The specification property for backward shifts

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We characterize when backward shift operators defined on Banach sequence spaces exhibit the strong specification property. In particular, within this framework, the specification property is equivalent to the notion of chaos introduced by Devaney.

Keywords: specification property; chaotic operators

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1. Introduction

A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, a notion of chaos stronger than Devaney's definition is the so called specification property. It was first introduced by Bowen [5]; since then, several kinds and degrees of this property have been stated [19], we will follow the definitions and terminology used in [1]. Some recent works on the specification property are [14, 16, 17].

Definition 1.1. A continuous map $f : X \to X$ on a compact metric space (X, d) has the strong specification property (SSP) if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \ldots, y_s\} \subset X$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \ldots, s-1$, there is a point $x \in X$ such that, for each positive integer $r \leq s$ and all integers i with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$

$$f^n(x) = x, \quad \text{where } n = N_\delta + k_s.$$

When the above property is satisfied for $s = 2$, then the dynamical system is said to satisfy the weak specification property (WSP).

Obviously, the SSP implies the WSP, and compact dynamical systems with the specification property are mixing and Devaney chaotic, among other basic properties (see, e.g., [8]).

Devaney chaos and mixing properties have been widely studied for linear operators on Banach and more general spaces [3, 4, 7, 9, 10, 12, 18]. The recent books [2] and

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[13] contain the basic theory, examples, and many results on chaotic linear dynamics.

We plan to study this strong specification property for bounded linear operators defined on separable Banach spaces. In this situation, the first crucial problem is that these spaces are never compact. The following definition can be considered the natural extension in this setting.

Definition 1.2. A bounded linear operator $T : X \to X$ on a separable Banach space X has the SSP if there exists an increasing sequence $(K_m)_m$ of T-invariant compact sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathrm{N}} K_m} = X$ such that for each $m \in \mathrm{N}$ the map $T|_{K_m}$ has the SSP, that is, for any $\delta > 0$ there is a positive integer $N_{\delta,m}$ such that for every $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset K_m$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for $1 \leq r \leq s-1$, there is a point $x \in K_m$ such that, for each positive integer $r \leq s$ and integers i with $j_r \leq i \leq k_r$, the following conditions hold:

$$\|T^i(x) - T^i(y_r)\| < \delta,$$

$$T^n(x) = x, \quad \text{where } n = N_{\delta,m} + k_s.$$

2. Strong specification property for backward shift operators

For a strictly positive sequence $(v_i)_i$ (weight sequence from now on), consider the Banach sequence spaces

$$\ell^p(v) := \left\{ (x_i)_i \in \mathrm{K}^{\mathrm{N}} : \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p \, v_i \right)^{1/p} < \infty \right\}, \ 1 \leq p < \infty,$$

$$c_0(v) := \left\{ (x_i)_i \in \mathrm{K}^{\mathrm{N}} : \lim_{i\to\infty} |x_i| \, v_i = 0, \ \|x\| := \sup_i |x_i| \, v_i \right\}.$$

On sequence spaces, the backward shift B is defined as $B((x_i)_i) := (x_{i+1})_i$, that is, $B(x_1, x_2, x_3, \dots) := (x_2, x_3, x_4, \dots)$. In order to have a bounded operator it is required that the weights satisfy

$$\sup_{i\in\mathrm{N}} \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

Theorem 2.1. *For a bounded backward shift operator B defined on $\ell^p(v)$, $1 \leq p < \infty$ (respectively, on $c_0(v)$) the following conditions are equivalent:*

(i) $\sum_{i=1}^{\infty} v_i < \infty$ *(respectively,* $\lim_{i\to\infty} v_i = 0$*).*
(ii) *B has SSP.*
(iii) *B is Devaney chaotic.*

Proof. To see (i) implies (ii) take the compact set $K = \{(x_i)_i \in \ell^p(v) : |x_i| \leq 1, \forall i\}$ and let $\delta > 0$ be fixed. There is N such that

$$\sum_{i\geq N} v_i < \frac{\delta}{2^p}.$$

Set $N_\delta = N + 1$. Take any $\{y_1, \dots, y_s\} \subset K$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \dots, s-1$. Consider $x = (x_i)_i$

defined as follows: for $1 \le i \le N_\delta + k_s$

$$x_i := \begin{cases} y_{1,i} \text{ if } i \in [1, j_2[\\ y_{2,i} \text{ if } i \in [j_2, j_3[\\ \vdots \\ y_{s,i} \text{ if } i \in [j_s, N_\delta + k_s] \end{cases},$$

that is, $(x_1, \dots, x_{N_\delta+k_s}) = (y_{1,1}, \dots, y_{1,j_2-1}, y_{2,j_2}, \dots, y_{2,j_3-1}, \dots, y_{s,j_s}, \dots, y_{s,N_\delta+k_s})$; and for any other index set $x_j := x_i$ if $j \equiv i \pmod{N_\delta + k_s}$. Clearly x is a periodic point belonging to K. For $r = 1, \dots, s-1$ and $j_r \le i \le k_r$ we have

$$\|B^i x - B^i y_r\|^p = \sum_{l \ge j_{r+1}} \left|x_l - y_{r,l}\right|^p v_{l-i} \le 2^p \sum_{l \ge j_{r+1}} v_{l-i} < \delta$$

since $l - i \ge j_{r+1} - i \ge j_{r+1} - k_r \ge N_\delta$. For $j_s \le i \le k_s$ we have

$$\|B^i x - B^i y_s\|^p = \sum_{l \ge m} \left|x_l - y_{s,l}\right|^p v_{l-i} \le 2^p \sum_{l \ge m} v_{l-i} < \delta$$

since $l - i \ge m - i \ge N_\delta + k_s - i \ge N_\delta$. The sequence of compact sets $(K_m := mK)_m$ satisfies the required properties so that B has SSP.

That condition (i) implies (iii) come from the characterizations of chaos for backward shift operators in weighted ℓ^p-spaces (see [15, Theorem 3.2] or [11, Theorem 8]).

The proof for the case $c_0(v)$ is similar using the supremum norm. □

Our next step is to take sequences indexed over the set of integers. For a strictly positive sequence $(v_i)_{i \in \mathbb{Z}}$, consider the Banach sequence spaces

$$\ell^p(v, \mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \|x\| := \left(\sum_{i=-\infty}^{\infty} |x_i|^p v_i \right)^{1/p} < \infty \right\}, \ 1 \le p < \infty,$$

$$c_0(v, \mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \lim_{|i| \to \infty} |x_i| v_i = 0, \ \|x\| := \sup_i |x_i| v_i \right\}.$$

The bilateral backward shift B is defined as $B((x_i)_i) := (x_{i+1})_i$, that is,

$$B(\dots, x_{-2}, x_{-1}, \overset{\triangledown}{x_0}, x_1, x_2, \dots) := (\dots, x_{-1}, x_0, \overset{\triangledown}{x_1}, x_2, x_3, \dots),$$

where the small triangle marks the coordinate corresponding to the index 0. In order to have a bounded operator it is required that the weights satisfy

$$\sup_{i \in \mathbb{Z}} \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

THEOREM 2.2. *For a bounded bilateral backward shift operator B defined on $\ell^p(v, \mathbb{Z})$, $1 \le p < \infty$ (respectively, on $c_0(v, \mathbb{Z})$) the following conditions are equivalent:*

(i) $\sum_{i=-\infty}^{\infty} v_i < \infty$ *(respectively,* $\lim_{|i| \to \infty} v_i = 0$*).*
(ii) *B has SSP.*
(iii) *B is Devaney chaotic.*

Proof. Take the compact set $K = \{(x_i)_i \in \ell^p(v, \mathbb{Z}) : |x_i| \le 1, \forall i\}$ and let $\delta > 0$ be fixed. There is a positive integer N such that

$$\sum_{|i|\geq N} v_i < \frac{\delta}{2^p}.$$

Set $N_\delta = 2N+1$. Take any $\{y_1, \ldots, y_s\} \subset K$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \ldots, s-1$. Consider $x = (x_i)_i$ defined as follows: for $-N \leq i \leq k_s + N$

$$x_i := \begin{cases} y_{1,i} \text{ if } i \in [-N, j_2 - N[\\ y_{2,i} \text{ if } i \in [j_2 - N, j_3 - N[\\ \vdots \\ y_{s,i} \text{ if } i \in [j_s - N, k_s + N] \end{cases},$$

that is,

$$(x_{-N}, \ldots, x_{k_s+N}) = \\ (y_{1,-N}, \ldots, y_{1,j_2-N-1}, y_{2,j_2-N}, \ldots, y_{2,j_3-N-1}, \ldots, y_{s,j_s-N}, \ldots, y_{s,k_s+N}),$$

and for any other index set $x_j := x_i$ if $j \equiv i \pmod{N_\delta + k_s}$. The rest of the proof goes verbatim to the one of Theorem 2.1. □

Some operators can be represented as a weighted backward shift operator $B_w(x_1, x_2, \ldots) := (w_2 x_2, w_3 x_3, \ldots)$ defined on a weighted $\ell^p(v)$ space. This case may be reduced to the non-weighted backward shift via topological conjugacy. Set

$$a_1 := 1,\ a_i := w_2 \ldots w_i,\ i > 1,$$

and consider $\ell^p(\bar{v})$ where

$$\bar{v}_i = \frac{v_i}{\prod_{j=2}^{i} |w_j^p|}, \text{ for all } i.$$

Take $\phi_a : \ell^p(v) \to \ell^p(\bar{v})$ defined as $\phi_a(x_1, x_2, \ldots) := (a_1 x_1, a_2 x_2, \ldots)$ to construct a commutative diagram $\phi_a \circ B_w = B \circ \phi_a$. Since ϕ_a is an isometry, by topological conjugacy, we have that B has SSP (is chaotic) on $\ell^p(\bar{v})$ if and only if B_w has SSP (is chaotic) on $\ell^p(v)$.

In this situation, the required condition to have $B_w : \ell^p(v) \to \ell^p(v)$ bounded is

$$\sup_{i\in\mathbb{N}} \left|w_{i+1}^p\right| \frac{v_i}{v_{i+1}} < \infty,$$

and the corresponding characterization is as follows.

Theorem 2.3. *For a bounded weighted backward shift operator B_w defined on $\ell^p(v)$, $1 \leq p < \infty$, (respectively, on $c_0(v)$) the following conditions are equivalent:*

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=2}^{i} |w_j^p|} < \infty$ *(respectively,* $\lim_{i\to\infty} \frac{v_i}{\prod_{j=2}^{i}|w_j|} = 0$*),*
(ii) *B_w has SSP.*
(iii) *B_w is Devaney chaotic.*

Similarly, for the bilateral case $B_w : \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ one should take

$$a_0 := 1,\ a_i := w_1 \ldots w_i,\ a_{-i} := \frac{1}{w_0 w_{-1} \ldots w_{-i+1}},\ i > 0,$$

and consider $\ell^p(\bar{v}, \mathrm{Z})$ where

$$\bar{v}_0 := v_0,\ \bar{v}_i = \frac{v_i}{\prod_{j=1}^{i} \left|w_j^p\right|},\ \bar{v}_{-i} = \prod_{j=0}^{-i+1} \left|w_j^p\right| v_{-i},\ i > 0.$$

The required condition to have $B_w : \ell^p(v, \mathrm{Z}) \to \ell^p(v, \mathrm{Z})$ bounded is

$$\sup_{i \in \mathrm{Z}} \left|w_{i+1}^p\right| \frac{v_i}{v_{i+1}} < \infty,$$

and the characterization for SSP in this bilateral case follows.

THEOREM 2.4. *For a bounded bilateral weighted backward shift operator B_w defined on $\ell^p(v, \mathrm{Z})$, $1 \leq p < \infty$, (respectively, on $c_0(v, \mathrm{Z})$) the following conditions are equivalent:*

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} \left|w_j^p\right|} < \infty$ *and* $\sum_{i=1}^{\infty} \prod_{j=0}^{-i+1} \left|w_j^p\right| v_{-i} < \infty$

(respectively, $\displaystyle\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j|} = \lim_{i\to\infty} \prod_{j=0}^{-i+1} |w_j|\, v_{-i} = 0$*).*

(ii) B_w *has SSP.*

(iii) B_w *is Devaney chaotic.*

3. Examples

3.1 Weighted backward shift operators on ℓ^p

For any bounded sequence $(w_i)_i$ with $w_i \neq 0$, its associated weighted backward shift $B_w : \ell^p \to \ell^p$ is a bounded operator. Since ℓ^p corresponds to $\ell^p(v)$ with $(v_i)_i = (1)_i$, we have that B_w defined on ℓ^p has the SSP property if and only if

$$\sum_{i=1}^{\infty} \prod_{j=2}^{i} \left|w_j^{-p}\right| < \infty.$$

The particular case $(w_i)_i = (\lambda)_i$ with $\lambda \in \mathrm{K}$ reads as $\lambda B : \ell^p \to \ell^p$ has the SSP if and only if $|\lambda| > 1$.

3.2 The operator of differentiation on Hilbert spaces of entire functions

Let $\gamma(z)$ be an admissible comparison entire function, that is, the Taylor coefficients $\gamma_i > 0$ for all $i \in \mathrm{N}_0$ and the sequence $(i\gamma_i/\gamma_{i-1})_i$ is monotonically decreasing. We consider the Hilbert space $E^2(\gamma)$ of power series

$$g(z) = \sum_{i=0}^{\infty} \hat{g}(i) z^i$$

for which

$$\|g\|_{2,\gamma}^2 := \sum_{i=0}^{\infty} \gamma_i^{-2} \left|\hat{g}(i)\right|^2 < \infty.$$

Chan and Shapiro studied some dynamical properties of the operator of differentiation and the translation operator on $E^2(\gamma)$ (see [6]).

It is clear that $E^2(\gamma)$ is isometric to $\ell^2(v)$ with $v = (v_i)_{i\in\mathrm{N}_0} = (\gamma_i^{-2})_{i\in\mathrm{N}_0}$ and with the identification $f \mapsto (f^{(i)}(0)/i!)_{i\in\mathrm{N}_0}$. Moreover, the operator of differentiation D turns out to be a weighted backward shift with weights $w = (w_i)_i = (i)_i$ or, equivalently, as a backward shift defined on $\ell^2(\bar{v})$, where

$$\bar{v}_i = \frac{1}{(\gamma_i\; i!)^2},\; i \geq 0.$$

Since $\gamma(z)$ is an admissible comparison entire function, it is easy to check that $\sup_{i\geq 0} \bar{v}_i/\bar{v}_{i+1} < \infty$ and B is a bounded operator on $\ell^2(\bar{v})$ (this is equivalent to saying that D is a bounded operator on $E^2(\gamma)$).

Applying Theorem 2.3 we have that $D : E^2(\gamma) \to E^2(\gamma)$ has SSP is and only if $\sum_{i=0}^{\infty} (\gamma_i i!)^{-2} < \infty$, in particular, if $\lim_i i\gamma_i/\gamma_{i-1} > 1$ then D has SSP.

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