{C})$ and the sequence $(C_n f)_n$ is $||\,||_v$-convergent to f when $f \in H_v^0(\mathbb{C})$ [4]. Clearly, changing the value of v on a compact interval does not change the spaces and gives an equivalent norm. By [6, Ex 2.2] the bidual of $H_v^0(\mathbb{C})$ is isometrically isomorphic to $H_v^\infty(\mathbb{C})$. When $v(r) = e^{-\alpha r}$ $(\alpha > 0)$ we write $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ for the weighted Banach spaces and denote their norm by $||\,||_\alpha$. The spaces $H_{a,\alpha}^\infty(\mathbb{C})$ and $H_{a,\alpha}^0(\mathbb{C})$, $(\alpha > 0, a \in \mathbb{R})$ correspond to the following weights: $v(r) = e^{-\alpha}, r \in [0,1[$, $v(r) = r^a e^{-\alpha r}, r \ge 1$, if $a < 0$ and $v(r) = (a/\alpha)^a e^{-a}, r \in [0, a/\alpha[$, $v(r) = r^a e^{-\alpha r}, r \ge a/\alpha$, if $a > 0$. In this case the norm will be denoted by $||\,||_{a,\alpha}$. Spaces of this type appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e.g. [4, 5, 7, 11, 19, 20] and the references therein.

Our main Theorem summarizes our results for the spaces $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$, where $v(r) = e^{-\alpha r}, \alpha > 0$.

Theorem 1.1. *(1) The differentiation operator D satisfies $||D^n||_\alpha = n!\left(\frac{e\alpha}{n}\right)^n$ for each $n \in \mathbb{N}$, hence it is power bounded if and only if $\alpha < 1$. The spectrum of D is the closed disc of radius α. It is uniformly mean ergodic on $H_\alpha^\infty(\mathbb{C})$ and $H_\alpha^0(\mathbb{C})$ if $\alpha < 1$, not mean ergodic if $\alpha > 1$, and it is not mean ergodic on $H_1^\infty(\mathbb{C})$ and not uniformly mean ergodic on $H_1^0(\mathbb{C})$.*

(2) The integration operator J is never hypercyclic on $H^0_\alpha(\mathbb{C})$ and it satisfies $||J^n||_\alpha = 1/\alpha^n$ for each $n \in \mathbb{N}$. Hence, it is power bounded if and only if $\alpha \geq 1$. The spectrum of J is the closed disc of radius $1/\alpha$. If $\alpha > 1$, then J is uniformly mean ergodic on $H^\infty_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$ and it is not mean ergodic on these spaces if $\alpha < 1$. If $\alpha = 1$, then J is not mean ergodic on $H^\infty_1(\mathbb{C})$, and mean ergodic but not uniformly mean ergodic on $H^0_1(\mathbb{C})$.

(3) The Hardy operator H is compact and has norm 1, its spectrum coincides with the set $\{1/n\} \cup \{0\}$, and it is power bounded and uniformly mean ergodic on $H^\infty_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$ for all $\alpha > 0$. Therefore, it is not hypercyclic on $H^0_\alpha(\mathbb{C})$.

It follows from [8, Corollary 2.6] that the differentiation operator on $H^0_\alpha(\mathbb{C})$ is not hypercyclic and has no periodic point different from 0 if $\alpha < 1$, it is hypercyclic and has a dense set of periodic points if $\alpha > 1$ and it is hypercyclic but has no periodic point different from 0 if $\alpha = 1$.

2. Preliminaries

Our first result is inspired by [8, Proposition 1.1].

Lemma 2.1. *Let $T : (\mathcal{H}(\mathbb{C}), \tau_{co}) \to (\mathcal{H}(\mathbb{C}), \tau_{co})$ be a continuous linear operator such that $T(\mathcal{P}) \subset \mathcal{P}$. The following conditions are equivalent:*

(i) $T(H^\infty_v(\mathbb{C})) \subset H^\infty_v(\mathbb{C})$,
(ii) $T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C})$ *is continuous,*
(iii) $T(H^0_v(\mathbb{C})) \subset H^0_v(\mathbb{C})$,
(iv) $T : H^0_v(\mathbb{C}) \to H^0_v(\mathbb{C})$ *is continuous.*

Moreover, if (i)-(iv) hold, then $||T||_{L(H^\infty_v(\mathbb{C}))} = ||T||_{L(H^0_v(\mathbb{C}))}$ and $\sigma_{H^\infty_v(\mathbb{C})}(T) = \sigma_{H^0_v(\mathbb{C})}(T)$.

Proof. The equivalences (i)-(ii) and (iii)-(iv) follow from the closed graph theorem. (ii) $\Rightarrow$ (iii) comes easily from the fact that the polynomials are dense in $H^0_v(\mathbb{C})$, $T(\mathcal{P}) \subset \mathcal{P}$ and $H^0_v(\mathbb{C})$ is closed in $H^\infty_v(\mathbb{C})$. Clearly $||T||_{L(H^0_v(\mathbb{C}))} \leq ||T||_{L(H^\infty_v(\mathbb{C}))}$. To show (iv) $\Rightarrow$ (ii), observe that given $f \in H^\infty_v(\mathbb{C})$, the sequence of the Cesàro means of its Taylor polynomials $(C_n f)_n$ belongs to $H^0_v(\mathbb{C})$, therefore $||T(C_n f)||_v \leq ||T||_{L(H^0_v(\mathbb{C}))} ||C_n f||_v \leq ||T||_{L(H^0_v(\mathbb{C}))} ||f||_v$. Hence, from the τ_{co}-compactness of the closed unit ball and the τ_{co}-continuity of T, we conclude $||Tf||_v \leq ||T||_{L(H^0_v(\mathbb{C}))} ||f||_v$. The assertion about the spectra is clear as the bi-transpose T'' of $T : H^0_v(\mathbb{C}) \to H^0_v(\mathbb{C})$ is $T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C})$ by [4] and [6]. It is a well-known fact that $\sigma(T) = \sigma(T') = \sigma(T'')$. □

As the three operators under consideration satisfy the hypothesis of the previous result, in what follows we will write $||T||_v$ instead of $||T||_{L(H^\infty_v(\mathbb{C}))} = ||T||_{L(H^0_v(\mathbb{C}))}$ and $\sigma_v(T)$ for the spectrum. The notation $||T||_{a,\alpha}, ||T||_\alpha, \sigma_{a,\alpha}(T)$ and $\sigma_\alpha(T)$ refers to the cases $v(r) = r^a e^{-\alpha r}$ and $v(r) = e^{-\alpha r}$, respectively.

Lemma 2.2. *$J - \lambda I$ is injective on $\mathcal{H}(\mathbb{C})$ for each $\lambda \in \mathbb{C}$.*

Proof. For $\lambda = 0$ this is trivial, since $DJ = I$. For $\lambda \neq 0$, the equation $Jf - \lambda f = 0$ implies $f - \lambda f' = 0$, hence $f(z) = Ce^{z/\lambda}$ and, as $f(0) = \frac{1}{\lambda} Jf(0) = 0$, we conclude $f = 0$. □

Proposition 2.3. *Let $T = D$ or $T = J$ and assume that $T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C})$ is continuous. The following conditions are equivalent:*

(i) $T : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is uniformly mean ergodic,*
(ii) $T : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ *is uniformly mean ergodic,*
(iii) $\lim_{N\to\infty} \frac{||T+\cdots+T^N||_v}{N} = 0$.

Moreover, if $1 \in \sigma_v(T)$*, then* T *is not uniformly mean ergodic.*

Proof. The implications (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear from Lemma 2.1. We show (ii) $\Rightarrow$ (iii).

Suppose first that $T = D$ is uniformly mean ergodic on $H_v^0(\mathbb{C})$. Since the polynomials are dense and the sequence $(\frac{1}{N}\sum_{n=1}^N D^n)_N$ converges pointwise to zero on $\mathcal{P}$, we have that

$$\lim_{N\to\infty} \frac{1}{N}||\sum_{n=1}^N D^n||_v = 0.$$

For $T = J$, we only have to prove that $\lim_N \frac{(J+\cdots+J^N)(f)}{N} = 0$ for each $f \in H_v^0(\mathbb{C})$. By assumption, the limit $\lim_N \frac{(J+\cdots+J^N)(f)}{N}$ exists. Moreover, $J(\lim_N \frac{(J+\cdots+J^N)(f)}{N}) = \lim_N(\frac{(J+\cdots+J^{N+1})(f)}{N+1}\frac{N+1}{N} - \frac{Jf}{N}) = \lim_N \frac{(J+\cdots+J^N)(f)}{N}$. By Lemma 2.2, the conclusion follows.

If T is uniformly mean ergodic, since $\lim_N \frac{||T+\cdots+T^N||}{N} = 0$, for N big enough the operator $I - \frac{1}{N}\sum_{n=1}^N T^n$ is invertible, i.e., $N \notin \sigma_v(p(T))$ for $p(z) = \sum_{n=1}^N z^n$, which, by the spectral mapping theorem, coincides with $p(\sigma_v(T))$. Thus, $1 \notin \sigma_v(T)$. □

The proof of the former proposition in fact shows that if J is mean ergodic on $H_v^\infty(\mathbb{C})$ or on $H_v^0(\mathbb{C})$, then $\lim_N \frac{(J+\cdots+J^N)(f)}{N} = 0$ for every f in the corresponding space.

An operator T on a Banach space X is said to be *Cesàro power bounded* if the sequence of the Cesàro means of its iterates is equicontinuous. By Lemma 2.1 and the Banach-Steinhaus theorem, D is Cesàro power bounded on $H_v^\infty(\mathbb{C})$ if and only if it is Cesàro power bounded on $H_v^0(\mathbb{C})$. Since the polynomials are dense in $H_v^0(\mathbb{C})$, the operator D is mean ergodic on $H_v^0(\mathbb{C})$ if and only if it is Cesàro power bounded. In this case, $P(f) = 0$ for every $f \in H_v^0(\mathbb{C})$.

The norms of the monomials played an important role in the characterization of hypercyclic differentiation operators in [8]. They are also important now.

Lemma 2.4. *Let* v *be a weight such that the differentiation operator* D *and the integration operator* J *are continuous on* $H_v^0(\mathbb{C})$ *and on* $H_v^\infty(\mathbb{C})$.

(i) If D *is power bounded (resp. uniformly mean ergodic) on* $H_v^\infty(\mathbb{C})$*, then* $\inf(\frac{||z^n||_v}{n!})_n > 0$ *(resp.* $(\frac{||z^n||_v}{(n-1)!})_n$ *tends to infinity).*

(ii) If J *is power bounded (resp. mean ergodic) on* $H_v^\infty(\mathbb{C})$*, then* $(\frac{||z^n||_v}{n!})_n$ *is bounded (resp.* $(\frac{||z^n||_v}{n!n})_n$ *tends to zero).*

Proof. This follows easily from the inequalities

$$v(0)n! \leq ||D^n||_v||z^n||_v$$

and

$$v(0)||J^n||_v \geq ||J^n(1)||_v = \frac{||z^n||_v}{n!}.$$

□

For weights $v(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r \geq r_0$ for some $r_0 \geq 0$ we have

$$||z^n||_v \approx (\frac{n+a}{e\alpha})^{n+a},$$

with equality for $v(r) = e^{-\alpha r}$, $r \geq 0$. It is enough to estimate the maximum of the function $g(r) = r^{n+a}e^{-\alpha r}$ and to have in mind that the weight v may change in a compact interval.

3. Main Results

3.1. **The integration operator.**

Proposition 3.1. *The operator J is never hypercyclic on $H_v^0(\mathbb{C})$ and it has no periodic points different from 0 in $H_v^\infty(\mathbb{C})$.*

Proof. By the very definition $Jf(0) = 0$ for every $f \in H_v^0(\mathbb{C})$. Therefore, $\mathrm{Im}(J)$, and a fortiori the orbit of an element, cannot be dense. Now suppose that $J^n f = f$ for some $f \neq 0$ and some $n \in \mathbb{N}$. We have $f - D^n f = 0$, therefore $f(z) = e^{\lambda z}$ for some $\lambda \in \mathbb{C}$ with $\lambda^n = 1$. But then, $J^n f(0) = 0$ whereas $f(0) \neq 0$. □

Proposition 3.2. *Let v be a weight such that $v(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$. Then, J is continuous with $||J||_v \leq 1/\alpha$. In particular, $\sigma_v(J) \subset (1/\alpha)\overline{\mathbb{D}}$. Moreover, $||J^n||_\alpha = 1/\alpha^n$ for all $n \in \mathbb{N}_0$.*

Proof. Fix $f \in H_v^\infty(\mathbb{C})$. We have

$$v(|z|)|Jf(z)| \leq |z| \int_0^1 v(t|z|)|f(tz)| \exp{(\alpha(t-1)|z|)}\, dt \leq ||f||_v \frac{1}{\alpha}.$$

Therefore $||J^n||_v \leq 1/\alpha^n$. The statement about $\sigma_v(J)$ follows immediately.
On the other hand, for $v(r) = e^{-\alpha r}$, we have

$$||J^n||_\alpha \geq \sup_{k\in\mathbb{N}} \frac{||J^n(z^k)||_\alpha}{||z^k||_\alpha} \geq \sup_{k\in\mathbb{N}} \frac{k!}{(k+n)!} \frac{||z^{k+n}||_\alpha}{||z^k||_\alpha}.$$

We can apply Stirling's formula to conclude

$$||J^n||_\alpha \geq \sup_{k\in\mathbb{N}} \frac{k!}{(k+n)!} \frac{(e\alpha/k)^k}{(e\alpha/(k+n))^{k+n}} \geq \sup_{k\in\mathbb{N}} \frac{(1+n/k)^k}{\alpha^n e^n} \geq \frac{1}{\alpha^n}.$$

□

Proposition 3.3. *Let v be a weight such that J is continuous on $H_v^\infty(\mathbb{C})$ and assume that $v(r)e^{\alpha r}$ is increasing. Then, $\sigma_v(J) \supset (1/\alpha)\mathbb{D}$.*

Proof. To see that $(1/\alpha)\mathbb{D} \subset \sigma_v(J)$ we show that $J - \lambda I$ is not surjective on $H_v^\infty(\mathbb{C})$ for $|\lambda| < \frac{1}{\alpha}$. For $\lambda = 0$, J is not surjective on any $H_v^\infty(\mathbb{C})$ (without any additional assumption) since $Jf(0) = 0$ for each f, hence the equation $J(f) = 1$ has no solution. Now assume that $\lambda \neq 0$ and that there is $f \in H_v^\infty(\mathbb{C})$ such that $Jf - \lambda f = 1$. Then, $f - \lambda f' = 0$ and, as $e^{z/\lambda} \notin H_v^\infty(\mathbb{C})$, we have $f \equiv 0$, therefore $Jf - \lambda f \neq 1$.

□

Corollary 3.4. *The spectrum of J satisfies*

$$\sigma_{a,\alpha}(J) = (1/\alpha)\overline{\mathbb{D}}.$$

Proof. We can apply Proposition 3.2 to conclude that J is continuous on $H^\infty_{a,\alpha}(\mathbb{C})$ and $H^0_{a,\alpha}(\mathbb{C})$. By Proposition 3.3, $\sigma_{a,\alpha}(J) \supset (1/\alpha)\mathbb{D}$. On the other hand, for each $\beta < \alpha$, the function $v(r)e^{\beta r}$ is decreasing in some interval $[r_0, \infty[$. Therefore, for an equivalent norm, $||J^n|| \leq \frac{1}{\beta^n}$ and thus the spectral radius $r(J)$ of J satisfies $r(J) \leq \frac{1}{\beta}$. Since $\beta < \alpha$ is arbitrary, the reverse inclusion holds. □

Theorem 3.5. *(i) J is power bounded on $H^\infty_v(\mathbb{C})$ and mean ergodic on $H^0_v(\mathbb{C})$ provided that $v(r)e^r$ is non increasing in some interval $[r_0, \infty[$. In particular, it is mean ergodic on $H^0_{a,1}(\mathbb{C})$ for every $a \leq 0$.*

(ii) If $r^a e^{-r} = O(v(r))$, with $a > 1/2$, then J is not power bounded on $H^\infty_v(\mathbb{C})$.

(iii) J is uniformly mean ergodic on $H^\infty_v(\mathbb{C})$ if for some $\alpha > 1$, $v(r)e^{\alpha r}$ is non increasing.

(iv) J is not uniformly mean ergodic on $H^0_v(\mathbb{C})$ if for all $\beta > 1$, $v(r)e^{\beta r}$ is increasing in some interval $[r_0, \infty[$. In particular J is not uniformly mean ergodic on $H^0_{a,1}(\mathbb{C})$ for all $a \in \mathbb{R}$.

(v) If $r^{3/2}e^{-r} = O(v(r))$, then J is not mean ergodic on $H^0_v(\mathbb{C})$. In particular, it is not mean ergodic in $H^0_{a,\alpha}(\mathbb{C})$ when $\alpha < 1$, $a \in \mathbb{R}$.

Proof. (i) The first statement follows from the estimates of the norm of J^n in Proposition 3.2.

Moreover, for each $k \in \mathbb{N}$,

$$||J^n(z^k)||_v = \frac{k!}{(n+k)!}||z^{n+k}||_v \leq$$

$$\leq v(0)e^{r_0}\frac{k!}{(n+k)!}||z^{n+k}||_1 = v(0)e^{r_0}\frac{k!}{(n+k)!}\left(\frac{n+k}{e}\right)^{n+k}.$$

By Stirling formula, this implies that the successive iterates tend to zero on the polynomials. As J is power bounded and the polynomials are a dense subset, $(J^n(f))_n$ converges to zero for each $f \in H^0_v(\mathbb{C})$. This implies that $\frac{1}{N}\sum_{n=1}^N J^n(f)$ also converges to 0 for each $f \in H^0_v(\mathbb{C})$ and J is mean ergodic on $H^0_v(\mathbb{C})$.

(ii) As $\frac{||z^n||_{a,1}}{n!} = O(\frac{||z^n||_v}{n!})$ and the sequence $(\frac{||z^n||_{a,1}}{n!})_n$ for $a > 1/2$ is unbounded, we conclude that J is not power bounded by Lemma 2.4.

(iii) The sequence $(||J^n||_v)_n$ tends to zero by Proposition 3.2, therefore

$$||\frac{1}{N}\sum_{n=1}^N J^n||_v \leq \frac{1}{N}\sum_{n=1}^N ||J^n||_v \to 0.$$

Hence, J is uniformly mean ergodic.

(iv) If for all $\beta > 1$, $v(r)e^{\beta r}$ is increasing in some interval $[r_0, \infty[$, $\sigma_v(J) \supset \overline{\mathbb{D}}$. In particular $1 \in \sigma_v(J)$. The conclusion follows from Proposition 2.3.

(v) By Stirling formula, the sequence $(\frac{||z^n||_{3/2,1}}{n\,n!})_n$ does not tend to zero and $||z^n||_{3/2,1} = O(||z^n||_v)$. By Lemma 2.4 (ii), J is not mean ergodic on $H^0_v(\mathbb{C})$. □

Corollary 3.6. *The integration operator J is uniformly mean ergodic on $H^\infty_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$ if $\alpha > 1$, and it is not mean ergodic on these spaces if $\alpha < 1$. If $\alpha = 1$, then J is not mean ergodic on $H^\infty_1(\mathbb{C})$, and mean ergodic but not uniformly mean ergodic on $H^0_1(\mathbb{C})$.*

Proof. All the statements but one follow from Theorem 3.5. It remains to show that J is not mean ergodic on $H_1^\infty(\mathbb{C})$. The space $H_1^\infty(\mathbb{C})$ is a Grothendieck Banach space with the Dunford-Pettis property since it is isomorphic to ℓ_∞ by Galbis [11] or Lusky [20]. As $||J^n||_1/n \to 0$, we can apply [17, Theorem 8] or [18, Theorem 5] to conclude that J is not mean ergodic in $H_1^\infty(\mathbb{C})$ because it is not uniformly mean ergodic by Theorem 3.5 (iv) and Proposition 2.3. $\square$

3.2. **The differentiation operator.** For the differentiation operator we mainly restrict our attention to the spaces $H_{a,\alpha}^\infty(\mathbb{C})$ and $H_{a,\alpha}^0(\mathbb{C})$.

Proposition 3.7. *The following holds for $a > 0$:*

$$||D^n||_{a,\alpha} = O\left(n!\left(\frac{e\alpha}{n-a}\right)^{n-a}\right)$$

and

$$n!\left(\frac{e\alpha}{n+a}\right)^{n+a} = O(||D^n||_{a,\alpha}).$$

For $a \leq 0$:

$$||D^n||_{a,\alpha} \approx n!\left(\frac{e\alpha}{n+a}\right)^{n+a},$$

and the equality holds for $a = 0$.

Proof. In case $a \leq 0$ the upper estimate follows from [2, Proposition 1]. For $a > 0$, we use Cauchy's formula for the n-th derivative and the fact that $(1+r)^a$ is submultiplicative to get, for all $n \in \mathbb{N}$,

$$|D^n f(z)|(1+|z|)^a \leq n!\frac{e^{\alpha(R+|z|)}(1+R)^a}{R^n}||f||_{a,\alpha}.$$

Taking the infimum over $R > 0$ we get the estimate.

For the lower estimate we use that

$$||D^n||_{a,\alpha} \geq \frac{n!v(0)}{||z^n||_{a,\alpha}} \approx n!\left(\frac{e\alpha}{n+a}\right)^{n+a}.$$

$\square$

Proposition 3.8. *The spectrum of D satisfies*

$$\sigma_{a,\alpha}(D) = \alpha\overline{\mathbb{D}}.$$

Proof. If $|\lambda| < \alpha$, the function $e_\lambda(z) := e^{\lambda z}$ belongs to $H_{a,\alpha}^0(\mathbb{C})$ and satisfies $De_\lambda = \lambda e_\lambda$. Therefore $D - \lambda I$ is not invertible, and thus, $\alpha\mathbb{D} \subset \sigma_{a,\alpha}(D)$. On the other hand, the spectral radius $r(D)$ of D satisfies $r(D) = \lim_{n\in\mathbb{N}} ||D^n||^{1/n}$. Using the Stirling's formula and the upper estimates for the norms in Proposition 3.7, $r(D) \leq \alpha$. $\square$

By [2, Proposition 4], $D - \lambda I$ *is not surjective on* $H_{a,\alpha}^\infty(\mathbb{C})$ *or on* $H_{a,\alpha}^0(\mathbb{C})$ *for* $|\lambda| = \alpha$. On the other hand, we get the following:

Proposition 3.9. *Let v be a weight such that D is continuous on $H_v^\infty(\mathbb{C})$ and that $v(r)e^{\alpha r}$ is non increasing. If $|\lambda| < \alpha$, then the operator $D - \lambda I$ is surjective on $H_v^\infty(\mathbb{C})$ and it even has a continuous linear right inverse. The same holds on $H_v^0(\mathbb{C})$. In particular, if $|\lambda| < \alpha$, $D - \lambda I$ has a continuous right inverse on $H_{a,\alpha}^\infty(\mathbb{C})$.*

Proof. Denote by $e_\lambda(z) := e^{\lambda z}$ and by $M_\lambda f(z) := (e_\lambda f)(z)$. We have

$$DM_{-\lambda} = e_{-\lambda}(D - \lambda I)$$

and

$$M_\lambda D = (D - \lambda I)M_\lambda.$$

Set $K_\lambda = M_\lambda J M_{-\lambda}$. More precisely,

$$K_\lambda f(z) = e^{\lambda z} \int_0^z e^{-\lambda \xi} f(\xi) d\xi, \ f \in \mathcal{H}(\mathbb{C}).$$

We have

$$(D - \lambda I)K_\lambda = (D - \lambda I)M_\lambda J M_{-\lambda} = M_\lambda D J M_{-\lambda} = M_\lambda M_{-\lambda} = I.$$

It remains to show that for $|\lambda| < \alpha$, $K_\lambda : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ and $K_\lambda : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ are continuous. Integrating along the segment between 0 and z we get

$$v(|z|)|K_\lambda f(z)| \leq |z| v(|z|) \int_0^1 e^{|\lambda||z|(1-t)} |f(tz)| dt \leq$$

$$|z| \int_0^1 e^{|\lambda||z|(1-t)} e^{-\alpha(1-t)|z|} v(tz)|f(tz)| dt \leq ||f||_v |z| \int_0^1 e^{(|\lambda|-\alpha)|z|(1-t)} dt \leq ||f||_v \frac{1}{\alpha - |\lambda|}$$

which finishes the proof for $H_v^\infty(\mathbb{C})$.

Now, since $K_\lambda : H_v^0(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous and the polynomials are dense in $H_v^0(\mathbb{C})$ it is enough to show that $K_\lambda(\mathcal{P}) \subset H_v^0(\mathbb{C})$. To this end,

$$K_\lambda(1) = -\frac{1}{\lambda} + \frac{1}{\lambda} e^{\lambda z} \in H_v^0(\mathbb{C})$$

for $|\lambda| < \alpha$. Integrating by parts

$$K_\lambda(z^n) = -\frac{1}{\lambda} z^n + \frac{n}{\lambda} K_\lambda(z^{n-1}), \ n \in \mathbb{N},$$

from where the conclusion follows. □

Proposition 3.10. *(i) For $\alpha > 1$ or $\alpha = 1$ and $a < 1/2$, D is not power bounded on $H_{a,\alpha}^\infty(\mathbb{C})$.*

(ii) If $v(r) = o(e^{-r})$, then D is not mean ergodic on $H_v^0(\mathbb{C})$. Consequently, D is not mean ergodic on $H_{a,\alpha}^0(\mathbb{C})$ if $\alpha > 1$ or if $\alpha = 1$ and $a < 0$.

(iii) For $\alpha < 1$, D is power bounded and uniformly mean ergodic on $H_{a,\alpha}^\infty(\mathbb{C})$.

(iv) D is not uniformly mean ergodic on $H_{a,1}^0(\mathbb{C})$, $a \in \mathbb{R}$.

Proof. (i) It is enough to observe that $||D^n||_{a,\alpha} \geq \frac{n! v(0)}{||z^n||_{a,\alpha}}$ which tends to infinity by Stirling formula.

(ii) If D is mean ergodic on $H_v^0(\mathbb{C})$, for each $f \in H_v^0(\mathbb{C})$

$$\frac{f' + f'' + \cdots f^{(N)}}{N} \to 0,$$

which is not the case for $e^z \in H_v^0(\mathbb{C})$.

(iii) By Proposition 3.7,

$$||D^n||_{a,\alpha} = O\left(n! \left(\frac{e\alpha}{n - |a|}\right)^{n-|a|}\right).$$

We apply Stirling's formula to get

$$||D^n||_{a,\alpha} = O\left(\frac{(n)^{n-|a|}}{(n-|a|)^{(n-|a|)}} e^{-|a|}\alpha^{n-|a|}n^{|a|+1/2}\right).$$

Therefore, for $\alpha < 1$, $\lim_{n\to\infty} ||D^n||_{a,\alpha} = 0$, hence

$$\lim_{n\to\infty} ||\frac{1}{n}\sum_{k=1}^{n} D^k||_{a,\alpha} \leq \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} ||D^k||_{a,\alpha} = 0.$$

(iv) In this case, $1 \in \sigma_{a,1}(D)$ and the conclusion follows from Proposition 2.3. □

Corollary 3.11. *D is not mean ergodic on $H_1^\infty(\mathbb{C})$.*

Proof. The space $H_1^\infty(\mathbb{C})$ is a Grothendieck Banach space with the Dunford-Pettis property. In fact it is isomorphic to ℓ_∞ by Galbis [11] or Lusky [20]. Since the operator of differentiation satisfies $||D^n||_1/n \to 0$, we can apply [17, Theorem 8] or [18, Theorem 5] to conclude that D is not mean ergodic on $H_1^\infty(\mathbb{C})$ because it is not uniformly mean ergodic by Propositions 3.10 (iv) and 2.3. □

We do not know if the differentiation operator is mean ergodic on the space $H_1^0(\mathbb{C})$. Related partial results can be seen in [7].

3.3. **The Hardy operator.**

Theorem 3.12. *Let v be an arbitrary weight. The Hardy operator $H : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is well defined and continuous with norm $\|H\|_v = 1$. Moreover, $H^2(H_v^\infty(\mathbb{C})) \subset H_v^0(\mathbb{C})$ and H^2 is compact. If the integration operator $J : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous, then H is compact.*

Proof. For every $f \in H_v^\infty(\mathbb{C})$ and $z \in \mathbb{C}$ we have

$$v(|z|)\Big|\frac{1}{z}\int_0^z f(\omega)d\omega\Big| \leq v(|z|)\sup_{|\omega|=z}|f(\omega)|.$$

Hence, $\|H\|_v \leq 1$. On the other hand, since $H(c) = c$ for every $c \in \mathbb{C}$, taking $g := 1/v(0) \in H_v^0(\mathbb{C})$, $\|g\|_v = 1$, we obtain $\|H\|_v = 1$.

Given $f = \sum_{k=0}^\infty a_k z^k \in H_v^\infty(\mathbb{C})$, the Cauchy inequalities imply $|a_k|\|z^k\|_v \leq \|f\|_v$ for every $k \in \mathbb{N}_0$. Then, as $H^2(f)(z) = \sum_{k=0}^\infty \frac{a_k}{(k+1)^2}z^k$, one has

$$\|H^2(f) - \sum_{k=0}^{N}\frac{a_k}{(k+1)^2}z^k\|_v \leq \sum_{k=N+1}^{\infty}\frac{|a_k|\|z^k\|_v}{(k+1)^2} \leq \|f\|_v \sum_{k=N+1}^{\infty}\frac{1}{(k+1)^2} \tag{3.1}$$

Therefore, $H^2(f)$ belongs to the closure of the polynomials, and thus, to $H_v^0(\mathbb{C})$.

The argument above also shows that the finite rank operators $H_N^2(\sum_{k=0}^\infty a_k z^k) := \sum_{k=0}^N \frac{a_k}{(k+1)^2}z^k$ are bounded on $H_v^\infty(\mathbb{C})$ and that

$$||H^2 - H_N^2||_v \leq \sum_{k=N+1}^{\infty}\frac{1}{(k+1)^2},$$

from where the compactness of H^2 follows.

Finally, suppose that the integration operator $J : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous. Then the Hardy operator $H : H_v^\infty(\mathbb{C}) \to H_v^0(\mathbb{C})$ is well defined, since for every $z \in \mathbb{C}$, $z \neq 0$,

$$v(|z|)|H(f)(z)| = v(|z|)\frac{1}{|z|}|J(f)(z)| \leq \frac{\|J\|}{|z|}\|f\|_v.$$

From here we also conclude that given a bounded set $B \subseteq H_v^\infty(\mathbb{C})$ and $\epsilon > 0$, there exists $R > 0$ such that, for every $f \in B$ and $|z| > R$, $v(|z|)|H(f)(z)| < \epsilon$, and the conclusion follows by [22, Lemma 2.1]. □

By the work of Harutyunyan and Lusky [13], the integration operator J is not continuous on $H_v^\infty(\mathbb{C})$ for $v(r) = \exp(-(\log r)^2)$. Moreover, by Lusky [19, Theorem 2.5.], the monomials constitute a basis of the space $H_v^0(\mathbb{C})$ and the norm of $H_v^\infty(\mathbb{C})$ is equivalent to $|| \sum_{k=0}^\infty a_k z^k||_v = \sup_k |a_k|||z^k||_v$. Moreover, $H_v^0(\mathbb{C})$ is isomorphic to c_0. In this example the Hardy Operator H maps $H_v^\infty(\mathbb{C})$ into $H_v^0(\mathbb{C})$ (just look at the Taylor expansion of the function). As $H_v^\infty(\mathbb{C})$ is canonically isometric to the bidual of $H_v^0(\mathbb{C})$ by [4] or [6], H is weakly compact as an operator on both spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$. Since $H_v^0(\mathbb{C})$ is isomorphic to c_0, H is compact on $H_v^0(\mathbb{C})$ (see e.g. [14, Corollary 17.2.6]). As H on $H_v^\infty(\mathbb{C})$ coincides with the bi-transpose, it follows that it is also compact.

Corollary 3.13. *The operator H is power bounded and uniformly mean ergodic on $H_v^\infty(\mathbb{C})$. Moreover, its spectrum is $\sigma(H) = \{\frac{1}{n}\}_{\mathbb{N}} \cup \{0\}$.*

Proof. As H is quasi-compact $\sigma(H) = \overline{\{\lambda : \lambda \,\text{is an eigenvalue of}\, H\}}$ and the eigenvalues of H are $\{\frac{1}{n} : n \in \mathbb{N}\}$. Clearly H is power bounded. The compactness of H^2 implies that $\mathrm{Im}(I - H)$ is closed. Now the conclusion follows from a criterion due to Lin [16] (see also [15, Theorem 2.2.1]).

□

Observe that contrary to what happens for the operators of integration J and of differentiation D, the Hardy operator H is mean ergodic and 1 belongs to the spectrum of H on the space $H_v^0(\mathbb{C})$. In this case, the Cesàro means of the iterates of H do not converge to zero on the polynomials. Being power bounded, H cannot be hypercyclic on $H_v^0(\mathbb{C})$. In fact, since $\delta_0(H^n f) = f(0)$ for each $f \in \mathcal{H}(\mathbb{C})$, H is not hypercyclic on $\mathcal{H}(\mathbb{C})$.

References

1. A.G. Arvanitidis, A.G. Siskakis, *Cesàro operators on the Hardy spaces of the half-plane*, arXiv:1006.1520v1.
2. A. Atzmon, B. Brive, *Surjectivity and invariant subspaces of differential operators on weighted Bergman spaces of entire functions*, Bergman spaces and related topics in complex analysis, 27-39, Contemp. Math., **404**, Amer. Math. Soc., Providence, RI, 2006.
3. F. Bayart, E. Matheron, *Dynamics of linear operators.* Cambridge Tracts in Mathematics, 179. Cambridge University Press, Cambridge, 2009.
4. K.D. Bierstedt, J. Bonet, A. Galbis, *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J. **40** (1993), no. 2, 271-297.
5. K.D. Bierstedt, J. Bonet, J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), 137-168.
6. K.D. Bierstedt, W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A **54** (1993), no. 1, 70-79.
7. O. Blasco, A. Galbis, *On Taylor coefficients of entire functions integrable against exponential weights*, Math. Nachr. **223** (2001), 5-21.

8. J. Bonet, *Dynamics of the differentiation operator on weighted spaces of entire functions*, Math.Z. **261** (2009) 649-657.
9. J. Bonet, A. Bonilla , *Chaos of the differentiation operator on weighted Banach spaces of entire functions*, Complex Anal. Oper. Theory, DOI: 10.1007/s11785-011-0134-5, to appear.
10. J. Bonet, W.J. Ricker, *Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions,* Arch. Math. **92**, (2009),428-437.
11. A. Galbis, *Weighted Banach spaces of entire functions,* Archiv Math. **62**, (1994), 58-64.
12. K.G. Grosse-Erdmann, A. Peris, *Linear Chaos*, Springer, Berlin, 2011.
13. A. Harutyunyan, W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. **184** (2008), 233-247.
14. H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
15. U. Krengel, *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.
16. M. Lin, *On the Uniform Ergodic Theorem*, Proc. Amer. Math. Soc. **43** (1974), 337-340.
17. H.P. Lotz, *Tauberian theorems for operators on L^∞ and similar spaces*, Functional Analysis: Surveys and Recent Results III, K.D. Bierstedt and B. Fuchssteiner (Eds.), North Holland, Amsterdam, 1984, pp. 117-133.
18. H.P. Lotz, *Uniform Convergence of Operators on L^∞ and Similar Spaces*, Math. Z. **190** (1985), 207-220.
19. W. Lusky, *On the Fourier series of unbounded harmonic functions*, J. London Math. Soc. **61** (2000), 568-580.
20. W. Lusky, *On the isomorphism classes of weighted spaces of harmonic and holomophic functions*, Studia Math. **175** (2006) 19-45.
21. R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.
22. A. Montes-Rodríguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc. **61** (2000), 872-884.
23. W. Rudin, *Functional Analysis*, Mc Graw-Hill, New York, 1973.

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, E-46071 Valencia, SPAIN
E-mail address: mabelme@upv.es,jbonet@mat.upv.es

Departamento de Análisis Matemático, Universidad de Valencia, E-46100 Burjassot
E-mail address: carmen.fdez-rosell@uv.es