MONTEL RESOLVENTS AND UNIFORMLY MEAN ERGODIC SEMIGROUPS OF LINEAR OPERATORS

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Abstract. For C_0–semigroups of continuous linear operators acting in a Banach space criteria are available which are equivalent to uniform mean ergodicity of the semigroup, meaning the existence of the limit (in the operator norm) of the Cesàro or Abel averages of the semigroup. Best known, perhaps, are criteria due to Lin, in terms of the range of the infinitesimal generator A being a closed subspace or, whether 0 belongs to the resolvent set of A or is a simple pole of the resolvent map $\lambda \mapsto (\lambda - A)^{-1}$. It is shown in the setting of locally convex spaces (even in Fréchet spaces), that neither of these criteria remain equivalent to uniform ergodicity of the semigroup (i.e., the averages should now converge for the topology of uniform convergence on bounded sets). Our aim is to exhibit new results dealing with uniform mean ergodicity of C_0–semigroups in more general spaces. A characterization of when a complete, barrelled space with a basis is Montel, in terms of uniform mean ergodicity of certain C_0–semigroups acting in the space, is also presented.

1. Introduction

Let $(T(t))_{t\geq 0}$ be a C_0–semigroup of continuous linear operators in a locally convex Hausdorff space X (briefly, lcHs). Ergodic theorems have a long tradition and are usually formulated for the Cesàro averages $C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt$ or the Abel averages $\lambda R_\lambda x = \lambda \int_0^\infty e^{-\lambda t}T(t)x\,dt$, for $x \in X$, where $r \to \infty$ and $\lambda \downarrow 0^+$, respectively. In the former case one speaks of the mean ergodicity of $(T(t))_{t\geq 0}$ and in the latter case of its Abel mean ergodicity. Particularly well developed is the theory and its applications when X is a Banach space (see, e.g., [8, Ch. 4], [13, Ch. VIII], [15, Ch. V], [18, Ch. XVIII], [24] and the references therein), both for the strong operator topology τ_s–convergence of $\lim_{r\to\infty} C(r)$, resp. $\lim_{\lambda\downarrow 0^+} \lambda R_\lambda$, and for their operator norm convergence. For certain aspects of the theory of mean ergodic semigroups of operators in *non–normable* spaces X (mainly for τ_s) we refer to [14], [24, Ch. 2], [31, Ch. III, §7] and the references therein. Further results, involving $\{C(r)\}_{r\geq 0}$, occur in [5], where geometric features of the underlying space X also play an important role. But for a few exceptions, there are not so many results available concerning the mean ergodicity of C_0–semigroups of operators in lcHs' when the averages are required to converge for the topology τ_b of *uniform convergence on the bounded subsets* of X. The aim of this paper is to develop this topic further.

Many criteria concerning the mean ergodicity of a τ_s–continuous C_0–semigroup $(T(t))_{t\geq 0}$ acting in a lcHs X involve its infinitesimal generator A. Under mild

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conditions this is a closed operator with a dense domain $D(A) \subseteq X$. For X a Banach space the resolvent set $\rho(A)$ of A is an open, non–empty subset of $\mathbb{C}$ and so the well developed spectral theory of closed operators in such spaces is available. In particular, the resolvent map $\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$ of A is holomorphic in $\rho(A)$ for the operator norm topology. For X non–normable, the spectral theory of closed operators A is much less developed. Even if A is the infinitesimal generator of a τ_s–continuous C_0–semigroup in a Fréchet space X and $D(A) = X$, it can happen that $\rho(A)$ fails to be open in $\mathbb{C}$ (see Remark 3.5(vii)) in which case the question of $R(\cdot, A)$ being holomorphic is not well–posed. In Section 3 we investigate and develop those aspects of spectral theory for closed operators (in spaces which may be non–normable) and, in particular, for infinitesimal generators, which are needed in later sections.

In Banach spaces there is a close connection between operator norm continuous, mean ergodic C_0–semigroups $(T(t))_{t\geq 0}$ and compactness of the resolvent operators $R(\lambda, A)$ of the infinitesimal generator A of $(T(t))_{t\geq 0}$, [15, Ch. V, §4]. For X a more general lcHs an appropriate analogue of $R(\lambda, A)$ being compact ìs that it maps bounded subsets of X to relatively compact subsets of X; such operators are called Montel, [12]. Section 4 investigates the connections between the operators $R(\lambda, A)$ being Montel (assuming $\rho(A) \neq \emptyset$), the τ_b–continuity of the map $t \mapsto T(t)$ in $[0, \infty)$ and of the individual operators $T(t)$, for $t \geq 0$, being Montel; this is made precise in the main result (Theorem 4.7).

Sections 3 and 4 treat some continuity and spectral properties of general C_0–semigroups of operators and their infinitesimal generators. These results are needed in Section 5 where we turn our attention to *mean ergodic* features of C_0–semigroups $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$, with X a sequentially complete lcHs and $\mathcal{L}(X)$ the vector space of all continuous linear operators from X into itself. Under mild conditions, the Cesàro averages $\{C(r)\}_{r\geq 0} \subseteq \mathcal{L}(X)$ exist as do the Abel averages $\{\lambda R(\lambda, A) : \lambda > 0\} \subseteq \mathcal{L}(X)$ where, for each real $\lambda > 0$, the resolvent operator $R(\lambda, A)$ coincides with the operator $R_\lambda x =: \int_0^\infty e^{-\lambda t} T(t)x\, dt$, for $x \in X$, mentioned above and which is defined via X–valued Riemann integrals. The central notions are the *uniform mean ergodicity* (resp. *uniform Abel mean ergodicity*) of $(T(t))_{t\geq 0}$, that is, τ_b-$\lim_{r\to\infty} C(r)$ (resp. τ_b-$\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$) exists. Here, convergence of the net $\{\lambda R(\lambda, A)\}$ for $\lambda \downarrow 0^+$ in $\mathcal{L}_b(X)$ (or $\mathcal{L}_s(X)$) is meant in the sense that there exists $\lambda_0 > 0$ such that $(0, \lambda_0] \subseteq \rho(A)$ and the interval $(0, \lambda_0]$ is considered as a directed set for the order $\leq$ induced from $\mathbb{R}$. A similar interpretation applies to convergence of the net $\{C(r)\}_{r\geq 0}$ for $r \to \infty$ (relative to the other order $\geq$ in $\mathbb{R}$). As already mentioned, in Banach spaces many results are available which imply or are equivalent to $(T(t))_{t\geq 0}$ being uniformly mean ergodic. But, for non–normable X, not so much is known. In Section 5 we present several new results in this direction. Example 5.8 makes it clear that *not* all Banach space results carry over automatically; new phenomena arise which are not present in Banach spaces. For instance, there exists an equicontinuous C_0–semigroup acting in a Fréchet space X which is uniformly mean ergodic (equivalently, uniformly Abel mean ergodic) but, unlike for Banach spaces, the range ImA of A *fails* to be closed in X. It can also happen that $0 \notin \rho(A)$ with 0 failing to be a simple pole of $R(\cdot, A)$, which is impossible in Banach spaces. Theorem 5.1 (where Montel resolvents arise) and Theorem 5.13 provide the most extensive results for a general lcHs X. The final two results (i.e., Theorems 5.16 and 5.17) deal with certain τ_b–continuous, mean ergodic C_0–semigroups in complete, barrelled lcHs' with a Schauder basis/decomposition.

2. Preliminaries

Let X be a lcHs with Γ_X always denoting a system of continuous seminorms determining the topology of X. The strong operator topology τ_s in $\mathcal{L}(X)$ (we write $\mathcal{L}(X,Y)$ for the space of all continuous linear operators from X into another lcHs Y) is determined by the family of seminorms $q_x(S) := q(Sx)$, for $S \in \mathcal{L}(X)$, with $x \in X$ and $q \in \Gamma_X$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b in $\mathcal{L}(X)$ is defined via the seminorms $q_B(S) := \sup_{x\in B} q(Sx)$, for $S \in \mathcal{L}(X)$, with $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$. The identity operator on X is denoted by I.

By X_σ we denote X with its weak topology $\sigma(X,X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X,X')$ (resp. $\beta(X',X)$) and we write X_β (resp. X'_β); see [22, §21.2] for the definition. The strong dual $(X'_\beta)'_\beta$ of X'_β is denoted by X''. By X'_σ we denote X' with its weak–star topology $\sigma(X',X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T^t \colon X' \to X'$ is defined by $\langle x, T^t x'\rangle = \langle Tx, x'\rangle$ for $x \in X$, $x' \in X'$. Then $T^t \in \mathcal{L}(X'_\sigma)$ and $T^t \in \mathcal{L}(X'_\beta)$, [23, p.134].

Definition 2.1. Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a 1–parameter family of operators. The map $t \mapsto T(t)$, for $t \in [0,\infty)$, is denoted by $T\colon [0,\infty) \to \mathcal{L}(X)$.

We say that $(T(t))_{t\geq 0}$ is a *semigroup* if it satisfies

(i) $T(s)T(t) = T(s+t)$ for all $s,t \geq 0$, with $T(0) = I$.

A semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous* if, for fixed $K > 0$, the set $\{T(t) : 0 \leq t \leq K\}$ is equicontinuous, i.e., given $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and $M > 0$ (depending on p and K) such that

$$p(T(t)x) \leq Mq(x)\,, \quad x \in X,\, t \in [0,K]. \tag{2.1}$$

A semigroup $(T(t))_{t\geq 0}$ is said to be a *C_0–semigroup* if it satisfies

(ii) $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_s(X)$.

If the C_0–semigroup $(T(t))_{t\geq 0}$ satisfies the additional condition that

(iii) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_s(X)$, for each $t_0 \geq 0$,

then it is called a *strongly continuous* C_0–semigroup.

A semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially equicontinuous* if there exists $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous, i.e.,

$$\forall p \in \Gamma_X\ \exists q \in \Gamma_X, M_p > 0 \text{ with } p(T(t)x) \leq M_p e^{at} q(x)\ \forall t \geq 0, x \in X. \tag{2.2}$$

If $a = 0$, then we simply say *equicontinuous*. Finally, a semigroup $(T(t))_{t\geq 0}$ is called *uniformly continuous* if $T\colon [0,\infty) \to \mathcal{L}_b(X)$ is continuous, i.e.,

(iv) $\lim_{t\to t_0} T(t) = T(t_0)$ in $\mathcal{L}_b(X)$, for each $t_0 > 0$ (with $t \to 0^+$ if $t_0 = 0$).

Given any locally equicontinuous C_0–semigroup $(T(t))_{t\geq 0}$ (resp. any locally equicontinuous, uniformly continuous C_0–semigroup) on a lcHs X, observe that condition (iii) (resp. condition (iv)) in Definition 2.1 is equivalent to $T(t) \to I$ in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) as $t \to 0^+$. This is a consequence of (i), namely, that $T(t_0+h) - T(t_0) = T(t_0)(T(h) - I)$ for each $t_0 > 0$ and all h such that $t_0 + h \geq 0$.

Remark 2.2. (i) Let X be a lcHs and $(T(t))_{t\geq 0}$ be an *equicontinuous* C_0–semigroup on X. For $p\in\Gamma_X$ define $\tilde{p}(x):=\sup_{t\geq 0}p(T(t)x)$, for $x\in X$. By Definition 2.1(i)–(iii) $\tilde{p}$ is well-defined, is a seminorm and satisfies

$$p(x)\leq\tilde{p}(x)\leq M_p q(x)\leq M_p\tilde{q}(x),\quad x\in X. \tag{2.3}$$

Hence, $\tilde{\Gamma}_X:=\{\tilde{p}:\ p\in\Gamma_X\}$ also generates the given lc–topology of X. Moreover, for $\tilde{p}\in\tilde{\Gamma}_X$, we have

$$\tilde{p}(T(t)x)=\sup_{s\geq 0}p(T(t)T(s)x)=\sup_{s\geq 0}p(T(t+s)x)\leq\tilde{p}(x),\quad x\in X,\, t\geq 0. \tag{2.4}$$

(ii) In [21, Prop. 1.1] it is shown that in a barrelled lcHs X every strongly continuous C_0–semigroup $(T(t))_{t\geq 0}$ is *locally equicontinuous*.

(iii) Every C_0–semigroup of operators in a Banach space is necessarily exponentially equicontinuous, [13, p.619]. For Fréchet spaces this need not be so. Indeed, in the sequence space $\omega=\mathbb{C}^{\mathbb{N}}$ (topology of coordinate convergence), $T(t)x:=(e^{nt}x_n)_{n=1}^{\infty}$, for $t\geq 0$ and $x=(x_n)_{n=1}^{\infty}\in\omega$, defines a C_0–semigroup which is not exponentially equicontinuous. As ω is a Montel space, $(T(t))_{t\geq 0}$ is also uniformly continuous.

If X is a *sequentially complete* lcHs and $(T(t))_{t\geq 0}$ is a locally equicontinuous C_0–semigroup on X, then the linear operator A defined by

$$Ax:=\lim_{t\to 0^+}\frac{T(t)x-x}{t},$$

for $x\in D(A):=\{x\in X:\ \lim_{t\to 0^+}\frac{T(t)x-x}{t}\text{ exists in }X\}$, is closed with $\overline{D(A)}=X$, [21, Propositions 1.3 & 1.4]. The operator $(A,D(A))$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$. Moreover, A and $(T(t))_{t\geq 0}$ commute, [21, Proposition 1.2(1)], i.e., for each $t\geq 0$ we have $\{T(t)x:x\in D(A)\}\subseteq D(A)$ and $AT(t)x=T(t)Ax$, for all $x\in D(A)$. Also known, [21, Proposition 1.2(2)], is that

$$T(t)x-x=\int_0^t T(s)Ax\,ds=\int_0^t AT(s)x\,ds,\quad x\in D(A), \tag{2.5}$$

and, [21, Corollary p.261], that

$$T(t)x-x=A\int_0^t T(s)x\,ds,\quad x\in X. \tag{2.6}$$

For each $x\in D(A)$ (resp. $x\in X$), the integrals occuring in (2.5) (resp. (2.6)) are Riemann integrals of an X–valued, continuous function on $[0,t]$; see [5, Appendix]. The closedness of A ensures that $\mathrm{Ker}\,A:=\{x\in D(A):Ax=0\}$ is a closed subspace of X. The range of A is the subspace $\mathrm{Im}A:=\{Ax:x\in D(A)\}$.

Recall that a linear map $S\colon X\to X$ is called *locally bounded* if $S(B)\in\mathcal{B}(X)$ for every $B\in\mathcal{B}(X)$. If $S\in\mathcal{L}(X)$, then S is necessarily locally bounded. In the event that X is bornological, every locally bounded linear map from X into itself is continuous.

Proposition 2.3. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on X whose infinitesimal generator A satisfies $D(A)=X$ with A a locally bounded map. Then $(T(t))_{t\geq 0}$ is uniformly continuous.*

Proof. Let $p \in \Gamma_X$ and $B \in \mathcal{B}(X)$. Then there exist $K > 0$ and $q \in \Gamma_X$ such that $p(T(t)x) \leq Kq(x)$, for $x \in X,\ t \in [0,1]$. This inequality and (2.5) imply that

$$p_B(T(t) - I) \leq \sup_{x\in B} \int_0^t p(T(s)Ax)\, ds \leq tK \sup_{x\in B} q(Ax), \quad t \in [0,1].$$

Since A is locally bounded, $\sup_{x\in B} q(Ax) < \infty$ and so $\lim_{t\to 0^+} T(t) = I$ in $\mathcal{L}_b(X)$. □

Remark 2.4. Since Banach spaces are bornological, Proposition 2.3 is well known in this setting, [15, p.15].

Let X be a sequentially complete lcHs and $A \in \mathcal{L}(X)$ be *power bounded*, i.e., $\{A^n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ is equicontinuous. It follows from Corollary 1 (and an examination of its proof) in [34, p.245] that $T(t) := e^{tA} = \sum_{n=0}^\infty \frac{t^n A^n}{n!}$, for $t \geq 0$, defines an exponentially equicontinuous C_0–semigroup; actually $(e^{-t}T(t))_{t\geq 0}$ is equicontinuous. In particular, $(T(t))_{t\geq 0}$ is also locally equicontinuous. So, Proposition 2.3 implies that $(T(t))_{t\geq 0}$ is necessarily uniformly continuous.

Concerning the converse of Proposition 2.3, it is known that the infinitesimal generator A of any uniformly continuous C_0–semigroup in a Banach space X satisfies $A \in \mathcal{L}(X)$, [13, Ch. VIII, Corollary 1.9]. For X a quojection (or, even prequojection) Fréchet space and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ an exponentially equicontinuous, uniformly continuous C_0–semigroup, it is also the case that its infinitesimal generator $A \in \mathcal{L}(X)$, [4, Theorem 3.3 & Proposition 3.4]. However, this is *not* the case for Fréchet spaces in general, [4, Example 3.1 & Proposition 3.2].

Let $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on a sequentially complete lcHs X. The operators

$$C(0) := I \text{ and } C(r)x := \frac{1}{r}\int_0^r T(t)x dt, \quad x \in X,\ r > 0, \tag{2.7}$$

are called the *Cesáro means* of $(T(t))_{t\geq 0}$. The integrals in (2.7) are X–valued Riemann integrals with respect to the locally convex topology of X; see [5], [20], [34], for example. The Cesáro means $\{C(r)\}_{r\geq 0}$ are well defined and belong to $\mathcal{L}(X)$, [5, Section 3]. If $(T(t))_{t\geq 0}$ is equicontinuous, then $\{C(r)\}_{r\geq 0}$ is also equicontinuous, [5, Section 3]. In case X is barrelled the Cesáro means exist in $\mathcal{L}(X)$ whenever the semigroup $(T(t))_{t\geq 0}$ is strongly continuous (via Remark 2.2(ii)).

3. Spectra of closed linear operators

The spectral theory of closed linear operators in Banach spaces is well developed. A traditional area of application is the theory of semigroups of linear operators, [8], [13], [15]. In particular, this applies to mean ergodic semigroups, [8, Ch. 4] [15, Ch.5, §4]. The extension of several classical Banach space results for strongly continuous, mean ergodic C_0–semigroups to the setting of lcHs' occur in [5], [14], [24, Ch.2], [31]. The spectral theory of continuous linear operators acting in non–normable lcHs' is well developed, especially in $\mathcal{L}_b(X)$, [7], [30], where the methods of lc–algebras are applicable. However, in an attempt to address *uniformly continuous*, mean ergodic semigroups in the *non–Banach* space setting one is confronted with the difficulty that the spectral theory of closed linear operators (not necessarily everywhere defined) in such a space is not nearly as satisfactory as for Banach spaces. The aim of this section is to present certain aspects of such a spectral theory (see also [32]) but, only to the extent needed in later sections dealing with operator semigroups and uniform mean ergodicity.

Let $A\colon D(A) \subseteq X \to X$ be a linear operator on a lcHs X. Whenever $\lambda \in \mathbb{C}$ is such that $(\lambda - A)\colon D(A) \to X$ is injective, the linear operator $(\lambda - A)^{-1}$ is understood to have domain $\mathrm{Im}(\lambda - A) := \{(\lambda - A)x : x \in D(A)\}$. Of course, $\mathrm{Im}(\lambda - A)^{-1} = D(A)$. The *resolvent set* of A is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} :\ (\lambda - A)\colon D(A) \to X \text{ is bijective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

and the *spectrum* of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. For $\lambda \in \rho(A)$ we also write $R(\lambda, A) := (\lambda - A)^{-1}$. Recall that A is called *closed* if the conditions $(x_\alpha)_\alpha \subseteq D(A)$ converges to x in X and $(Ax_\alpha)_\alpha$ converges to y in X imply that $x \in D(A)$ and $y = Ax$. We point out for A closed, that also $\lambda - A$ is closed for all $\lambda \in \mathbb{C}$ and that $(\lambda - A)^{-1}$ is closed whenever $\lambda - A$ is injective.

For fixed λ, $\mu \in \rho(A)$, it follows from the definition that

$$\begin{aligned} R(\lambda, A) &= R(\lambda, A)(\mu - A)R(\mu, A) = R(\lambda, A)[(\mu - \lambda)I + (\lambda - A)]R(\mu, A) \\ &= (\mu - \lambda)R(\lambda, A)R(\mu, A) + R(\mu, A), \end{aligned}$$

from which we obtain the *resolvent equation*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \tag{3.1}$$

Remark 3.1. (i) Let $A\colon D(A) \subseteq X \to X$ be a linear operator on a lcHs X. If $\rho(A) \neq \emptyset$, then A is a closed operator. Indeed, for any fixed net $(x_\alpha)_\alpha \subseteq D(A)$ satisfying $x_\alpha \to x$ and $Ax_\alpha \to y$ in X we have that $\lambda x_\alpha - Ax_\alpha \to \lambda x - y$ in X for every $\lambda \in \mathbb{C}$. In case $\lambda \in \rho(A)$ we also have

$$R(\lambda, A)(\lambda x_\alpha - Ax_\alpha) = x_\alpha, \quad \forall \alpha. \tag{3.2}$$

Via the continuity of $R(\lambda, A)$ and passing to the limits in (3.2), we obtain

$$R(\lambda, A)(\lambda x - y) = x. \tag{3.3}$$

This implies $x \in D(A)$ so that also $R(\lambda, A)(\lambda x - Ax) = x$. Combining this identity with (3.3) we get $R(\lambda, A)(Ax - y) = 0$. Hence, $y = Ax$ as $R(\lambda, A)$ is injective.

(ii) If $A\colon D(A) \to X$ is a closed linear operator and $\lambda \in \mathbb{C}$ satisfies $(\lambda - A)\colon D(A) \to X$ is bijective, then $(\lambda - A)^{-1}\colon X \to D(A) \subseteq X$ is closed. So, for X a Fréchet space, the Closed Graph Theorem ensures $(\lambda - A)^{-1} \in \mathcal{L}(X)$. If the Closed Graph Theorem is not available in a lcHs X, then it is necessary to *assume*, as in the above definition of $\rho(A)$, that $(\lambda - A)^{-1} \in \mathcal{L}(X)$.

Proposition 3.2. *Let $\lambda \in \mathbb{C}$ and $A\colon D(A) \to X$ be a closed linear operator in a complete lcHs X. If $(\lambda - A)\colon D(A) \to X$ is injective with a continuous inverse $(\lambda - A)^{-1}\colon \mathrm{Im}(\lambda - A) \to X$, then $\mathrm{Im}(\lambda - A)$ is a closed subspace of X.*

If, in addition, $\mathrm{Im}(\lambda - A)$ is dense in X, then $\lambda \in \rho(A)$.

Proof. The continuity of $(\lambda - A)^{-1}\colon \mathrm{Im}(\lambda - A) \to X$ implies, for each $p \in \Gamma_X$, that there exist $M_p > 0$ and $q \in \Gamma_X$ satisfying $p((\lambda - A)^{-1}y) \le M_p q(y)$, for $y \in \mathrm{Im}(\lambda - A)$, or equivalently, that

$$p(x) \le M_p q((\lambda - A)x), \quad x \in D(A). \tag{3.4}$$

Let $y \in \overline{\mathrm{Im}(\lambda - A)}$. Then $y = \lim_\alpha(\lambda - A)x_\alpha$ for some net $(x_\alpha)_\alpha \subseteq D(A)$. It follows from (3.4) that $(x_\alpha)_\alpha \subseteq D(A)$ is Cauchy in X. By completeness there is $x \in X$ with $x = \lim_\alpha x_\alpha$. Since $(\lambda - A)\colon D(A) \to X$ is closed, it follows $x \in D(A)$ and $y = (\lambda - A)x$, i.e., $y \in \mathrm{Im}(\lambda - A)$. So, $\mathrm{Im}(\lambda - A)$ is closed in X.

If, in addition, $\mathrm{Im}(\lambda - A)$ is dense in X, then actually $\mathrm{Im}(\lambda - A) = X$, i.e., $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and so $\lambda \in \rho(A)$. □

Remark 3.3. (i) Proposition 3.2 ensures that, for a closed linear operator $A\colon D(A) \to X$ (with $D(A) \subseteq X$) in a *complete* lcHs X we have

$$\begin{aligned}\rho(A) \quad = \quad & \Big\{\lambda \in C : \ (\lambda - A)\colon D(A) \to X \text{ is injective}, \ \overline{\mathrm{Im}(\lambda - A)} = X \\ & \quad \text{and } (\lambda - A)^{-1}\colon \mathrm{Im}(\lambda - A) \to X \text{ is continuous}\Big\}.\end{aligned}$$

(ii) In [34, Ch.VIII, p.209] the resolvent set of a linear operator $A\colon D(A) \to X$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $\mathrm{Im}(\lambda - \mathrm{A})$ is dense in X and $(\lambda - A)$ has a continuous inverse belonging to $\mathcal{L}(\mathrm{Im}(\lambda - \mathrm{A}), \mathrm{D(A)})$. Let us denote this resolvent set by $\rho_Y(A)$. Clearly, we always have $\rho(A) \subseteq \rho_Y(A)$. In case the space X is complete and A is closed, it follows from Proposition 3.2 that, for each $\lambda \in \rho_Y(A)$, we have $\mathrm{Im}(\lambda - \mathrm{A}) = \mathrm{X}$ and so $\lambda \in \rho(A)$. That is, $\rho_Y(A) = \rho(A)$ whenever A is closed and X is complete.

Proposition 3.4. *Let $A\colon D(A) \to X$ be a closed linear operator in a sequentially complete lcHs X with $\rho(A) \neq \emptyset$. Let $\mathcal{U} \subseteq \rho(A)$ be non-empty.*

(i) *Assume that, for each $\lambda \in \mathcal{U}$, there exists an open neighbourhood $V(\lambda) \subseteq \mathbb{C}$ of λ with $V(\lambda) \subseteq \mathcal{U}$ such that the set $R(V(\lambda)) := \{R(\mu, A) : \mu \in V(\lambda)\}$ is equicontinuous in $\mathcal{L}(X)$. Then $\mathcal{U}$ is open in $\mathbb{C}$, the resolvent map $R\colon \lambda \to R(\lambda, A)$ is holomorphic from $\mathcal{U}$ into $\mathcal{L}_b(X)$, and*

$$\frac{d^n}{d\lambda^n}R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}, \quad n \in \mathbb{N}, \ \lambda \in \mathcal{U}. \tag{3.5}$$

In particular, the resolvent map R is continuous from $\mathcal{U}$ into $\mathcal{L}_b(X)$.

(ii) *In addition, let $\mathcal{L}_b(X)$ be sequentially complete. Then, under the assumptions of* (i)*, for each $\mu \in \mathcal{U}$ one has the series expansion*

$$R(z, A) = \sum_{n=0}^{\infty}(\mu - z)^n R(\mu, A)^{n+1} \tag{3.6}$$

in $\mathcal{L}_b(X)$, for all z in some open disc with centre μ and contained in $\mathcal{U}$.

(iii) *Assume, for each $\lambda \in \mathcal{U}$, that there exists $M_\lambda > 0$ satisfying*

$$p(R(\lambda, A)x) \leq M_\lambda p(x), \quad \forall p \in \Gamma_X, \ x \in X. \tag{3.7}$$

Then the assumptions of (i) *are satisfied and, for each $\mu \in \mathcal{U}$, an open disc with centre μ for which* (3.6) *holds can be chosen with radius $1/M_\mu$.*

Proof. (i) The assumptions clearly imply that $\mathcal{U}$ is open in $\mathbb{C}$.

We first prove the continuity of $R\colon \mathcal{U} \to \mathcal{L}_b(X)$. So, fix $\lambda \in \mathcal{U}$ and a continuous seminorm p_B in $\mathcal{L}_b(X)$, i.e., $p \in \Gamma_X$ and $B \in \mathcal{B}(X)$. By assumption there exists an open neighbourhood $V(\lambda) \subseteq \mathbb{C}$ of λ with $V(\lambda) \subseteq \mathcal{U}$ such that $R(V(\lambda)) := \{R(\mu, A) : \mu \in V(\lambda)\}$ is *equicontinuous* in $\mathcal{L}(X)$. So, corresponding to p there exist $M_p > 0$ and $q \in \Gamma_X$ such that $p(R(\mu, A)x) \le M_p q(x)$, for $\mu \in V(\lambda)$, $x \in X$. Using the resolvent equation (3.1) it follows that

$$\begin{aligned} p_B(R(\lambda, A) - R(\mu, A)) &= \sup_{x \in B} p(R(\lambda, A)x - R(\mu, A)x) \\ &= |\mu - \lambda| \sup_{x \in B} p(R(\mu, A)R(\lambda, A)x) \le \alpha M_p |\mu - \lambda|, \quad \mu \in V(\lambda), \end{aligned}$$

where $\alpha := \sup_{x \in B} q(R(\lambda, A)x) < \infty$ as $R(\lambda, A)(B) \in \mathcal{B}(X)$ via the continuity of $R(\lambda, A)$. This inequality ensures that if $\mu_n \to \lambda$ in $\mathcal{U}$, then $R(\mu_n, A) \to R(\lambda, A)$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

Using again the resolvent equation (3.1), we have that

$$\frac{R(\mu, A) - R(\lambda, A)}{\mu - \lambda} = -R(\mu, A)R(\lambda, A), \quad \lambda, \mu \in \rho(A), \lambda \ne \mu. \tag{3.8}$$

This formula together with the continuity of the resolvent map $R\colon \mathcal{U} \to \mathcal{L}_b(X)$ imply that $\lambda \to R(\lambda, A)$ is holomorphic in $\mathcal{U}$. In particular, (3.8) also implies that $\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}$, for $\lambda \in \mathcal{U}$, $n \in \mathbb{N}$, where we need to use the equicontinuity of $\{R(\mu, A)^k : \mu \in V(\lambda)\} \subseteq \mathcal{L}(X)$, for each $k \in \mathbb{N}$.

(ii) Let $f\colon \mathcal{U} \to \mathcal{L}_b(X)$ be holomorphic and $\mu \in \mathcal{U}$ be fixed. Suppose that $D \subseteq \mathcal{U}$ is an open disc centred at μ and that C_0 is a circle centred at μ with radius r_0 such that $C_0 \subseteq D$. Fix any z inside C_0 and write $r := |z - \mu| < r_0$. If s is any point on a circle C_1 centred at μ with radius $r_1 \in (r, r_0)$, then the theory of integration for continuous vector–valued (in this case $\mathcal{L}_b(X)$–valued) functions defined on a compact interval in $\mathbb{R}$ (in this case $[0, 2\pi]$, which is used to parameterise the curve C_1 in $\mathbb{C}$ via $\theta \mapsto \mu + r_1 e^{i\theta}$) yields, by an argument analogous to the case when f is $\mathbb{C}$–valued (see, e.g., [10, §52]) that

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{(s - \mu)^{n+1}} ds = \frac{f^{(n)}(\mu)}{n!}, \quad n = 0, 1, 2, \ldots.$$

One can then argue as for $\mathbb{C}$–valued functions (e.g. [10, pp.145–147]) to establish that the power series expansion (in $\mathcal{L}_b(X)$) of f is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\mu)}{n!} (z - \mu)^n, \quad |z - \mu| < r_0, \tag{3.9}$$

provided $\lim_{n\to\infty} R_N(z) := \lim_{N\to\infty} \frac{(z-\mu)^N}{2\pi i} \int_{C_1} \frac{f(s)}{(s-z)(s-\mu)^N} ds = 0$ in $\mathcal{L}_b(X)$. To see that this is the case, recall that $|z - \mu| = r$ and $|s - \mu| = r_1$ and hence, $|s - z| \ge |s - \mu| - |z - \mu| = r_1 - r$. So, if $q \in \Gamma_X$ and $B \in \mathcal{B}(X)$, then it follows from [5, Proposition 11(vii)] applied in the sequentially complete lcHs $\mathcal{L}_b(X)$ that

$$\begin{aligned} q_B(R_N(z)) &= \frac{r^N}{2\pi} q_B\left(\int_{C_1} \frac{f(s)}{(s - z)(s - \mu)^N} ds\right) \le \frac{r^N}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r) r_1^N} \sup_{s \in C_1} q_B(f(s)) \\ &= \frac{r_1}{(r_1 - r)} \cdot \left(\frac{r}{r_1}\right)^N \sup_{s \in C_1} q_B(f(s)). \end{aligned}$$

But, $f(C_1)$ is compact in $\mathcal{L}_b(X)$ and $q_B\colon \mathcal{L}_b(X) \to [0, \infty)$ is continuous and so $\sup_{s \in C_1} q_B(f(s)) < \infty$. Since $\frac{r}{r_1} < 1$, we can conclude that $q_B(R_N(z)) \to 0$ as $N \to \infty$. That is, $R_N(z) \to 0$ in $\mathcal{L}_b(X)$ for each z in the interior of C_0 and so (3.9) is indeed valid.

For the particular case when f is the resolvent function $R\colon \lambda \to R(\lambda, A)$ of A, the identities (3.5) and (3.9) yield (3.6) for all z satisfying $|z-\mu| < r_0$.

(iii) Fix $\mu \in \mathcal{U}$. Then, for any $\lambda \in \mathbb{C}$, we can write

$$\begin{aligned} \lambda - A &= \mu - A + \lambda - \mu = (\mu - A) - (\mu - \lambda)R(\mu, A)(\mu - A) \\ &= [I - (\mu - \lambda)R(\mu, A)](\mu - A), \end{aligned}$$

as an identity on $D(A)$. This operator is bijective if and only if $[I - (\mu - \lambda)R(\mu, A)]\colon X \to D(A)$ is bijective. Now, by (3.7) we have

$$\begin{aligned} p((\mu-\lambda)^n R(\mu, A)^n x) &\leq |\lambda - \mu|^n M_\mu p(R(\mu, A)^{n-1}x) \\ &\leq |\lambda - \mu|^n M_\mu^n p(x), \quad x \in X,\, p \in \Gamma_X. \end{aligned} \tag{3.10}$$

This inequality ensures that if $\lambda \in \mathbb{C}$ satisfies $|\lambda - \mu| < M_\mu^{-1}$, then the series $\sum_{n=0}^\infty (\mu - \lambda)^n R(\mu, A)^n x$ converges absolutely for every $x \in X$ and hence, converges in X (by sequential completeness of X). In case $|\lambda - \mu| < M_\mu^{-1}$ the inverse of $\lambda - A$ is then the linear operator from X to X given by $R_\lambda\colon x \to \sum_{n=0}^\infty (\mu-\lambda)^n R(\mu, A)^{n+1} x$ which, by (3.10), satisfies

$$p(R_\lambda x) \leq \frac{M_\mu}{1 - |\mu - \lambda| M_\mu} p(x), \quad x \in X,\, p \in \Gamma_X, \tag{3.11}$$

i.e., $R_\lambda \in \mathcal{L}(X)$. Hence,

$$R_\lambda = R(\lambda, A) = R(\mu, A)[I - (\mu - \lambda)R(\mu, A)]^{-1} = \sum_{n=0}^\infty (\mu - \lambda)^n R(\mu, A)^{n+1}.$$

In particular, (3.11) ensures that $\{R(\lambda, A) : \lambda \in V(\mu)\} \subseteq \mathcal{L}(X)$ is equicontinuous, where $V(\mu) := \{\lambda \in \mathbb{C} : |\lambda - \mu| < 1/2M_\mu\}$ is contained in $\mathcal{U}$. So, we have shown that the assumptions of (i) are satisfied.

Of course, for any fixed λ belonging to $D(\mu) := \{\lambda \in \mathbb{C} : |\lambda - \mu| < 1/M_\mu\}$ we see from (3.11) that $R(\lambda, A) \in \mathcal{L}(X)$, i.e., $D(\mu) \subseteq \mathcal{U}$. But, for *equicontinuity* of $R(V(\lambda)) \subseteq \mathcal{L}(X)$ we require a smaller radius for $V(\lambda)$, e.g., $1/2M_\mu$. □

Remark 3.5. (i) When $\mathcal{L}_b(X)$ has additional completeness properties, the expansion (3.6) in $\mathcal{L}_b(X)$ is well known; see, for example, [16, pp.493–503], [19, §16.7].

(ii) If X is sequentially complete and barrelled, then both $\mathcal{L}_s(X)$ and $\mathcal{L}_b(X)$ are sequentially complete, [11, Proposition 1.8 & Remark 1.9]. This is relevant for Proposition 3.4(ii).

(iii) Let X be a Banach space and $A\colon D(A) \to X$ be any closed operator such that $\rho(A) \neq \emptyset$. Then, for each $\lambda \in \rho(A)$, we see in (3.7) that $M_\lambda := \|R(\lambda, A)\| < \infty$ can be chosen, [15, p.240]. For an example where $\rho(A) = \emptyset$, see [15, p.241] for instance. Such an operator A cannot be the infinitesimal generator of a C_0–semigroup, [15, Ch.II, Theorem 1.10(ii)].

(iv) If $(A, D(A))$ is the infinitesimal generator of any *equicontinuous* C_0–semigroup $(T(t))_{t\geq 0}$ on a sequentially complete lcHs X, then (3.7) is always satisfied at each point $\lambda \in \mathbb{C}_+ \subseteq \rho(A)$, for some $M_\lambda > 0$, where $\mathbb{C}_+ := \{\mu \in \mathbb{C} : \mathrm{Re}(\mu) > 0\}$. Indeed, for such an operator A, we first claim that $\mathbb{C}_+ \subseteq \rho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \quad \mathrm{Re}(\lambda) > 0,\, x \in X. \tag{3.12}$$

To see this, by Remark 2.2(i), we may assume that each $p \in \Gamma_X$ satisfies

$$p(T(t)x) \leq p(x), \quad x \in X, \, t \geq 0. \tag{3.13}$$

For each $x \in X$, the integral $\int_0^\infty e^{-\lambda t}T(t)x\,dt = \lim_{n\to\infty}\int_0^n e^{-\lambda t}T(t)x\,dt$ exists as an improper Riemann integral whenever $\mathrm{Re}(\lambda) > 0$. Indeed, $t \mapsto e^{-\lambda t}T(t)x$, for $t \in [0,n]$, is a continuous X–valued function and so, as noted above, the sequential completeness of X ensures that the integral $\int_0^n e^{-\lambda t}T(t)x\,dt \in X$ exists as a limit of X–valued Riemann sums. The convergence of the improper integral follows from the sequential completeness of X, the equicontinuity of $(T(t))_{t\geq 0}$, the inequalities (using (3.13)) $p(e^{-\lambda t}T(t)x) \leq e^{-\mathrm{Re}(\lambda)t}p(x)$, for $t \geq 0$, and the inequalities (again using (3.13))

$$p\left(\int_m^n e^{-\lambda t}T(t)x\,dt\right) \leq \int_m^n p(e^{-\lambda t}T(t)x)dt \leq \frac{e^{-\mathrm{Re}(\lambda)m} - e^{-\mathrm{Re}(\lambda)n}}{\mathrm{Re}(\lambda)}p(x), \quad n > m,$$

for all $p \in \Gamma_X$. Putting $m = 0$ and letting $n \to \infty$ gives

$$p\left(\int_0^\infty e^{-\mathrm{Re}(\lambda)t}T(t)x\,dt\right) \leq \frac{1}{\mathrm{Re}(\lambda)}p(x), \quad x \in X, \; p \in \Gamma_X. \tag{3.14}$$

In particular, (3.14) implies that the linear map $R_\lambda \colon x \to \int_0^\infty e^{-\lambda t}T(t)x\,dt$, for $x \in X$, satisfies $R_\lambda \in \mathcal{L}(X)$. It follows from Theorem 1 and Corollary 1 of [34, pp. 240–241], that actually $R_\lambda = (\lambda - A)^{-1}$ and so $\lambda \in \rho(A) \subseteq \rho_Y(A)$ whenever $\lambda \in \mathbb{C}_+$ (cf. Remark 3.3(ii)). This establishes (3.12).

It follows from (3.14) that (3.7) holds with $M_\lambda := 1/\mathrm{Re}(\lambda)$, for each $\lambda \in \mathbb{C}_+$. In particular, $R(\cdot, A) \colon \mathbb{C}_+ \to \mathcal{L}_b(X)$ is holomorphic; see Proposition 3.4(iii) with $\mathcal{U} = \mathbb{C}_+$. We point out that (3.7) may *not* hold for all $\lambda \in \rho(A)$; see (vi) and (vii) below.

(v) We note that (3.7) may fail to hold at *every* point $\lambda \in \rho(A)$ for an exponentially equicontinuous C_0–semigroup, even if its infinitesimal generator $A \in \mathcal{L}(X)$. Consider $X = \omega = \mathbb{C}^{\mathbb{N}}$ equipped with the seminorms $p_k(x) = \max_{1\leq j\leq k}|x_j|$, for $x = (x_1, x_2, \ldots) \in X$, for each $k \in \mathbb{N}$; see Section 2. Then X is a Fréchet space. The unit right shift $A \in \mathcal{L}(X)$ is given by $A(x) := (0, x_1, x_2, \ldots)$, for $x = (x_1, x_2, \ldots) \in X$. For $\lambda = 0$ we see that $(\lambda - A)$ is not surjective, i.e., $0 \in \sigma(A)$. If $\lambda \neq 0$, then $(\lambda - A)$ is injective and a direct calculation shows that

$$R(\lambda, A)(y) = \left(\frac{1}{\lambda}y_1, \, \frac{1}{\lambda}y_2 + \frac{1}{\lambda^2}y_1, \, \frac{1}{\lambda}y_3 + \frac{1}{\lambda^2}y_2 + \frac{1}{\lambda^3}y_1, \ldots\right), \quad y \in X, \tag{3.15}$$

and hence, $R(\lambda, A) \in \mathcal{L}(X)$. Accordingly, $\sigma(A) = \{0\}$ and $\rho(A) = \mathbb{C} \setminus \{0\}$. Set $d_\lambda^{(n)} := \max\left\{\frac{1}{|\lambda|}, \frac{1}{|\lambda|^2}, \ldots, \frac{1}{|\lambda|^n}\right\}$ for $\lambda \neq 0$, $n \in \mathbb{N}$. It turns out that

$$p_n(R(\lambda, A)y) \leq n d_\lambda^{(n)} p_n(x), \quad y \in X. \tag{3.16}$$

Moreover, $nd_\lambda^{(n)}$ is the smallest constant for which (3.16) holds; to see this consider the vector $y^{(n)} \in X$ with $y_j^{(n)} = 1$ for $1 \leq j \leq n$ and, $y_j^{(n)} = 0$ for $j > n$. That is, $\sup_{p_n(x)\leq 1} p_n(R(\lambda, A)x) = nd_\lambda^{(n)}$, for each $n \in \mathbb{N}$. Since

$$nd_\lambda^{(n)} = \begin{cases} \frac{n}{|\lambda|} & \text{if } |\lambda| \geq 1 \\ \frac{n}{|\lambda|^n} & \text{if } 0 < |\lambda| < 1, \end{cases}$$

it follows that $M_\lambda := \sup_{n\in\mathbb{N}} n d_\lambda^{(n)} = \infty$, for every $\lambda \neq 0$. Thus, for no $\lambda \neq 0$ does there exist $M_\lambda \in (0,\infty)$ satisfying (3.7).

The claim is that the semigroup $T(t) := e^{tA}$, $t \geq 0$, is *exponentially equicontinuous*. Indeed, direct calculation gives (via the power series) that

$$e^{tA}x = \left(x_1,\, x_2 + tx_1,\, x_3 + tx_2 + \frac{t^2}{2!}x_1,\, x_4 + tx_3 + \frac{t^2}{2!}x_2 + \frac{t^3}{3!}x_1, \ldots\right), \quad x \in X,$$

for each $t \geq 0$. From the definition of p_n we deduce, for $t \geq 0$ and $n \in \mathbb{N}$, that

$$p_n(e^{tA}x) \leq n\left(1 + t + \ldots + \frac{t^n}{n!}\right) p_n(x) \leq n e^t p(x), \quad x \in X.$$

Accordingly, $\{e^{tA}\}_{t\geq 0}$ is *exponentially equicontinuous*; see also Remark 2.4.

So, here the infinitesimal generator A of the exponentially equicontinuous C_0–semigroup $\{e^{tA}\}_{t\geq 0}$ satisfies $\rho(A) \neq \emptyset$ but, condition (3.7) fails to hold for *every* $\lambda \in \rho(A)$. Nevertheless, the set $\rho(A)$ is still open, i.e., that (3.7) holds for every $\lambda \in \rho(A)$ is sufficient but not necesssary for $\rho(A)$ to be open.

Such an example as just given in the Fréchet space ω cannot occur in a Banach space X. Indeed, *every* strongly continuous C_0–semigroup $(T(t))_{t\geq 0}$ in a Banach space is exponentially equicontinuous, [15, Ch.I, Proposition 5.5], and its infinitesimal generator A is a closed operator, [15, Ch.II, Theorem 1.4]. So, we see from part (iii) of this Remark that (3.7) is satisfied for every $\lambda \in \rho(A)$, where it was also noted that $\rho(A) \neq \emptyset$, [15, Ch.II, Theorem 1.10(ii)].

Returning to the example in ω we observe that, for *every* $\lambda \in \rho(A)$, i.e., $|\lambda| > 0$, the open neighbourhood $V(\lambda) := \{\mu \in \mathbb{C} :\ |\mu - \lambda| < \frac{|\lambda|}{2}\}$ of λ does have the property that $R(V(\lambda)) \subseteq \mathcal{L}(\omega)$ is *equicontinuous* (i.e., the assumption of Proposition 3.4(i) is satisfied). Indeed, each $\mu \in V(\lambda)$ satisfies $\frac{1}{|\mu|} \leq \frac{2}{|\lambda|}$. Fix $k \in \mathbb{N}$ and $x \in \omega$. Then it follows from (3.15) that

$$p_k(R(\mu, A)x) \leq p_k(x) \sum_{j=1}^{k} \left(\frac{2}{|\lambda|}\right)^j, \quad \mu \in V(\lambda).$$

Accordingly, $\{R(\mu, A)x :\ \mu \in V(\lambda)\}$ is a bounded set in ω, for each $x \in \omega$. Since ω is barrelled, we can conclude that $R(V(\lambda))$ is equicontinuous, for each $\lambda \in \rho(A)$.

(vi) The equicontinuity assumption in Proposition 3.4(i) is not always satisfied, even if $\rho(A) \neq \emptyset$ is open. Part of the following example (without details) is stated in [27, Example 2]. Let $X := \{f \in C^\infty([0,1]) :\ f^{(k)}(0) = 0\ \forall k \in \mathbb{N}_0\}$ be the Fréchet–Montel space equipped with the increasing sequence of norms

$$p_n(f) := \max_{0\leq k\leq n} \sup_{t\in[0,1]} |f^{(k)}(t)|, \quad f \in X,\ n \in \mathbb{N}_0.$$

The differentiation operator $Df := f'$, for $f \in X$, clearly belongs to $\mathcal{L}(X)$. Define

$$V_\lambda f \colon x \mapsto -e^{\lambda x} \int_0^x e^{-\lambda t} f(t)\, dt, \quad x \in [0,1], \tag{3.17}$$

for each $f \in X$ and $\lambda \in \mathbb{C}$. Then $V_\lambda f \in X$ and the linear map $V_\lambda \colon f \mapsto V_\lambda f$, for $f \in X$, belongs to $\mathcal{L}(X)$. Direct calculation verifies that $R(\lambda, D) = V_\lambda$ and so $\rho(D) = \mathbb{C}$. According to [16, p.512], the resolvent map $R(\cdot, D)$ is entire from $\mathbb{C}$ into $\mathcal{L}_b(X)$.

Consider now $A := V_0$ so that $A \in \mathcal{L}(X)$. Direct calculation shows that $R(0,A) = (0-A)^{-1} = D$ and that

$$R(\lambda, A) = -\frac{1}{\lambda} V_{-\frac{1}{\lambda}} D, \quad \lambda \neq 0. \tag{3.18}$$

So, also $\rho(A) = \mathbb{C}$. To see that $R(\cdot, A)$ is *not* equicontinuous in any bounded open disc about 0 it suffices to exhibit $g \in X$ satisfying $\sup_{-r<\lambda<0} p_0(R(\lambda,A)g) = \infty$ for every $r > 0$. To this effect, choose any $g \in X \setminus \{0\}$ satisfying $g \geq 0$ and $g' \geq 0$. Then, for $r > 0$ and $\lambda \in (-r, 0)$ it follows from (3.17) and (3.18) that

$$p_0(R(\lambda,A)g) = \frac{1}{|\lambda|} \sup_{x\in[0,1]} \int_0^x e^{(t-x)/\lambda} g'(t)\,dt \geq \frac{1}{|\lambda|} \sup_{x\in[0,1]} \int_0^x g'(t)\,dt = \frac{1}{|\lambda|} p_0(g)$$

with $p_0(g) > 0$. So, $R(\cdot, A)$ indeed fails to be equicontinuous in $\{z \in \mathbb{C} : |z| < r\}$ for every $r > 0$.

It should be noted that A generates a C_0–semigroup. Indeed, for each $f \in X$ and $m \in \mathbb{N}_0$, direct calculation yields

$$(A^m f)^{(j)} = (-1)^j A^{m-j} f, \quad 1 \leq j \leq m,$$

and also that

$$(A^m f)^{(k)} = (-1)^k f^{(k-m)}, \quad k \geq m.$$

It follows that $p_0(A^m f) \leq p_0(f)$, for $f \in X$ and $m \in \mathbb{N}_0$, and also that

$$p_{k+1}(A^m f) \leq p_k(f), \quad k \in \mathbb{N}.$$

Accordingly, A is power bounded and so it is the infinitesimal generator of the exponentially equicontinuous, uniformly continuous C_0–semigroup $(e^{At})_{t\geq 0}$ with $T(t) := e^{-t}e^{At}$, for $t \geq 0$, being an *equicontinuous*, uniformly continuous C_0–semigroup; see Remark 2.4. Recall that $B := A - I$ is the infinitesimal generator of $(T(t))_{t\geq 0}$. Moreover, $R(\lambda, B) = R(\lambda+1, A)$ for all $\lambda \in \mathbb{C} = \rho(A) = \rho(B)$. Let $\lambda_0 := -1 \in \rho(B)$ and U_{λ_0} be any open disc centred at λ_0. Then $W_0 := \{1 + \mu : \mu \in U_{\lambda_0}\}$ is an open disc centred at $0 = 1 + \lambda_0$ and so $\{R(\mu, A) : \mu \in W_0\}$ is not equicontinuous. But, $\{R(\mu,A) : \mu \in W_0\} = \{R(\lambda, B) : \lambda \in U_{\lambda_0}\}$ and so we can conclude that $\{R(\lambda,B) : \lambda \in U_{\lambda_0}\}$ fails to be equicontinuous for every disc U_{λ_0} centred at λ_0. According to Proposition 3.4(iii), (3.7) fails to hold for λ_0.

(vii) The resolvent set $\rho(A)$ is not always an open subset of $\mathbb{C}$, even for the infinitesimal generator A of an equicontinuous C_0–semigroup. Let $X = \omega$ be the Fréchet space as in part (v) and $\Gamma := \{\gamma_n\}_{n\in\mathbb{N}}$ be any (fixed) countable, dense subset of $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$. Then the linear operator $Ax := (\gamma_n x_n)_{n=1}^\infty$, for $x = (x_n)_{n=1}^\infty \in X$, belongs to $\mathcal{L}(X)$. Direct calculation shows $\sigma(A) = \Gamma$ and, for $\lambda \notin \Gamma$, that

$$R(\lambda, A)x = \left(\frac{x_n}{(\lambda - \gamma_n)}\right)_{n=1}^\infty, \quad x \in X,$$

belongs to $\mathcal{L}(X)$. So, $\rho(A) = \mathbb{C} \setminus \Gamma$ surely fails to be open. Moreover, A is the infinitesimal generator of the equicontinuous, uniformly continuous (as X is Montel) C_0–semigroup $(T(t))_{t\geq 0}$ given by $T(t)x := (e^{\gamma_n t} x_n)_{n=1}^\infty$, for $x \in X$, $t \geq 0$.

The claim is that (3.7) fails to hold for *every* $\lambda \in \rho(A)$ satisfying $\mathrm{Re}(\lambda) < 0$. This is equivalent to showing that

$$\sup_{k\in\mathbb{N}} \sup_{p_k(x)\leq 1} p_k(R(\lambda, A)x) = \infty. \tag{3.19}$$

To this effect, let $x := (1, 1, \ldots)$ so that $p_k(x) = 1$ for all $k \in \mathbb{N}$. Moreover, $p_k(R(\lambda, A)x) = \max_{1\leq j\leq k} |\lambda - \gamma_j|^{-1}$, for $k \in \mathbb{N}$. Choose a subsequence $\{\gamma_{n_r}\}_{r=1}^{\infty}$ with $\gamma_{n_r} \to \lambda$ as $r \to \infty$. Then, for each $m \in \mathbb{N}$, there is $n_{r(m)} \in \mathbb{N}$ such that $|\lambda - \gamma_{n_{r(m)}}|^{-1} \geq m$ and hence, $p_{n_{r(m)}}(R(\lambda, A)x) \geq m$. This clearly implies (3.19).

Let $(A, D(A))$ be a linear operator in a lcHs X such that $\rho(A) \neq \emptyset$, in which case A is closed by Remark 3.1(i). For each $\lambda \in \rho(A)$ we have $(\lambda I - A)R(\lambda, A) = I$ on X and so

$$\lambda R(\lambda, A)x - x = AR(\lambda, A)x, \quad x \in X. \tag{3.20}$$

Lemma 3.6. *Let $(A, D(A))$ be a linear operator in a lcHs X with $\rho(A) \neq \emptyset$.*

(i) *For each $\lambda \in \rho(A)$ it is the case that*

$$\operatorname{Ker} A = \{x \in D(A) : \lambda R(\lambda, A)x = x\}. \tag{3.21}$$

(ii) *The subspace* ImA*, hence also* $\overline{\text{Im}A}$*, is invariant for each operator in* $\{\lambda R(\lambda, A) : \lambda \in \rho(A)\}$.

Proof. (i) If x belongs to the right–side of (3.21), then it follows from (3.20) and the fact that $x \in D(A)$ that $R(\lambda, A)Ax = 0$. By injectivity of $R(\lambda, A)$ also $Ax = 0$, i.e., $x \in \operatorname{Ker} A$. The reverse containment follows directly from (3.20).

(ii) Let $u = Ax$ with $x \in D(A)$. Then, for $\lambda \in \rho(A)$,

$$\lambda R(\lambda, A)u = A(\lambda R(\lambda, A)x) \in \text{Im}A$$

and so ImA is $\lambda R(\lambda, A)$–invariant. By continuity of $\lambda R(\lambda, A)$, the same is true of $\overline{\text{Im}A}$. □

Lemma 3.7. *Let $(A, D(A))$ be a linear operator in a lcHs X such that, for some $\lambda_0 > 0$, we have $(0, \lambda_0] \subseteq \rho(A)$ with $\{\lambda R(\lambda, A) : 0 < \lambda \leq \lambda_0\}$ equicontinuous in $\mathcal{L}(X)$. Then the resolvent map $R(\cdot, A) : (0, \lambda_0] \to \mathcal{L}_b(X)$ is continuous. Hence, also $\lambda \mapsto \lambda R(\lambda, A)$ is continuous from $(0, \lambda_0]$ into $\mathcal{L}_b(X)$.*

Proof. Fix $\mu \in (0, \lambda_0]$. By (3.1), for each $\lambda \in (0, \lambda_0]$, we have

$$R(\lambda, A) - R(\mu, A) = \frac{(\mu - \lambda)}{\lambda} R(\mu, A)[\lambda R(\lambda, A)]. \tag{3.22}$$

For $\lambda \in (\frac{\mu}{2}, \lambda_0]$ we have $\left|\frac{\mu-\lambda}{\lambda}\right| \leq \frac{2|\mu-\lambda|}{\mu}$. Given $B \in \mathcal{B}(X)$, it follows from [23, (1) p.137] that $C := \cup_{0<\lambda\leq\lambda_0} \lambda R(\lambda, A)(B)$ belongs to $\mathcal{B}(X)$ and hence, also $H := \frac{2}{\mu} R(\mu, A)(C) \in \mathcal{B}(X)$. Fix $p \in \Gamma_X$. Then (3.22) yields

$$p_B(R(\lambda, A) - R(\mu, A)) \leq |\mu - \lambda| p_H(I), \quad \lambda \in (\frac{\mu}{2}, \lambda_0],$$

from which it follows that $\tau_b - \lim_{\lambda\to\mu} R(\lambda, A) = R(\mu, A)$. □

Lemma 3.8. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq0} \subseteq \mathcal{L}(X)$ be a locally equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$. Suppose there exists $\lambda_0 > 0$ with $(0, \lambda_0] \subseteq \rho(A)$.*

(i) *If $x \in X$ satisfies $\lim_{\lambda\downarrow0^+} \lambda R(\lambda, A)x = 0$, then $x \in \overline{\text{Im}A}$.*

(ii) *Suppose, in addition, that $\{\lambda R(\lambda, A) : 0 < \lambda \leq \lambda_0\}$ is equicontinuous. Then*

$$\overline{\text{Im}A} = \{x \in X : \lim_{\lambda\downarrow0^+} \lambda R(\lambda, A)x = 0\}. \tag{3.23}$$

Proof. (i) It follows from (3.20) that $x = \lim_{\lambda\downarrow 0^+} AR(\lambda, A)(-x)$, from which it is clear that $x \in \overline{\mathrm{Im}A}$.

(ii) Fix $x \in X$ and $t \geq 0$. According to (2.6) we have $y := \int_0^t T(s)x\,ds \in D(A)$ with $T(t)x - x = Ay$. It follows that

$$\lambda R(\lambda, A)(T(t)x - x) = \lambda R(\lambda, A)[(A - \lambda I)y + \lambda y] = -\lambda y + \lambda^2 R(\lambda, A)y. \quad (3.24)$$

But, $\{\lambda R(\lambda, A)y : 0 < \lambda \leq \lambda_0\} \in \mathcal{B}(X)$ and so $\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)(T(t)x - x) = 0$. Moreover, by [5, Remark 5(iii)] we have (even without the hypothesis $(0, \lambda_0] \subseteq \rho(A)$) that

$$\overline{\mathrm{Im}A} = \overline{\mathrm{span}\{T(t)x - x : t \geq 0, x \in X\}}. \quad (3.25)$$

So, the equicontinuous family $\{\lambda R(\lambda, A) : 0 < \lambda \leq \lambda_0\}$ converges to 0 at each point of a dense subset of $\overline{\mathrm{Im}A}$, with $\overline{\mathrm{Im}A}$ being $\lambda R(\lambda, A)$–invariant for each $0 < \lambda \leq \lambda_0$; see Lemma 3.6(ii). It follows from [23, (1) p.138] that $\{\lambda R(\lambda, A) : 0 < \lambda \leq \lambda_0\}$ converges to 0 at every point of $\overline{\mathrm{Im}A}$. Combined with part (i) this yields (3.23). □

In conclusion, let $(A, D(A))$ be a closed linear operator in a lcHs X such that $\rho(A) \neq \emptyset$. A point $\lambda_0 \in \sigma(A)$ is called a *simple pole* of $R(\cdot, A)$ if there exist a punctured disc $D(\lambda_0, r) := \{z \in \mathbb{C} : 0 < |z - \lambda_0| < r\} \subseteq \rho(A)$, for some $r > 0$, and $P \in \mathcal{L}(X)$ such that $\lambda \mapsto R(\lambda, A) - (\lambda - \lambda_0)^{-1}P$ has a holomorphic, $\mathcal{L}_b(X)$–valued extension from $D(\lambda_0, r)$ to the open disc $\{\lambda_0\} \cup D(\lambda_0, r)$. Simple poles arise in criteria for determining when an individual operator from $\mathcal{L}(X)$ is uniformly ergodic in $\mathcal{L}_b(X)$; see [13, Ch. VIII, §8] for X a Banach space and [33, §4.1] for X a lcHs. In Section 5 we will have occasion to use simple poles in the study of uniformly mean ergodic C_0–semigroups, where A will be the corresponding infinitesimal generator.

4. Montel resolvents and semigroups of operators

The infinitesimal generator A of a C_0–semigroup $(T(t))_{t\geq 0}$ in a Banach space X is said to have compact resolvent if $R(\lambda, A)$ is compact for some (hence, all) $\lambda \in \rho(A)$, [15, Ch. II, Definition 4.24]. The C_0–semigroup $(T(t))_{t\geq 0}$ is called immediately compact if each operator $T(t)$, for $t > 0$, is compact, [15, Ch. II, Definition 4.23]. The relationship between these two notions is well known, [15, Ch. II, Theorem 4.29], and is closely connected to operator–norm continuity of $t \mapsto T(t)$. In turn, such connections are crucial for the theory of operator–norm continuous, *mean ergodic* C_0–semigroups, [15, Ch. V, Sect. 4]. In attempting to extend this theory to a non–normable lcHs X one is confronted with the question: When should an operator $S \in \mathcal{L}(X)$ be called compact? According to Grothendieck, [17, Ch. 5, Part 2], (see also [26, Ch. 3], [27]), this is defined via the existence of some 0–neighbourhood $\mathcal{U} \subseteq X$ such that $S(\mathcal{U})$ is a relatively compact subset of X; for X a Banach space this reduces to the traditional definition. Unfortunately, this notion of compactness is somewhat restrictive when attempting to apply it to uniformly continuous, mean ergodic C_0–semigroups in non–normable spaces. An alternative notion could be that S maps bounded subsets of X to relatively compact subsets of X (which, for X a Banach space, is also equivalent to S being compact). Such operators $S \in \mathcal{L}(X)$, called Montel, were introduced and studied in [12] (see also [9]) and are more suitable for the treatment of uniformly continuous, mean ergodic C_0–semigroups in non–normable lcHs'. In this section we develop the relevant results for uniformly continuous C_0–semigroups (of interest in their own right); their application to mean ergodicity occurs in the following section.

Given lcHs' X, Y, an operator $S \in \mathcal{L}(X,Y)$ is called *Montel* if $S(B) \subseteq Y$ is relatively compact for every $B \in \mathcal{B}(X)$. Every compact operator (in the sense of Grothendieck) is Montel but, not conversely; consider the identity operator on an infinite–dimensional Montel lcHs. The Montel operators form a 2–sided ideal within the class of all continuous operators between lcHs'.

A linear operator $A\colon D(A) \subseteq X \to X$ in a lcHs X with $\rho(A) \neq \emptyset$ is said to have *Montel resolvent* if $R(\lambda, A)$ is Montel for some $\lambda \in \rho(A)$. It then follows from the ideal property of Montel operators and (3.1) that $R(\mu, A)$ is Montel for every $\mu \in \rho(A)$.

Lemma 4.1. *Let $A\colon D(A) \subseteq X \to X$ be a linear operator on a lcHs X. Define a system of seminorms $\{p^A\}_{p\in\Gamma_X}$ on $D(A)$ by*

$$p^A(x) := p(x) + p(Ax), \quad x \in D(A), \tag{4.1}$$

for each $p \in \Gamma_X$. Then $X[A] := (D(A), \{p^A\}_{p\in\Gamma_X})$ is a lcHs and the canonical inclusion $i\colon X[A] \hookrightarrow X$ is continuous. Moreover, if $\rho(A) \neq \emptyset$, then for any fixed $\lambda \in \rho(A)$ the system of seminorms $\{p_\lambda\}_{p\in\Gamma_X}$ on $D(A)$ defined by

$$p_\lambda(x) := p((\lambda - A)x), \quad x \in D(A),\ p \in \Gamma_X, \tag{4.2}$$

is equivalent to $\{p^A\}_{p\in\Gamma_X}$, i.e., also generates the lc–topology of $X[A]$.

If X is complete (resp. quasicomplete, sequentially complete) and A is closed, then $X[A]$ is also complete (resp. quasicomplete, sequentially complete).

Proof. It is routine to verify that $X[A]$ is a lcHs. Clearly the inclusion $i\colon X[A] \hookrightarrow X$ is continuous as $p \leq p^A$, for each $p \in \Gamma_X$.

Fix $\lambda \in \rho(A)$, so that $R(\lambda, A) \in \mathcal{L}(X)$. Then, for any fixed $p \in \Gamma_X$, there exist $M > 0$ and $q \in \Gamma_X$ with $q \geq p$ such that $p(R(\lambda, A)x) \leq Mq(x)$, for $x \in X$. Thus, for each $x \in D(A)$, we have

$$\begin{aligned}
p^A(x) &= p(x) + p(Ax) = p(R(\lambda,A)(\lambda - A)x) + p((A-\lambda)x + \lambda x) \\
&\leq Mq((\lambda - A)x) + p((\lambda - A)x) + p(\lambda x) \\
&\leq (M+1)q((\lambda - A)x) + |\lambda| p(x) \\
&= (M+1)q((\lambda - A)x) + |\lambda| p(R(\lambda,A)(\lambda - A)x) \\
&\leq (M+1)q((\lambda - A)x) + |\lambda| Mq((\lambda - A)x) \\
&= [(1+|\lambda|)M + 1]q((\lambda - A)x) = [(1+|\lambda|)M + 1]q_\lambda(x).
\end{aligned}$$

On the other hand, for each $x \in D(A)$, we always have that

$$p_\lambda(x) = p((\lambda - A)x) \leq |\lambda| p(x) + p(Ax) \leq \max\{1, |\lambda|\} p^A(x).$$

Suppose X is complete. Let $\{x_\alpha\} \subseteq X[A] = D(A)$ be a Cauchy net. It follows from (4.1) that $\{x_\alpha\}$ (resp. $\{Ax_\alpha\}$) is Cauchy in X and so there exists $x \in X$ (resp. $y \in X$) with $x_\alpha \to x$ (resp. $Ax_\alpha \to y$) in X. By closedness of A we conclude that $x \in D(A)$ and $Ax = y$. Then (4.1) implies $x_\alpha \to x$ in $X[A]$. So, $X[A]$ is complete. The argument for X quasicomplete or sequentially complete is analogous. □

For Banach spaces the following result occurs in [15, Ch. II, Proposition 4.25].

Proposition 4.2. *Let X be a lcHs and $(A, D(A))$ be a closed linear operator on X with $\rho(A) \neq \emptyset$. The following assertions are equivalent.*

(i) The operator A has Montel resolvent.

(ii) The canonical inclusion $i\colon X[A] \hookrightarrow X$ is Montel.

Proof. Fix any $\mu \in \rho(A)$. By Lemma 4.1 the system $\{p_\mu\}_{p\in\Gamma_X}$ defined according to (4.2) generates the lc–topology of $X[A]$. The operator $R(\mu, A)\colon X \to X[A]$ is then a topological isomorphism with continuous inverse $(\mu - A)\colon X[A] \to X$. Indeed, $p_\mu(R(\mu, A)x) = p((\mu - A)R(\mu, A)x) = p(x)$, for $x \in X$, $p \in \Gamma_X$, and also $p((\mu - A)x) = p_\mu(x)$, for $x \in D(A)$, $p \in \Gamma_X$.

(i)$\Rightarrow$(ii). Suppose that $R(\lambda, A)$ is Montel for some $\lambda \in \rho(A)$. Then, for every $B \in \mathcal{B}(X[A])$ we have that $(\lambda - A)(B) \in \mathcal{B}(X)$ as $(\lambda - A) \in \mathcal{L}(X[A], X)$ and hence, $B = R(\lambda, A)(\lambda - A)(B)$ is relatively compact in X.

(ii)$\Rightarrow$(i). Fix any $\lambda \in \rho(A)$. Let $B \in \mathcal{B}(X)$. Since $R(\lambda, A) \in \mathcal{L}(X, X[A])$, we have $R(\lambda, A)(B) \in \mathcal{B}(X[A])$ and hence, $R(\lambda, A)(B)$ is relatively compact in X (as $i \in \mathcal{L}(X[A], X)$ is Montel). So, $R(\lambda, A)$ is Montel. $\square$

Lemma 4.3. *Let X be a lcHs and let $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0–semigroup on X. If $T(t_0)$ is Montel for some $t_0 > 0$, then $T(t)$ is Montel for all $t \geq t_0$ and the map $t \to T(t)$ is continuous from $[t_0, \infty)$ into $\mathcal{L}_b(X)$.*

Proof. For every $t > t_0$, the identity $T(t) = T(t - t_0)T(t_0)$ together with the operator ideal property for Montel operators implies that $T(t)$ is Montel.

Fix $s \geq 0$ and a compact set $K \subseteq X$. It follows from the strong continuity of $(T(t))_{t\geq 0}$ (cf. Section 2), the equicontinuity of $\{T(t) : t \in [0, s + 1]\} \subseteq \mathcal{L}(X)$ and [23, (2) p.139], that $\lim_{h\to 0} T(s + h)x = T(s)x$ in X (for $h \to 0^+$ if $s = 0$) uniformly for $x \in K$. Let $B \in \mathcal{B}(X)$ and $t \geq t_0$. Since $T(t_0)$ is Montel, the set $L := \overline{T(t_0)(B)} \subseteq X$ is compact. For $q \in \Gamma_X$ we have (for $s := t - t_0$) that

$$\begin{aligned} q_B(T(t) - T(r)) &= q_B([T(t - t_0) - T(r - t_0)]T(t_0)) \\ &\leq \sup_{x\in L} q([T(t - t_0) - T(r - t_0)]x) = \sup_{x\in L} q([T(s) - T(s + (r - t))]x). \end{aligned}$$

Since $h := (r - t) \to 0$ as $r \to t$ ($h \to 0^+$ as $r \to t_0^+$, if $t = t_0$) it follows that $\lim_{r\to t} T(r) = T(t)$ in $\mathcal{L}_b(X)$; for $r \to t_0^+$ if $t = t_0$. $\square$

The following observation will be needed later.

Lemma 4.4. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be an exponentially equicontinuous C_0–semigroup in X with infinitesimal generator $(A, D(A))$. Then*

$$\mathbb{C}_{a^+} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > a\} \subseteq \rho(A) \tag{4.3}$$

for all $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous. In particular, $\rho(A) \neq \emptyset$ and $R(\cdot, A)\colon \mathbb{C}_{a^+} \to \mathcal{L}_b(X)$ is holomorphic.

Proof. Recall (cf. Section 2) that $(T(t))_{t\geq 0}$ is strongly continuous. Fix any $a \geq 0$ such that $(e^{-at}T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is equicontinuous. Then $(A - aI, D(A))$ is the infinitesimal generator of the equicontinuous C_0–semigroup $t \mapsto S(t) := e^{-at}T(t)$, for $t \geq 0$. The argument for equicontinuous C_0–semigroups given in Remark 3.5(iv) can be adapted to show that (4.3) holds and $R(\lambda, A) \in \mathcal{L}(X)$ is given by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt = \int_0^\infty e^{-(\lambda-a)t}S(t)x\,dt, \quad x \in X,\ \operatorname{Re}\lambda > a, \tag{4.4}$$

where the integral exists as an improper, X–valued Riemann integral. For the case when X is a Fréchet space, see [28, pp.165–166]; the general case follows the same lines.

It follows from (4.4) that $R(\lambda, A) = R(\lambda - a, A - a)$ for $\lambda \in \mathbb{C}_{a+}$. Then Remark 3.5(iv) implies that $R(\cdot, A)\colon \mathbb{C}_{a+} \to \mathcal{L}_b(X)$ is holomorphic. $\square$

Remark 4.5. In Lemma 4.4 it is not possible to weaken the exponential equicontinuity of $(T(t))_{t\geq 0}$ to local equicontinuity. To see this, consider $X := \mathbb{C}^{\mathbb{C}}$, i.e., the linear space of all functions $f\colon \mathbb{C} \to \mathbb{C}$, equipped with the topology of pointwise convergence. The seminorms in Γ_X can be chosen as $f \mapsto \max_{u\in F} |f(u)|$, for $f \in X$, where F runs through the family of all finite subsets of $\mathbb{C}$. Then X is a complete Montel lcHs. Let $\psi(u) := u$, for $u \in \mathbb{C}$, and define $A \in \mathcal{L}(X)$ by $Af := \psi f$, for $f \in X$. Given any $\lambda \in \mathbb{C}$, the element $f_\lambda := \chi_{\{\lambda\}} \in X \setminus \{0\}$ satisfies $Af_\lambda = \lambda f_\lambda$. Hence, $\sigma(A) = \mathbb{C}$ and so $\rho(A) = \emptyset$. For each $t \geq 0$, define $T(t) \in \mathcal{L}(X)$ by

$$T(t)f\colon u \mapsto e^{t\psi(u)} f(u) = e^{tu} f(u), \quad u \in \mathbb{C},$$

for each $f \in X$. Then $(T(t))_{t\geq 0}$ is a locally equicontinuous, uniformly continuous (as X is a Montel space) C_0–semigroup. Direct calculation (or an appeal to Lemma 4.4) shows that $(T(t))_{t\geq 0}$ is not exponentially equicontinuous.

For X a Banach space, the following result occurs in [15, p.119, Lemma 4.28].

Lemma 4.6. *Let X be a quasicomplete lcHs and $(T(t))_{t\geq 0}$ be an exponentially equicontinuous C_0–semigroup on X with infinitesimal generator $(A, D(A))$. If, for some $t_0 > 0$, the map $t \to T(t)$ is continuous at t_0 as an $\mathcal{L}_b(X)$–valued function and $R(\lambda, A)T(t_0)$ is Montel for some $\lambda \in \rho(A)$, then the operator $T(t)$ is Montel for every $t \geq t_0$.*

Proof. By exponential equicontinuity of $(T(t))_{t\geq 0}$ there is $a \geq 0$ such that $\{e^{-at}T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is equicontinuous. By Lemma 4.4 we see that $\mu := (a+1) \in \rho(A)$ and so $0 \in \rho(B)$, where $B := (A - \mu)$. Moreover, $D(B) = D(A)$ and B is the infinitesimal generator of $S(t) := e^{-\mu t}T(t)$ for $t \geq 0$. Since $S(t) = e^{-t}(e^{-at}T(t))$, for $t \geq 0$, it follows that $(S(t))_{t\geq 0}$ is an equicontinuous C_0–semigroup. Clearly, $t \mapsto S(t)$ is continuous at t_0 as an $\mathcal{L}_b(X)$–valued function and, via (3.1), we have that

$$R(0,B)S(t_0) = e^{-\mu t_0} R(\mu, A)T(t_0) = e^{-\mu t_0}[R(\lambda, A) + (\lambda - \mu)R(\mu, A)R(\lambda, A)]T(t_0)$$

which implies that $R(0, B)S(t_0)$ is Montel (as $R(\lambda, A)T(t_0)$ is Montel).

Consider the operators $V(t)$ defined by $V(t)x := \int_0^t S(s)x\, ds$ for every $x \in X$ and $t \geq 0$. Since $V(t) = tC(t)$, for $t \geq 0$, where $(C(t))_{t\geq 0}$ are the Cesáro means of $(S(t))_{t\geq 0}$, it follows from Section 2 that $V(t) \in \mathcal{L}(X)$, for all $t \geq 0$. Moreover, by (2.6) we have $BV(t) = S(t)x - x$, for $t \geq 0,\ x \in X$, and hence,

$$V(t) = R(0, B)(I - S(t)), \quad t \geq 0.$$

It follows that $V(t_0 + h) - V(t_0) = R(0, B)S(t_0)(I - S(h))$, $h \geq 0$, and hence, that the operators $V(t_0 + h) - V(t_0)$ are Montel as $R(0, B)S(t_0)$ is Montel. Observe, for each $h > 0$, that

$$\frac{(V(t_0 + h) - V(t_0))x}{h} - S(t_0)x = \frac{1}{h}\int_{t_0}^{t_0+h} (S(u) - S(t_0))x\, du, \quad x \in X.$$

Given $H \in \mathcal{B}(X)$ and $p \in \Gamma_X$, it follows from the previous formula that

$$p_H\left(\frac{(V(t_0+h)-V(t_0))}{h} - S(t_0)\right) \leq \frac{1}{h}\int_{t_0}^{t_0+h} p_H(S(u)-S(t_0))\,du \qquad (4.5)$$

for $h > 0$. Since $t \mapsto S(t)$ is continuous at t_0 as an $\mathcal{L}_b(X)$–valued function, given $\varepsilon > 0$ there is $\delta > 0$ such that $p_H(S(u) - S(t_0)) < \varepsilon$ for all $u \in [t_0, t_0+\delta]$. It follows from (4.5) that $p_H\left(\frac{(V(t_0+h)-V(t_0))}{h} - S(t_0)\right) < \varepsilon$ for all $h \in (0, \delta]$. Accordingly, $\lim_{h\to 0^+} \frac{(V(t_0+h)-V(t_0))}{h} = S(t_0)$ in $\mathcal{L}_b(X)$. The quasicompleteness of X ensures that the precompact (=totally bounded) and the relatively compact sets in X coincide, [22, pp.308–309]. Thus $S(t_0)$, being the limit in $\mathcal{L}_b(X)$ of Montel operators, is also Montel; see (3) on p.201 of [23]. Hence, also $T(t) = T(t-t_0)T(t_0) = T(t-t_0)e^{\mu t_0}S(t_0)$ is Montel, for each $t \geq t_0$. □

Let X be a lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be a semigroup. Then $(T(t))_{t\geq 0}$ is called *immediately Montel* if $T(t)$ is Montel for every $t > 0$ and $(T(t))_{t\geq 0}$ is called *eventually Montel* if there exists $t_0 > 0$ such that $T(t)$ is Montel for every $t > t_0$.

For Banach spaces the following result occurs in [15, p.119, Theorem 4.29].

Theorem 4.7. *Let X be a quasicomplete lcHs such that $\mathcal{L}_b(X)$ is sequentially complete and $(T(t))_{t\geq 0}$ be an exponentially equicontinuous C_0–semigroup on X. Then the following assertions are equivalent.*

(i) *$(T(t))_{t\geq 0}$ is immediately Montel.*

(ii) *$(T(t))_{t\geq 0}$ is continuous from $[0,\infty)$ into $\mathcal{L}_b(X)$ and its infinitesimal generator has Montel resolvent.*

Proof. (i)⇒(ii). Let $(A, D(A))$ be the infinitesimal generator of $(T(t))_{t\geq 0}$ and choose any $a \geq 0$ such that $\{e^{-at}T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is equicontinuous. Fix any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) > a$. By Lemma 4.3 (with $t_0 := 0$) the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous, i.e., continuous from $[0,\infty)$ into $\mathcal{L}_b(X)$. Fix $n \in \mathbb{N}$. By continuity of $t \mapsto e^{-\lambda t}T(t)$ from $[0,n]$ into the sequentially complete lcHs $\mathcal{L}_b(X)$ the Riemann integral $\int_0^n e^{-\lambda t}T(t)dt \in \mathcal{L}(X)$ is the limit of a *sequence* of Riemann sums of the form $\sum_{j=1}^k e^{-\lambda\xi_j}(t_j - t_{j-1})T(\xi_j)$, for some partition $0 = t_0 < t_1 < \ldots < t_k = n$ of $[0,n]$ and points $\xi_j \in (t_{j-1}, t_j]$ for $1 \leq j \leq k$; cf. proof of Theorem 10 in [5]. Since each such Riemann sum is Montel, also the $\mathcal{L}_b(X)$–limit $\int_0^n e^{-\lambda t}T(t)dt$ is Montel, [23, (3), p.201]. Hence, also the $\mathcal{L}_b(X)$–limit $\int_0^\infty e^{-\lambda t}T(t)dt = \lim_{n\to\infty}\int_0^n e^{-\lambda t}T(t)dt$ is Montel. Since the limit also exists in $\mathcal{L}_s(X)$, where it coincides with the resolvent operator $R(\lambda, A)$ (cf. (4.4)), we can conclude that $R(\lambda, A)$ is Montel. Hence, A has Montel resolvent.

(ii)⇒(i). Since $\rho(A) \neq \emptyset$ (cf. Lemma 4.4), there is (by assumption) some $\lambda \in \rho(A)$ with $R(\lambda, A)$ Montel. For any $t_0 > 0$ the operator $R(\lambda, A)T(t_0)$ is also Montel and hence, $T(t)$ is Montel for all $t \geq t_0$ as $(T(t))_{t\geq 0}$ is continuous from $[0,\infty)$ into $\mathcal{L}_b(X)$ (cf. Lemma 4.6). As t_0 is arbitrary, $T(t)$ is Montel for every $t > 0$, i.e., $(T(t))_{t\geq 0}$ is immediately Montel. □

Our final result is well known for Banach spaces, [15, p.318, Corollary 2.15].

Lemma 4.8. *Let X be a quasicomplete lcHs and $(T(t))_{t\geq 0}$ be an equicontinuous C_0–semigroup on X with infinitesimal generator $(A, D(A))$. Each of the following properties implies $\{T(t)x : t \geq 0\}$ is relatively compact in X, for every $x \in X$.*

(i) *A has Montel resolvent.*
(ii) *$(T(t))_{t\geq 0}$ is eventually Montel.*

Proof. According to Remark 2.2(i), the equicontinuity of $(T(t))_{t\geq 0}$ ensures the existence of a system Γ_X such that, for each $p \in \Gamma_X$, we have

$$p(T(t)x) \leq p(x), \quad x \in X,\, t \geq 0. \tag{4.6}$$

In case (i), fix any $\lambda_0 \in \rho(A)$. Then $R(\lambda_0, A)(B)$ is relatively compact in X for all $B \in \mathcal{B}(X)$. Fix $x \in D(A) = R(\lambda_0, A)(X)$, in which case $x = R(\lambda_0, A)y$ for some $y \in X$. It follows that

$$\{T(t)x :\ t \geq 0\} = \{R(\lambda_0, A)T(t)y :\ t \geq 0\} = R(\lambda_0, A)(\{T(t)y :\ t \geq 0\})$$

is relatively compact in X as $\{T(t)y :\ t \geq 0\} \in \mathcal{B}(X)$ by (4.6). Since $D(A)$ is dense in X (cf. Section 2), for any $z \in X \setminus D(A)$ there exists a net $(x_\alpha)_\alpha \subseteq D(A)$ such that $x_\alpha \to z$ in X. Then, given $p \in \Gamma_X$ and $\varepsilon > 0$, there exists α_0 such that $p(x_{\alpha_0} - z) < \varepsilon$. Hence, by (4.6), $\sup_{t\geq 0} p(T(t)(x_{\alpha_0} - z)) < \varepsilon$. It follows that

$$\{T(t)z :\ t \geq 0\} \subseteq \varepsilon U_p + \{T(t)x_{\alpha_0} :\ t \geq 0\},$$

where $U_p := \{y :\ p(y) < 1\}$. Since $\{T(t)x_{\alpha_0} :\ t \geq 0\}$ is relatively compact, X is quasicomplete and p, ε are arbitrary, this inclusion implies that $\{T(t)x :\ t \geq 0\}$ is also relatively compact.

In case (ii) there is $t_0 > 0$ with $T(t)$ Montel for $t \geq t_0$. Furthermore,

$$\{T(t)x :\ t \geq 0\} = \{T(t)x :\ t \in [0, t_0]\} \cup T(t_0)(\{T(s)x :\ s \geq 0\}), \quad x \in X.$$

Continuity of $t \mapsto T(t)x$ on $[0, t_0]$ implies $\{T(t)x :\ t \in [0, t_0]\}$ is compact. Also, $\{T(s)x :\ s \geq 0\} \in \mathcal{B}(X)$ via equicontinuity of $(T(t))_{t\geq 0}$. Hence, $T(t_0)(\{T(s)x :\ s \geq 0\})$ is relatively compact in X. So, $\{T(t)x :\ t \geq 0\}$ is relatively compact. $\square$

5. Mean ergodicity and uniformly continuous C_0–semigroups

In Banach spaces, various criteria for *uniform* mean ergodicity of a C_0–semigroup $(T(t))_{t\geq 0}$ are known; see [8, Ch.4, §3], [15, Ch.V, §4], [24, Ch.2], [25], and the references therein. Let $(A, D(A))$ be the infinitesimal generator of $(T(t))_{t\geq 0}$. Fundamental features involved in determining such criteria involve a combination of the existence (in the operator norm) of $\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$, closedness of the subspace ImA in X, and whether $0 \in \rho(A)$ or 0 is a simple pole of the resolvent $R(\cdot, A)$ relative to the operator norm. In this section, several of these criteria are extended to lcHs' but, *not* all. New phenomena arise which are not present in the Banach space setting and these lead to certain inherent problems. In view of the difficulties encountered in Section 3 with the spectral theory of closed linear operators in non–normable spaces, this is not totally unexpected. Some of the basic techniques for Banach spaces which are crucial for establishing various uniform mean ergodic theorems (eg., if $\|R(\lambda, A)\| \to 0$ as $\lambda \to 0$, then $\|R(\lambda, A)\| < 1$ for λ small enough, or the inequality $\mathrm{dist}(\lambda, \sigma(A)) \geq 1/\|R(\lambda, A)\|$ for $\lambda \in \rho(A)$, or that $\rho(A)$ is always open and is the natural domain in which $R(\cdot, A)$ is holomorphic) are simply not available in more general spaces. Nevertheless, many positive results remain valid.

We recall the closed subspace of X, [24, p.77], given by

$$\mathrm{Fix}(T(\cdot)) := \{x \in X : T(t)x = x,\ \forall t \geq 0\}.$$

For Banach spaces, the following result occurs in [15, Ch. V, Corollary 4.8].

Theorem 5.1. *Let X be a quasicomplete lcHs and $(T(t))_{t\geq 0}$ be an equicontinuous C_0-semigroup on X whose infinitesimal generator $(A, D(A))$ has Montel resolvent. Then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

Proof. Since $(T(t))_{t\geq 0}$ is equicontinuous and A has Montel resolvent, Lemma 4.8 implies, for every $x \in X$, that $B[x] := \{T(t)x : \ t \geq 0\}$ is relatively compact in X. Then $B[x]$ is also relatively $\sigma(X, X')$–compact and so, relatively countably $\sigma(X, X')$–compact, [5, Remark 6(i)]. By Remark 5(i) and Proposition 3 of [5] the semigroup $(T(t))_{t\geq 0}$ is mean ergodic, i.e., there exists a projection $P \in \mathcal{L}(X)$ such that τ_s-$\lim_{r\to\infty} C(r) = P$. To complete the proof we show that $C(r) \to P$ in $\mathcal{L}_b(X)$ as $r \to \infty$.

Since A has Montel resolvent, it follows from Proposition 4.2 that the canonical inclusion $i\colon X[A] \hookrightarrow X$ is Montel (recall $X[A] = (D(A), \{p^A\}_{p\in\Gamma_X})$). On the other hand, the linear operator $V : x \mapsto \int_0^1 T(t)x\,dt$, for $x \in X$, is continuous from X into $X[A]$. Indeed, choose Γ_X to satisfy (4.6). Recalling the identity (2.6), it follows from (4.1) and (4.6) that

$$p^A(Vx) = p\left(\int_0^1 T(t)x\,dt\right) + p\left(A\int_0^1 T(t)x\,dt\right) \leq p(x) + p(T(1)x - x) \leq 3p(x),$$

for all $x \in X$ and $p \in \Gamma_X$. As $V = i \circ V$, we can conclude that $V\colon X \to X$ is a Montel operator. Moreover, since P is a projection onto $\mathrm{Fix}(T(\cdot))$, [5, Remark 4(ii)], we have $PV = P$ because of $PVx = \int_0^1 T(t)Px\,dt = \int_0^1 Px\,dt = Px$, for $x \in X$. So, $(C(r) - P)V = C(r)V - P$ for every $r > 0$. Using the facts that V is Montel and that τ_s-$\lim_{r\to\infty} C(r) = P$, we obtain that $(C(r) - P)V \to 0$ in $\mathcal{L}_b(X)$, [23, (2) p.139], i.e., that $C(r)V - P \to 0$ in $\mathcal{L}_b(X)$ as $r \to \infty$. On the other hand, by applying [5, Proposition 11] we can adapt the formulae on p.341 of [15] to the lc–setting to yield, for $r > 0$ and $x \in X$, that

$$C(r)Vx - C(r)x = \frac{1}{r}\int_0^1 \left(\int_r^{r+s} T(t)x\,dt - \int_0^s T(t)x\,dt\right) ds\,. \tag{5.1}$$

Given $B \in \mathcal{B}(X)$ and $p \in \Gamma_X$, it follows from (4.6) that, for each $r > 0$,

$$\sup_{x\in B} p\left(\frac{1}{r}\int_0^1 \left(\int_r^{r+s} T(t)x\,dt - \int_0^s T(t)x\,dt\right) ds\right) \leq \frac{1}{r}\int_0^1 2s \sup_{x\in B} p(x)\,ds = \frac{p_B(I)}{r}.$$

It then follows from (5.1) that τ_b-$\lim_{r\to\infty}(C(r)V - C(r)) = 0$. But, for $r > 0$, we have $C(r) - P = (C(r) - C(r)V) + (C(r)V - P)$ and so τ_b-$\lim_{r\to\infty} C(r) = P$. □

In order to proceed further we require two preliminary results. The first one follows the lines of [8, p.31, Proposition 1.4.5]; see also p.112 in [8]

Lemma 5.2. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ a locally equicontinuous C_0-semigroup on X with infinitesimal generator $(A, D(A))$ such that*

$$\left\{e^{-at}\int_0^t T(s)\,ds : t \geq 0\right\} \subseteq \mathcal{L}(X) \text{ is equicontinuous,} \tag{5.2}$$

for some $a \geq 0$. Then $\mathbb{C}_{a^+} \subseteq \rho(A)$ and

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}T(t)\,dt = \lambda\int_0^\infty se^{-\lambda s}C(s)\,ds, \quad \lambda \in \mathbb{C}_{a^+}. \tag{5.3}$$

Proof. Fix $\lambda \in \mathbb{C}_{a^+}$ and $x \in X$. We show that the improper Riemann integral $\int_0^\infty e^{-\lambda t}T(t)x\,dt$ exists in X. Define $F\colon [0,\infty) \to X$ by $F(t) := \int_0^t T(s)x\,ds$. Let $p \in \Gamma_X$. Then for $t,\ s \in [0,\infty)$ we have

$$p(F(t) - F(s)) \leq \left| \int_s^t p(T(u)x)\,du \right| \leq |t-s|K_x$$

for some $K_x > 0$, which exists via the local equicontinuity of $(T(t))_{t\geq 0}$ on an interval $[0,N]$ chosen large enough to contain, say t, and all $s \in [0,\infty)$ satisfying $|t-s| \leq 1$. It is then clear that F is continuous. Integrating by parts yields

$$\begin{aligned} \int_0^t e^{-\lambda s}T(s)x\,ds &= [e^{-\lambda s}F(s)]_{s=0}^{s=t} + \lambda \int_0^t e^{-\lambda s}F(s)\,ds \\ &= e^{-\lambda t}F(t) + \lambda \int_0^t e^{-\lambda s}F(s)\,ds. \end{aligned} \tag{5.4}$$

For a fixed $p \in \Gamma_X$, via (5.2) there exist $M_p > 0$ and $q \in \Gamma_X$ such that

$$p(e^{-\lambda t}F(t)) \leq M_p e^{(a - \mathrm{Re}(\lambda))t} q(x), \quad t \geq 0. \tag{5.5}$$

Since $\mathrm{Re}(a-\lambda) < 0$ it follows from (5.5) that $e^{-\lambda t}F(t) \to 0$ in X as $t \to \infty$. Also, the sequential completeness of X and the inequalities

$$p\left(\int_m^n e^{-\lambda s}F(s)\,ds \right) \leq M_p q(x) \int_m^n e^{\mathrm{Re}(a-\lambda)s}\,ds, \quad m < n \text{ in } \mathbb{N},$$

show that the improper Riemann integral $\int_0^\infty e^{-\lambda s}F(s)\,ds$ exists in X. It is then immediate from (5.4) that also the improper Riemann integral $\int_0^\infty e^{-\lambda s}T(s)x\,ds = \lim_{n\to\infty} \int_0^n e^{-\lambda s}T(s)x\,ds$ exists in X and equals $\lambda \int_0^\infty e^{-\lambda s}F(s)\,ds = \lambda \int_0^\infty s e^{-\lambda s}C(s)x\,ds$. Combining (5.4) and (5.5) we also obtain that

$$\begin{aligned} p\left(\int_0^t e^{-\lambda s}T(s)x\,ds \right) &\leq p(e^{-\lambda t}F(t)) + M_p|\lambda|q(x) \int_0^t e^{\mathrm{Re}(a-\lambda)s}\,ds \\ &= p(e^{-\lambda t}F(t)) + M_p|\lambda|q(x) \frac{1 - e^{\mathrm{Re}(a-\lambda)t}}{\mathrm{Re}(\lambda - a)}, \end{aligned}$$

for $t \geq 0$. Letting $t \to \infty$ yields $p\left(\int_0^\infty e^{-\lambda s}T(s)x\,ds\right) \leq M_p \frac{|\lambda|}{\mathrm{Re}(\lambda-a)} q(x)$. As the previous inequality holds for every $x \in X$ and $p \in \Gamma_X$, the operator $R(\lambda)\colon x \mapsto \int_0^\infty e^{-\lambda s}T(s)x\,ds$ belongs to $\mathcal{L}(X)$ whenever $\lambda \in \mathbb{C}_{a^+}$. According to Remark 3.5(iv) (see also the proof of Lemma 4.4) we also have $R(\lambda) = R(\lambda, A)$. ☐

Remark 5.3. Suppose that $(T(t))_{t\geq 0}$ is exponentially equicontinuous of some *positive* order, i.e., satisfies (2.2) for some $a > 0$. Then, in the notation of (2.2) we have, for each $t \geq 0$, that

$$\begin{aligned} p\left(\int_0^t T(s)x\,ds \right) &\leq \int_0^t p(T(s)x)\,ds \leq M_p q(x) \int_0^t e^{as}\,ds \\ &= M_p q(x) \left[\frac{e^{as}}{a} \right]_{s=0}^{s=t} \leq \frac{2M_p}{a} e^{at} q(x), \end{aligned}$$

for each $x \in X$. So, (5.2) is necessarily satisfied. If $(T(t))_{t\geq 0}$ is equicontinuous, then for $\beta > 0$ we have $p\left(\int_0^t T(s)x\,ds\right) \leq M_p q(x) \int_0^t ds \leq M_p e^{\beta t} q(x)$ and so (5.2) is satisfied for every $a > 0$. In this case we recover Remark 3.5(iv) from Lemma 5.2.

Lemma 5.4. *Let X be a lcHs such that $\mathcal{L}_s(X)$ is sequentially complete and $K\colon [0,\infty) \to \mathbb{C}$ be a continuous, Lebesgue integrable function with*

$$\int_0^\infty K(y) y^{-i\xi} dy \neq 0, \quad \xi \in \mathbb{R}. \tag{5.6}$$

Suppose that $\Phi\colon [0,\infty) \to \mathcal{L}_s(X)$ is a bounded, continuous function and that there exists $P \in \mathcal{L}(X)$ such that

$$\tau_s - \lim_{\lambda \to 0^+} \lambda \int_0^\infty K(\lambda t)\Phi(t)\, dt = \left(\int_0^\infty K(t)\, dt \right) P. \tag{5.7}$$

Then the following limit exists in $\mathcal{L}_s(X)$:

$$\lim_{\lambda \to 0^+} \lambda \int_0^\infty \chi_{[0,1]}(\lambda t)\Phi(t)\, dt = P. \tag{5.8}$$

If $\mathcal{L}_b(X)$ is sequentially complete, $\Phi\colon [0,\infty) \to \mathcal{L}_b(X)$ is bounded and continuous, and the limit (5.7) *exists in $\mathcal{L}_b(X)$ for some $P \in \mathcal{L}(X)$, then also the limit* (5.8) *exists in $\mathcal{L}_b(X)$.*

Proof. Since $t \mapsto K(\lambda t)\Phi(t)$ is a continuous $\mathcal{L}_s(X)$–valued function on $[0,\infty)$, for fixed $\lambda > 0$, the Riemann integral $\int_0^n K(\lambda t)\Phi(t)dt \in \mathcal{L}(X)$ is defined for each $n \in \mathbb{N}$. The existence of the improper Riemann integral $\int_0^\infty K(\lambda t)\Phi(t)dt \in \mathcal{L}(X)$ follows as $\{\int_0^n K(\lambda t)\Phi(t)dt\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}_s(X)$. This can be seen from the estimates

$$p\left(\int_m^n K(\lambda t)\Phi(t)x\, dt \right) \leq \int_m^n |K(\lambda t)| p(\Phi(t)x)\, dt \leq \alpha_{x,p} \int_m^n |K(\lambda t)|\, dt,$$

for each $n > m$ in $\mathbb{N}$ and all $x \in X$, $p \in \Gamma_X$, where $\alpha_{x,p} := \sup_{t \geq 0} p(\Phi(t)x) < \infty$ as Φ has bounded range in $\mathcal{L}_s(X)$.

Fix $\alpha > 0$ and define $K_\alpha(t) := K(\alpha t)$ for all $t \geq 0$. Then, with $\mu = \lambda\alpha$ we have, for each $x \in X$, that

$$\begin{aligned} &p\left(\lambda \int_0^\infty K_\alpha(\lambda t)\Phi(t)x\, dt - \left(\int_0^\infty K_\alpha(t)\, dt \right) Px \right) \\ &\qquad = \frac{1}{\alpha} p\left(\mu \int_0^\infty K(\mu t)\Phi(t)x\, dt - \left(\int_0^\infty K(s)\, ds \right) Px \right). \end{aligned}$$

Let $\lambda \to 0^+$ (i.e., $\mu \to 0^+$) and apply (5.7) to conclude $\tau_s\text{-}\lim_{\lambda \to 0^+} \lambda \int_0^\infty K_\alpha(\lambda t)\Phi(t)\, dt = \left(\int_0^\infty K_\alpha(t)\, dt\right) P$. It follows immediately that also

$$\tau_s - \lim_{\lambda \to 0^+} \lambda \int_0^\infty f(\lambda t)\Phi(t)\, dt = \left(\int_0^\infty f(t)\, dt \right) P, \tag{5.9}$$

for every $f \in \operatorname{span}\{K_\alpha : \alpha > 0\}$.

Observe that $t \mapsto \chi_{[0,1]}(\lambda t)\Phi(t) = \chi_{[0,\lambda^{-1}]}(\lambda t)\Phi(t)$, for $t \in [0,\infty)$, is improper Riemann integrable in $\mathcal{L}_s(X)$ with $\int_0^\infty \chi_{[0,1]}(\lambda t)\Phi(t)dt = \int_0^{1/\lambda} \Phi(t)dt$. By Wiener's Theorem, [6, Theorem 3.10], there exists a sequence $\{\sigma_n\}_{n=1}^\infty \subseteq$

span$\{K_\alpha : \alpha > 0\}$ of Lebesgue integrable (continuous) functions such that $\sigma_n \to \chi_{[0,1]}$ in $L^1([0,\infty))$. For $x \in X$ and $p \in \Gamma_X$ fixed, we have

$$p\left(\lambda \int_0^\infty \chi_{[0,1]}(\lambda t)\Phi(t)x\,dt - Px\right)$$

$$\leq p\left(\lambda \int_0^\infty [\chi_{[0,1]}(\lambda t) - \sigma_n(\lambda t)]\Phi(t)x\,dt\right) \tag{5.10}$$

$$+p\left(\lambda \int_0^\infty \sigma_n(\lambda t)\Phi(t)x\,dt - \left(\int_0^\infty \sigma_n(t)\,dt\right)Px\right) \tag{5.11}$$

$$+p\left(\int_0^\infty [\sigma_n(t) - \chi_{[0,1]}(t)]Px\,dt\right). \tag{5.12}$$

It follows via (5.9) that (5.11) tends to 0 as $\lambda \to 0^+$, for $n \in \mathbb{N}$. The estimate

$$p\left(\lambda \int_0^\infty [\chi_{[0,1]}(\lambda t) - \sigma_n(\lambda t)]\Phi(t)x\,dt\right) \leq \alpha_{x,p} \int_0^\infty |\chi_{[0,1]}(\lambda t) - \sigma_n(\lambda t)|\,d(\lambda t)$$

together with $\sigma_n \to \chi_{[0,1]}$ in $L^1([0,\infty))$ imply the right–side of (5.10) tends to 0 as $n \to \infty$. Also the inequality

$$p\left(\int_0^\infty [\sigma_n(t) - \chi_{[0,1]}(t)]Px\,dt\right) \leq p(Px) \int_0^\infty |\sigma_n(t) - \chi_{[0,1]}(t)|\,dt$$

implies that (5.12) tends to 0 as $n \to \infty$. Accordingly, the left–side of the chain of inequalities (5.10)–(5.12) tends to 0 as $\lambda \to 0^+$.

Since the seminorms $S \mapsto p(Sx)$, for $S \in \mathcal{L}(X)$, generate τ_s as we vary $x \in X$ and $p \in \Gamma_X$, the identity (5.8) follows.

For the case when $\mathcal{L}_b(X)$ is sequentially complete and $\Phi\colon [0,\infty) \to \mathcal{L}_b(X)$ is bounded and continuous a similar proof applies. □

The following result, connecting mean ergodicity of $(T(t))_{t\geq 0}$ with its Abel–mean ergodicity, is inspired by [8, p.265, Proposition 4.3.4].

Theorem 5.5. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be a locally equicontinuous C_0-semigroup on X with infinitesimal generator $(A, D(A))$.*

(i) *If $(T(t))_{t\geq 0}$ is mean ergodic (resp. uniformly mean ergodic) and $\{C(r)\}_{r\geq 0}$ is equicontinuous, then $\mathbb{C}_{0^+} \subseteq \rho(A)$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic (resp. uniformly Abel mean ergodic).*

(ii) *If $\mathcal{L}_s(X)$ is sequentially complete, $(T(t))_{t\geq 0}$ is a bounded set in $\mathcal{L}_s(X)$ and $(T(t))_{t\geq 0}$ is Abel mean ergodic (resp. uniformly Abel mean ergodic with $\mathcal{L}_b(X)$ sequentially complete), then $(T(t))_{t\geq 0}$ is mean ergodic (resp. uniformly mean ergodic).*

Proof. (i) By hypothesis $\{C(r)\}_{r\geq 0}$ converges in $\mathcal{L}_s(X)$ to some $P \in \mathcal{L}(X)$. By equicontinuity of $\{C(r)\}_{r\geq 0}$ we have

$$\forall p \in \Gamma_X\ \exists M_p > 0,\ q \in \Gamma_X \text{ with } p(C(r)x) \leq M_p q(x),\ \forall x \in X,\ r \geq 0. \tag{5.13}$$

Equivalently, via (2.7), we have

$$p\left(\int_0^r T(s)x\,ds\right) \leq M_p r q(x), \quad \forall x \in X,\ r \geq 0. \tag{5.14}$$

Given $a > 0$ we have $r < \frac{e^{ar}}{a}$ for $r \geq 0$. Then (5.14) yields $p\left(\int_0^r T(s)x\,ds\right) \leq \frac{M_p}{a}e^{ar}q(x)$, for $x \in X$, $r \geq 0$, which is precisely (5.2) and so Lemma 5.2 implies that $\mathbb{C}_{a+} \subseteq \rho(A)$. Consequently, $\mathbb{C}_{0+} \subseteq \rho(A)$. If $\mathrm{Re}(\lambda) > 0$, then $a := \mathrm{Re}(\lambda)/2 > 0$ and we have (cf. (5.3)) that $\lambda R(\lambda, A)x = \lambda^2 \int_0^\infty se^{-\lambda s}C(s)x\,ds$, for $x \in X$. Via this identity and $\lambda^2 \int_0^\infty se^{-\lambda s}ds = 1$ (for all $\lambda \in (0, \infty)$) we have, for each $p \in \Gamma_X$, that

$$p(\lambda R(\lambda, A)x - Px) \leq \int_0^\infty \lambda^2 se^{-\lambda s}p(C(s)x - Px)ds \leq \int_0^\infty te^{-t}p(C(t/\lambda)x - Px)\,dt, \tag{5.15}$$

for all $x \in X$ and $\lambda \in (0, \infty)$. By the mean ergodicity of $(T(t))_{t\geq 0}$, for every $x \in X$ we have $p(C(t/\lambda)x - Px) \to 0$ pointwise for t in $[0, \infty)$ as $\lambda \downarrow 0^+$, with $\sup_{t\geq 0,\, \lambda>0} p(C(t/\lambda)x - Px) < \infty$ by (5.13). Hence, by the dominated convergence theorem, (5.15) implies that $\lim_{\lambda\downarrow 0^+} p(\lambda R(\lambda, A)x - Px) = 0$. Accordingly, τ_s-$\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A) = P$ exists.

Suppose now that $(T(t))_{t\geq 0}$ is uniformly mean ergodic. Fix $B \in \mathcal{B}(X)$. By (5.15) we have $p_B(\lambda R(\lambda, A) - P) \leq \int_0^\infty te^{-t}p_B(C(t/\lambda) - P)dt$, for $\lambda \in (0, \infty)$. Since X is sequentially complete, $\mathcal{L}_s(X)$ and $\mathcal{L}_b(X)$ have the same bounded sets, [23, (3), p.135] and so $\sup_{t\geq 0,\ \lambda>0} p_B(C(t/\lambda) - P) < \infty$. As $p_B(C(t/\lambda) - P) \to 0$ pointwise for t in $[0, \infty)$ with $\lambda \downarrow 0^+$ (by uniform mean ergodicity of $(T(t))_{t\geq 0}$), the dominated convergence theorem yields $\lim_{\lambda\downarrow 0^+} p_B(\lambda R(\lambda, A) - P) = 0$, i.e., $\lambda R(\lambda, A) \to P$ in $\mathcal{L}_b(X)$ as $\lambda \downarrow 0^+$.

(ii) By assumption there is $P \in \mathcal{L}(X)$ such that $\lambda R(\lambda, A) \to P$ in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) as $\lambda \downarrow 0^+$. For $r > 0$ set $\lambda := \frac{1}{r}$. Since the improper Riemann integral

$$x \mapsto \lambda \int_0^\infty \chi_{[0,1]}(\lambda s)T(s)x\,ds = \lambda \int_0^{\frac{1}{\lambda}} T(s)x\,ds = C(r)x, \quad x \in X,$$

exists in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) it suffices to show that

$$\lambda \int_0^\infty \chi_{[0,1]}(\lambda s)T(s)\,ds \to P \text{ in } \mathcal{L}_s(X) \text{ (resp. in } \mathcal{L}_b(X)) \text{ as } \lambda \downarrow 0^+. \tag{5.16}$$

To this effect, observe that the continuous function $K(s) := e^{-s}$, for $s \geq 0$, is Lebesgue integrable and $\int_0^\infty K(y)y^{-i\xi}\,dy = \int_0^\infty e^{-y}y^{-i\xi}\,dy = \Gamma(1 - i\xi) \neq 0$, for $\xi \in \mathbb{R}$, where Γ is the Euler gamma function. Let $\Phi\colon [0, \infty) \to \mathcal{L}_s(X)$ (resp. $\mathcal{L}_b(X)$) be the bounded, continuous function $T(\cdot)$. Then $\lambda \int_0^\infty K(\lambda t)\Phi(t)\,dt = \lambda \int_0^\infty e^{-\lambda t}T(t)\,dt = \lambda R(\lambda, A)$, for $\lambda \in (0, \infty)$, converges to P in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$) as $\lambda \downarrow 0^+$. So, Lemma 5.4 yields that (5.16) does indeed hold. $\square$

Remark 5.6. (i) Concerning part (i) of Theorem 5.5 we note that the equicontinuity of $\{C(r)\}_{r\geq 0}$ is automatic whenever X is barrelled. For, in this case, the convergence of $\{C(r)\}_{r\geq 0}$ in $\mathcal{L}_s(X)$ as $r \to \infty$ ensures its boundedness in $\mathcal{L}_s(X)$. Indeed, let $P := \tau_s$-$\lim_{r\to\infty} C(r)$. Fix $x \in X$ and $p \in \Gamma_X$. Then there exists $r_0 > 0$ such that $p(Px - C(r)x) \leq 1$, for all $r \geq r_0$, and so $\sup_{r\geq r_0} p(C(r)x) < \infty$. Since $r \mapsto C(r)x$ is continuous on $[0, r_0]$, [5, Lemma 1], also $\sup_{0\leq r\leq r_0} p(C(r)x) < \infty$. It then follows that $\sup_{r\geq 0} p(C(r)x) < \infty$, i.e., $\{C(r)x\}_{r\geq 0} \in \mathcal{B}(X)$.

(ii) The boundedness of $(T(t))_{t\geq 0}$ in $\mathcal{L}_s(X)$ *cannot* be omitted in part (ii) of Theorem 5.5. Indeed, let $X := \mathbb{C}^2$ and $T(t) := e^{it}\begin{pmatrix} 1 & t \\ 0 & t \end{pmatrix}$, for $t \geq 0$. Then

$\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$ exists in $\mathcal{L}_b(X)$ but, $(T(t))_{t\geq 0}$ fails to be mean ergodic, [8, p.266].

For Banach spaces the following result, [15, Ch. V, Theorem 4.10], [25], due to M. Lin, is fundamental; see also [8, Proposition 4.3.15].

Theorem 5.7. *Let X be a Banach space and $(T(t))_{t\geq 0}$ be a uniformly bounded (i.e., equicontinuous) C_0-semigroup on X with infinitesimal generator $(A, D(A))$. Then the following assertions are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is uniformly mean ergodic.*
(ii) $(T(t))_{t\geq 0}$ *is uniformly Abel mean ergodic.*
(iii) $\mathrm{Im}A$ *is a closed subspace of X.*
(iv) $0 \in \rho(A)$ *or* 0 *is a simple pole of the resolvent map* $R(\cdot, A)$ *of* A.

If $(T(t))_{t\geq 0}$ is equicontinuous, then also $\{C(r)\}_{r\geq 0}$ is equicontinuous; see Section 2. Consequently, Theorem 5.5 asserts that (i)$\Leftrightarrow$(ii) in Theorem 5.7 carries over to the setting of X a lcHs (under mild restrictions). The same is *not* true of conditions (iii) and (iv) in Theorem 5.7, even for Fréchet spaces.

Example 5.8. Let $X := s$ be the nuclear Fréchet space of all rapidly decreasing sequences $x = (x_i)_{i=1}^\infty \in \mathbb{C}^{\mathbb{N}}$, i.e., for which $p_n(x) := \sup_{j\in\mathbb{N}} j^n|x_j|$, for $x = (x_j)_{j=1}^\infty \in s$, is finite for each $n \in \mathbb{N}$. Then $\Gamma_X = \{p_n\}_{n=1}^\infty$ is an increasing sequence of seminorms determining the topology of s. Consider the operator $B \in \mathcal{L}(s)$ given by $Bx := ((1 - 2^{-j})x_j)_{j=1}^\infty$, for $x = (x_j)_{j=1}^\infty \in s$. It is shown in [1, Example 2.17] that B is power bounded. According to Remark 2.4 the C_0–semigroup $(e^{tB})_{t\geq 0}$ is exponentially equicontinuous and $T(t) := e^{-t}e^{tB}$, for $t \geq 0$, is an *equicontinuous*, uniformly continuous C_0–semigroup with infinitesimal generator $A := B - I$ (and $D(A) = X$). Since s is Montel, $(T(t))_{t\geq 0}$ is *uniformly mean ergodic*, [5, Corollary 2(ii)], i.e., condition (i) of Theorem 5.7 is satisfied.

Observe that $Ax = (-2^{-j}x_j)_{j=1}^\infty$, for $x = (x_j)_{j=1}^\infty \in s$. By considering the standard unit basis vectors $\{e_n\}_{n=1}^\infty$ of s it is routine to check that each $\lambda_j := -2^{-j}$, for $j \in \mathbb{N}$, is an eigenvalue of A and hence, $\{\lambda_j\}_{j=1}^\infty \subseteq \sigma(A)$. It is shown in [1, Example 2.17] that A is not surjective and so also $0 \in \sigma(A)$. For each $\lambda \notin \{0\} \cup \{\lambda_j\}_{j=1}^\infty$ it can be verified that the linear map $R_\lambda \colon x \mapsto \left(\frac{1}{\lambda+2^{-j}}x_j\right)_{j=1}^\infty$, for $x \in s$, belongs to $\mathcal{L}(s)$ and satisfies $R_\lambda(\lambda - A) = I = (\lambda - A)R_\lambda$, i.e., $R(\lambda, A) = R_\lambda$. Hence, $\sigma(A) = \{0\} \cup \{\lambda_j\}_{j=1}^\infty$ and so $\rho(A)$ is surely open in $\mathbb{C}$.

Clearly, $0 \notin \rho(A)$. Moreover, $0 \in \sigma(A)$ is *not* a simple pole of $R(\cdot, A)$ since there is *no* punctured disc, centred in 0, which is contained in $\rho(A)$. So, for the assertions in Theorem 5.7, we see that (i)$\not\Rightarrow$(iv).

Since the basis $\{e_n\}_{n=1}^\infty \subseteq \mathrm{Im}A$, we see that $\mathrm{Im}A$ is dense in s. But, A is not surjective and so $\mathrm{Im}A$ is not closed in X (if so, it would be equal to X). So, for the assertions of Theorem 5.7, we also have (i)$\not\Rightarrow$(iii). □

In relation to Theorem 5.7, Example 5.8 shows that entirely new phenomena arise in non–normable lcHs' which are simply not present for Banach spaces. We proceed to formulate some analogues which do hold in lcHs'. In view of [5, Corollary 2(ii)], the following result is mainly of interest in non–Montel spaces.

Proposition 5.9. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup with* $\mathrm{Im}A$ *closed in X, where $(A, D(A))$ is its* *infinitesimal generator. Then $(T(t))_{t\geq 0}$ is mean ergodic iff it is uniformly mean ergodic.*

Proof. Suppose that $(T(t))_{t\geq 0}$ is mean ergodic, i.e., there is $P \in \mathcal{L}(X)$ such that τ_s-$\lim_{r\to\infty} C(r) = P$. It is known that P is a projection and satisfies

$$\mathrm{Im}P = \mathrm{Fix}(T(\cdot)) \quad \text{and} \quad \ker P = \overline{\mathrm{span}\{x - T(t)x : t \geq 0,\ x \in X\}},$$

[5, Remark 4(ii)]. It follows from these identities and Remark 5(iii) of [5] that $\mathrm{Im}P = \ker A$ and $\ker P = \overline{\mathrm{Im}A}$. Since $\mathrm{Im}A$ is closed, we have the direct decomposition

$$X = \ker A \oplus \mathrm{Im}A. \tag{5.17}$$

Noting that $T(t)x = x$, for $t \geq 0$, whenever $x \in \ker A = \mathrm{Fix}(T(\cdot))$, we have

$$C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt = x, \quad \forall x \in \ker A. \tag{5.18}$$

Observe that $Y := \mathrm{Im}A$ is invariant for $\{rC(r)\}_{r\geq 0}$. Indeed, if $y = Ax \in Y$ (with $x \in D(A)$), then (2.5) and (2.6) imply that

$$rC(r)y = \int_0^r T(t)Ax\,dt = A\int_0^r T(t)x\,dt \in Y. \tag{5.19}$$

So, the restriction $rC_Y(r)$ of $rC(r)$ to Y, for each $r \geq 0$, belongs to $\mathcal{L}(Y)$. Let Γ_X satisfy (4.6). If $y = Ax \in Y$, then (2.5), (4.6) and (5.19) yield

$$p(rC_Y(r)y) = p(T(r)x - x) \leq 2p(x), \quad r \geq 0, p \in \Gamma_X.$$

Hence, $\{rC_Y(r)\}_{r\geq 0}$ is bounded in $\mathcal{L}_s(Y)$. But, Y is sequentially complete and so $\{rC_Y(r)\}_{r\geq 0}$ is also bounded in $\mathcal{L}_b(Y)$, [23, (3), p.135]. Accordingly,

$$\tau_b - \lim_{r\to\infty} C_Y(r) = \tau_b - \lim_{r\to\infty} \frac{1}{r}(rC_Y(r)) = 0. \tag{5.20}$$

To complete this part of the proof we show that

$$\tau_b - \lim_{r\to\infty} C(r) = P\,. \tag{5.21}$$

Fix $B \in \mathcal{B}(X)$. If $u \in B$, then $u = Pu \oplus (I-P)u$ with $Pu \in \mathrm{Im}P = \ker A$ and so, by (5.18), we have $C(r)Pu = Pu$, for $r \geq 0$. It follows that

$$(C(r) - P)u = (C(r) - P)(Pu \oplus (I-P)u) = C(r)(I-P)u.$$

Moreover, $(I-P)u \in \ker P = \mathrm{Im}A = Y$ which shows that $D := (I-P)(B) \subseteq Y$, i.e., $D \in \mathcal{B}(Y)$. Given $p \in \Gamma_X$ we have

$$p((C(r) - P)u) = p(C(r)(I-P)u) \leq \sup_{y\in D} p(C(r)y) = p_D(C_Y(r)),$$

for all $r \geq 0$, i.e., $p_B(C(r) - P) \leq p_D(C_Y(r))$, for all $r \geq 0$ with $D \in \mathcal{B}(Y)$. Then (5.21) follows from (5.20). Hence, $(T(t))_{t\geq 0}$ is uniformly mean ergodic.

Conversely, if $(T(t))_{t\geq 0}$ is uniformly mean ergodic, then it is surely mean ergodic (even without $\mathrm{Im}A$ being closed). □

We proceed to extend Proposition 5.9 for which some preliminaries are needed. A directed family $\{A(\alpha)\}_{\alpha\in\Lambda} \subseteq \mathcal{L}(X)$, which we assume *commutes* with the C_0–semigroup $(T(t))_{t\geq 0}$, is an *ergodic net* for $(T(t))_{t\geq 0}$ if:

(E1) $A(\alpha)x \in \overline{\mathrm{co}\{T(t)x : t \geq 0\}}$, for all $\alpha \in \Lambda$ and $x \in X$.
(E2) $\{A(\alpha)\}_{\alpha\in\Lambda}$ is equicontinuous in $\mathcal{L}(X)$.

(E3) For each $t \geq 0$ we have $\tau_s - \lim_\alpha A(\alpha)(T(t) - I) = 0$.

This is a version of a more general definition due to W.F. Eberlein, [14], [24, Ch. 2]. Perhaps, the most familiar ergodic net is the directed family $\{C(r)\}_{r\geq 0}$ of all Cesàro operators (under suitable conditions on X and $(T(t))_{t\geq 0}$), [5]. There is an alternate ergodic net available.

Proposition 5.10. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0}$ be an equicontinuous C_0–semigroup on X with infinitesimal generator $(A, D(A))$. Then the directed family $\{\lambda R(\lambda, A) : 0 < \lambda \leq 1\}$ is an ergodic net for $(T(t))_{t\geq 0}$.*

Proof. According to Remark 3.5(iv) we have $\mathbb{C}_+ \subseteq \rho(A)$ and, via (3.12), that $R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt$, for all real $\lambda > 0$ and $x \in X$.

To verify (E1) suppose that $\lambda R(\lambda, A)x \notin \overline{\mathrm{co}\{T(t)x : t \geq 0\}}$ for some $x \in X$ and $\lambda > 0$. Let $X_{\mathbb{R}}$ denote X considered as a vector space over $\mathbb{R}$. Then there exists $u \in (X_{\mathbb{R}})'$ and $\beta \in \mathbb{R}$ such that $\langle y, u\rangle < \beta < \langle \lambda R(\lambda, A)x, u\rangle$, for all $y \in \overline{\mathrm{co}\{T(t)x : t \geq 0\}} =: B_T[x]$, [19, p.131 Theorem 7.3.4]. So,

$$\sup_{y\in B_T[x]} \langle y, u\rangle \leq \beta < \langle \lambda R(\lambda, A)x, u\rangle.$$

Choose $s > 0$ with $\beta + s < \langle \lambda R(\lambda, A)x, u\rangle$. Then (3.12) yields

$$\beta + s < \langle \lambda R(\lambda, A)x, u\rangle = \lambda \int_0^\infty e^{-\lambda t}\langle T(t)x, u\rangle\, dt \leq \lambda \int_0^\infty e^{-\lambda t} \beta\, dt = \beta;$$

contradiction. So, (E1) holds.

Concerning (E2), for each $p \in \Gamma_X$ we may assume that (3.13) holds (because of the equicontinuity of $(T(t))_{t\geq 0}$). Then (3.12) yields

$$p(\lambda R(\lambda, A)x) \leq \lambda \int_0^\infty e^{-\lambda t} p(T(t)x)\, dt \leq \lambda p(x) \int_0^\infty e^{-\lambda t}\, dt = p(x),$$

for each $x \in X$ and $\lambda > 0$. So, $\{\lambda R(\lambda, A) : 0 < \lambda \leq 1\}$ is equicontinuous.

To verify (E3), let $x \in X$ and $t \geq 0$. Then, with $y := \int_0^t T(s)x\, ds \in D(A)$, it follows from (3.24) that $\lambda R(\lambda, A)(T(t) - I)x = -\lambda y + \lambda^2 R(\lambda, A)y$. Hence, (E2) ensures that $\lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)(T(t) - I)x = 0$, i.e., (E3) holds. □

Combining Proposition 5.10 with Eberlein's Theorem, [14, Theorem 3.1], [24, Ch. 2, Theorem 1.5, p.76], yields the following fact.

Corollary 5.11. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup. For every $x, y \in X$ the following assertions are equivalent.*

(i) *$y \in \mathrm{Fix}(T(\cdot))$ and $y \in \overline{\mathrm{co}\{T(t)x : t \geq 0\}}$.*
(ii) *$y = \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)x$ in X.*
(iii) *$y = \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)x$ in X_σ.*
(iv) *y is a $\sigma(X, X')$ cluster point of the net $\{\lambda R(\lambda, A)x : 0 < \lambda \leq 1\}$.*

We can now present a characterization of Abel mean ergodicity.

Proposition 5.12. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$. Then $(T(t))_{t\geq 0}$ is Abel mean ergodic if and only if*

$$\{\lambda R(\lambda, A)x : 0 < \lambda \leq 1\} \textit{ is relatively countably } \sigma(X, X')\textit{-compact}, \ \forall x \in X. \tag{5.22}$$

Proof. Suppose $(T(t))_{t\geq 0}$ is Abel mean ergodic. Then $P = \tau_s - \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$ exists. According to Lemma 3.7, $\lambda \mapsto \lambda R(\lambda, A)$ has a continuous, $\mathcal{L}_s(X)$–valued extension to $[0, 1]$ if we define its value at $\lambda = 0$ to be P. So, for each $x \in X$, the set $\{\lambda R(\lambda, A)x : 0 \leq \lambda \leq 1\}$ is compact in X which implies that (5.22) is valid.

Now assume that (5.22) holds and fix $x \in X$. Choose any sequence $\lambda_n \downarrow 0^+$ in $(0, 1]$. Then $\{\lambda_n R(\lambda_n, A)x\}_{n=1}^\infty$ has a cluster point in X_σ, say y. Let $\mathcal{U}$ be any neighbourhood of y in X_σ and fix any $\mu \in (0, 1]$. Choose N satisfying $0 < \lambda_N < \mu$. Select $n_0 > N$ with $\lambda_{n_0} R(\lambda_{n_0}, A)x \in \mathcal{U}$. Then $\lambda_{n_0} < \mu$ and so y is also a cluster point of $\{\lambda R(\lambda, A)x : 0 < \lambda \leq 1\}$ in X_σ. By Corollary 5.11 if we set $Px := y$, then $Px = \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)x$. Equicontinuity of $\{\lambda R(\lambda, A)x : 0 < \lambda \leq 1\}$ guarantees that the linear map $x \mapsto Px$, for $x \in X$, belongs to $\mathcal{L}(X)$, i.e., $P = \tau_s - \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$. So, $(T(t))_{t\geq 0}$ is Abel mean ergodic. □

If X is a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ is an equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$, then it follows from [5, Remark 5(iii)] and Lemma 3.6(i) above that

$$\operatorname{Ker} A = \operatorname{Fix}(T(\cdot)) = \{x \in D(A) : \lambda R(\lambda, A)x = x\}, \quad 0 < \lambda \leq 1. \tag{5.23}$$

Combining (5.23) with both (3.23) and (3.25) yields

$$\operatorname{Ker} A \cap \overline{\operatorname{Im} A} = \{0\}. \tag{5.24}$$

We can now formulate and establish one of the main results.

Theorem 5.13. *Let X be a sequentially complete lcHs and $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ be an equicontinuous C_0–semigroup with infinitesimal generator $(A, D(A))$. The following assertions are equivalent.*

(i) *The condition* (5.22) *is satisfied.*
(ii) $(T(t))_{t\geq 0}$ *is Abel mean ergodic.*
(iii) $(T(t))_{t\geq 0}$ *is mean ergodic.*

Suppose, in addition, that $\operatorname{Im} A$ *is closed in X. Then* (i)–(iii) *are also equivalent to each of the following assertions.*

(iv) $(T(t))_{t\geq 0}$ *is uniformly mean ergodic.*
(v) $(T(t))_{t\geq 0}$ *is uniformly Abel mean ergodic.*

Proof. The equivalence (i)⇔(ii) is Proposition 5.12. By the discussion prior to Example 5.8 we see that (ii)⇔(iii) follows from Theorem 5.5.

So, suppose additionally that $\operatorname{Im} A$ is closed. Then (iii)⇔(iv) is Proposition 5.9. Since (v)⇒(ii), it remains to establish (ii)⇒(v).

By (ii), $P := \tau_s - \lim_{\lambda\downarrow 0^+} \lambda R(\lambda, A)$ exists and is a projection; see the proof of Proposition 5.9. Clearly P commutes with $\lambda R(\lambda, A)$, for $\lambda \in (0, 1]$. Given $x \in X$, Corollary 5.11 implies that $Px \in \operatorname{Fix}(T(\cdot))$, i.e., $\operatorname{Im} P \subseteq \operatorname{Fix}(T(\cdot))$. On the other hand, if $x \in \operatorname{Fix}(T(\cdot))$, then (5.23) implies that $Px = x$, i.e., $x \in \operatorname{Im} P$. So (cf. also (5.23)), we have

$$\operatorname{Im} P = \operatorname{Fix}(T(\cdot)) = \operatorname{Ker} A. \tag{5.25}$$

By Lemma 1.8 of [24, p.78], applied to the $T(\cdot)$–ergodic net $\{\lambda R(\lambda, A) : 0 < \lambda \leq 1\}$, we have (cf. also (3.25))

$$\operatorname{Ker} P = \overline{\operatorname{span}\{T(t)x - x : t \geq 0, x \in X\}} = \overline{\operatorname{Im} A}. \tag{5.26}$$

It is immediate from (3.21) that

$$\lambda R(\lambda, A)x = x, \quad \forall x \in \operatorname{Ker} A, \ \lambda \in (0,1]. \tag{5.27}$$

By Lemma 3.6(ii), the restriction $\lambda R_Y(\lambda, A)$ of $\lambda R(\lambda, A)$ to the sequentially complete lcHs $Y := \overline{\mathrm{Im} A} = \mathrm{Im} A$ belongs to $\mathcal{L}(Y)$, for each $\lambda \in (0,1]$. Given $p \in \Gamma_X$, by equicontinuity of $\{\lambda R(\lambda, A) : 0 < \lambda \leq 1\}$ there exist $M_p > 0$ and $q \in \Gamma_X$ such that

$$p(\lambda R(\lambda, A)x) \leq M_p q(x), \quad \forall x \in X, \ \lambda \in (0,1]. \tag{5.28}$$

Fix $y \in Y$, so that $y = Ax$ for some $x \in D(A)$. For $\lambda \in (0,1]$ we have

$$R_Y(\lambda, A)y = R(\lambda, A)[(A - \lambda I)x + \lambda x] = -x + \lambda R(\lambda, A)x$$

and so (5.28) implies $p(R_Y(\lambda, A)y) \leq p(x) + M_p q(x)$. So, $\{R_Y(\lambda, A) : 0 < \lambda \leq 1\}$ is bounded in $\mathcal{L}_s(Y)$ and hence, by sequential completeness of Y, also in $\mathcal{L}_b(Y)$, [23, (3), p.135]. Accordingly,

$$\tau_b - \lim_{\lambda \downarrow 0^+} \lambda R_Y(\lambda, A) = 0. \tag{5.29}$$

Fix $B \in \mathcal{B}(X)$. If $u \in B$, then $u = Pu \oplus (I - P)u$ with $Pu \in \operatorname{Ker} A$ (see (5.25)) and so, via (5.27), we have $\lambda R(\lambda, A)Pu = Pu$ for $0 < \lambda \leq 1$. It follows that $(\lambda R(\lambda, A) - P)u = \lambda R(\lambda, A)(I - P)u$ for $0 < \lambda \leq 1$. By (5.26), $(I - P)u \in \operatorname{Ker} P = Y$ which shows $D := (I - P)(B) \subseteq Y$, i.e., $D \in \mathcal{B}(Y)$. Given $p \in \Gamma_X$, for each $\lambda \in (0,1]$ we have $p([\lambda R(\lambda, A) - P]u) = p(\lambda R(\lambda, A)(I - P)u) \leq p_D(\lambda R_Y(\lambda, A))$. So, $p_B(\lambda R(\lambda, A) - P) \leq p_D(\lambda R_Y(\lambda, A))$ for $\lambda \in (0,1]$ and with $D \in \mathcal{B}(Y)$. Then (5.29) implies $P = \tau_b - \lim_{\lambda \downarrow 0^+} \lambda R(\lambda, A)$, i.e., $(T(t))_{t \geq 0}$ is uniformly Abel mean ergodic. □

For X a Banach space, the implication (ii)⇒(v) in Theorem 5.13 (assuming $\mathrm{Im} A$ closed) occurs as part of [8, Proposition 4.3.15].

Recall that a lcHs X is *semi–reflexive* iff every bounded subset of X is relatively $\sigma(X, X')$–compact. Such spaces X are necessarily quasicomplete, [19, p.229], and every equicontinuous C_0–semigroup on X is automatically mean ergodic, [5, Corollary 2(i)]. Combining this with Proposition 5.9 gives the following fact.

Corollary 5.14. *Let X be a semi–reflexive lcHs. Then every equicontinuous C_0–semigroup on X whose infinitesimal generator has closed range is necessarily uniformly mean ergodic.*

We now formulate a result dealing with condition (iv) of Theorem 5.7.

Proposition 5.15. *Let X be a lcHs such that $\mathcal{L}_b(X)$ is sequentially complete, $(T(t))_{t \geq 0}$ be a locally equicontinuous C_0–semigroup on X which is τ_b–bounded and $(A, D(A))$ be the infinitesimal generator of $(T(t))_{t \geq 0}$. Either of the following two conditions ensures that $(T(t))_{t \geq 0}$ is uniformly mean ergodic.*

(i) *$R(\cdot, A)$ exists and is τ_b–bounded in some neighbourhood of* 0.
(ii) $0 \in \sigma(A)$ *and is a simple pole of $R(\cdot, A)\colon \rho(A) \to \mathcal{L}_b(X)$.*

Proof. (i) Choose $r > 0$ such that $D_r := \{\mu \in \mathbb{C} : |\mu| < r\} \subseteq \rho(A)$ and $\{R(\lambda, A) : \lambda \in D_r\}$ is bounded in $\mathcal{L}_b(X)$. Given $B \in \mathcal{B}(X)$ and $p \in \Gamma_X$, there is $M > 0$ such that $p_B(R(\lambda, A)) \leq M$, for all $|\lambda| < r$, and hence, $p_B(\lambda R(\lambda, A)) \leq |\lambda| M$, for all $|\lambda| < r$. It follows that $\tau_b\text{-}\lim_{\lambda \downarrow 0^+} \lambda R(\lambda, A) = 0$ and the desired conclusion follows from Theorem 5.5(ii).

(ii) Choose $r > 0$, $P \in \mathcal{L}(X)$ and a holomorphic function $H\colon D_r \to \mathcal{L}_b(X)$ such that $R(\lambda, A) = \lambda^{-1}P + H(\lambda)$, for all $0 < |\lambda| < r$. Fix any $0 < r_1 < r$, in which case $\{H(\lambda) : |\lambda| \leq r_1\}$ is τ_b–bounded (even τ_b–compact). Moreover,

$$\lambda R(\lambda, A) = P + \lambda H(\lambda), \quad 0 < |\lambda| \leq r_1.$$

Since τ_b-$\lim_{\lambda \downarrow 0^+} \lambda H(\lambda) = 0$, it follows that τ_b-$\lim_{\lambda \downarrow 0^+} \lambda R(\lambda, A) = P$. Again Theorem 5.5(ii) implies that $(T(t))_{t \geq 0}$ is uniformly mean ergodic. □

Conditions (i)–(ii) in Proposition 5.15 are by no means necessary. The operator $A \in \mathcal{L}(s)$ of Example 5.8 generates an equicontinuous C_0–semigroup which is uniformly mean ergodic but, satisfies neither of (i)–(ii) in Proposition 5.15. The same is true for the operator $A \in \mathcal{L}(\omega)$ in Remark 3.5(vii); its uniform mean ergodicity is a consequence of ω being Montel, [5, Corollary 2(ii)].

A decomposition of a lcHs X is a sequence $(X_n)_n$ of closed, non–trivial subspaces of X such that $X_i \cap X_j = \{0\}$, for $i \neq j$, and each $x \in X$ can be expressed uniquely in the form $x = \sum_{j=1}^{\infty} y_j$ with $y_j \in X_j$, for $j \in \mathbb{N}$. This induces a sequence of projections $(Q_n)_n$ defined by $Q_n x := y_n$ where $x = \sum_{j=1}^{\infty} y_j$ with $y_j \in X_j$ for each $j \in \mathbb{N}$. These projections are pairwise orthogonal (i.e., $Q_n Q_m = 0$ if $n \neq m$) and $Q_n(X) = X_n$ for $n \in \mathbb{N}$. If, in addition, each $Q_n \in \mathcal{L}(X)$, for $n \in \mathbb{N}$, then we speak of a *Schauder decomposition* of X. In particular, if each space $X_n = \operatorname{span}\{x_n\}$ of the Schauder decomposition is 1–dimensional, for $n \in \mathbb{N}$, then $\{x_n\}_{n=1}^{\infty}$ is called a *Schauder basis* of X, in which case every $x \in X$ has a unique expansion of the form $x = \sum_{j=1}^{\infty} \alpha_j x_j$. A Schauder decomposition (of projections) $(Q_n)_n \subseteq \mathcal{L}(X)$ is said to have *property* (M) if τ_b-$\lim_{n\to\infty}(\sum_{j=1}^{n} Q_j) = I$.

Theorem 5.16. *Let X be a complete barrelled lcHs which admits a Schauder decomposition without property (M). Then there exists an equicontinuous, mean ergodic, uniformly continuous C_0–semigroup $(T(t))_{t\geq 0}$ on X which is not uniformly mean ergodic.*

Proof. Let $\{Q_j\}_{j=1}^{\infty} \subseteq \mathcal{L}(X)$ denote a Schauder decomposition without property (M) and define the closed subspaces $X_j := Q_j(X)$ for all $j \in \mathbb{N}$. By Lemma 3.2 in [3] there exist a bounded sequence $(z_j)_j \subseteq X$ and $p_0 \in \Gamma_X$ with $z_j \in X_{j+1}$ and $p_0(z_j) > 1/2$ for all $j \in \mathbb{N}$.

Now, setting $\alpha_k := 1 - 2^{-k}$ for all $k \in \mathbb{N}$, by the proof of Theorem 3.6 in [3] the linear map $T\colon X \to X$ defined by $Tx := \sum_{k=1}^{\infty} \alpha_k Q_k x$, for $x \in X$, belongs to $\mathcal{L}(X)$, is power bounded and mean ergodic with $\operatorname{Ker}(I - T) = \{0\}$ and $\operatorname{Ker}(I - T^t) = \{0\}$ but, is not uniformly mean ergodic. Moreover, T also satisfies

$$T^m x = \sum_{k=1}^{\infty} \alpha_k^m Q_k x, \quad x \in X,\ m \in \mathbb{N}. \tag{5.30}$$

Since T is power bounded, the system of seminorms $\{\overline{p} : p \in \Gamma_X\}$ defined by $\overline{p}(x) := \sup_{n\geq 0} p(T^n x)$, for $x \in X$ and $p \in \Gamma_X$, also generates the lc–topology of X and we have $\overline{p}(T^n x) = \sup_{m\geq 0} p(T^m T^n x) = \sup_{h\geq n} p(T^h x) \leq \overline{p}(x)$, for $x \in X$, $p \in \Gamma_X$. It follows, for each $t \geq 0$, that the operator $T(t) := \sum_{m=0}^{\infty} \frac{t^m T^m}{m!}$

is well defined in $\mathcal{L}(X)$ and that

$$S(t) := e^{-t}T(t) = e^{-t}\sum_{m=0}^{\infty}\frac{t^m T^m}{m!}, \quad t \geq 0, \tag{5.31}$$

is an equicontinuous, uniformly continuous C_0–semigroup on X with infinitesimal generator $(T-I, X)$; see Remark 2.4. Since $\{0\} = \mathrm{Ker}(T-I) = \mathrm{Fix}(S(\cdot))$ and $\{0\} = \mathrm{Ker}(T-I)^t = \mathrm{Fix}(S^t(\cdot))$ (cf. proof of Theorem 5 in [5]), the semigroup $(S(t))_{t\geq 0}$ is mean ergodic by [5, Remark 6(iv), Fact], i.e., there exists a projection $P \in \mathcal{L}(X)$ such that $C(r) \to P$ in $\mathcal{L}_s(X)$ as $r \to \infty$.

To see that $(S(t))_{t\geq 0}$ is not uniformly mean ergodic, we proceed as follows. For any $r > 0$, the formula (5.31) yields for $(S(t))_{t\geq 0}$ that

$$C(r)x := \frac{1}{r}\int_0^r S(t)x\,dt = \sum_{m=0}^{\infty}\left(\frac{1}{m!}\frac{1}{r}\int_0^r t^m e^{-t}\,dt\right)T^m x, \quad x \in X,$$

with $\int_0^r t^m e^{-t}dt = m! - e^{-r}(r^m + mr^{m-1} + \ldots + m(m-1)\cdots 2r + m!)$. Therefore, if $x \in X_j$ for a fixed $j \in \mathbb{N}$, we have by (5.30) that

$$C(r)x = \left(\sum_{m=0}^{\infty}\left[\frac{1}{r} - \frac{e^{-r}}{rm!}(r^m + mr^{m-1} + \ldots + m(m-1)\cdots 2r + m!)\right]\alpha_j^m\right)x,$$

for all $r > 0$. Observing that $\sum_{m=0}^{\infty}\alpha_j^m = \frac{1}{1-\alpha_j}$ and (with $D^k := \frac{d^k}{dr^k}$) that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(r^m + mr^{m-1} + m(m-1)r^{m-2} + \ldots + m(m-1)\cdots 2r + m!\right)\frac{\alpha_j^m}{m!} \\
&\quad = \sum_{m=0}^{\infty}\sum_{k=0}^{m}D^k(r^m)\frac{\alpha_j^m}{m!} = \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}D^k(r^m)\frac{\alpha_j^m}{m!} = \sum_{k=0}^{\infty}D^k(\sum_{m=0}^{\infty}\frac{r^m\alpha_j^m}{m!}) \\
&\quad = \sum_{k=0}^{\infty}D^k(e^{r\alpha_j}) = \sum_{k=0}^{\infty}\alpha_j^k e^{r\alpha_j} = \frac{e^{r\alpha_j}}{1-\alpha_j},
\end{aligned}$$

we obtain

$$C(r)x = \left(\frac{e^{(\alpha_j-1)r} - 1}{r(\alpha_j - 1)}\right)x. \tag{5.32}$$

Since $\alpha_j - 1 = -2^{-j}$, it follows that $C(r)x \to 0$ in X as $r \to \infty$. So, $Px = 0$ for all $x \in X_j$ and $j \in \mathbb{N}$. Since $\cup_{j=1}^{\infty}X_j$ is dense in X and $P \in \mathcal{L}(X)$, we obtain that $P = 0$ on X, i.e., $C(r) \to 0$ in $\mathcal{L}_s(X)$ as $r \to \infty$.

Suppose that $C(r) \to 0$ in $\mathcal{L}_b(X)$ as $r \to \infty$. In particular, since $(z_j)_j$ is a bounded sequence in X, we have that

$$\lim_{r\to\infty}\sup_{j\in\mathbb{N}}\overline{p_0}(C(r)z_j) = 0. \tag{5.33}$$

But $\overline{p_0} \geq p_0$ and hence, for all $j \in \mathbb{N}$, we obtain from (5.32) that

$$\overline{p_0}(C(2^j)z_j) = \overline{p_0}(z_j)\frac{e^{(\alpha_j-1)2^j} - 1}{2^j(\alpha_j - 1)} > \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

Since $2^j \to \infty$ as $j \to \infty$, this contradicts (5.33). $\square$

For Banach spaces (which are Montel iff they are finite–dimensional) our final result occurs in [29, Theorem 2.6].

Theorem 5.17. *Let X be a complete barrelled lcHs with a Schauder basis. Then X is Montel if and only if every equicontinuous, uniformly continuous C_0–semigroup on X is uniformly mean ergodic.*

Proof. Let X be Montel. Then [5, Corollary 2(ii)] implies that every equicontinuous, uniformly continuous C_0–semigroup on X is uniformly mean ergodic.

Conversely, suppose that X is not Montel. Observe that the Schauder decomposition $\{Q_n\}_{n=1}^{\infty} \subseteq \mathcal{L}(X)$ induced by the basis of X has the property that each space $X_n := Q_n(X)$, for $n \in \mathbb{N}$, is Montel because $\dim X_n = 1$ for all $n \in \mathbb{N}$. By [2, Theorem 3.7(iii)] $\{Q_n\}_{n=1}^{\infty}$ does not satisfy property (M) and hence, Theorem 5.16 ensures that there exists an equicontinuous, mean ergodic, uniformly continuous C_0–semigroup on X which is not uniformly mean ergodic. $\square$

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