The final publication is available at

 http://dx.doi.org/10.1017/S0004972700018943

Additional Information

NILPOTENT-LIKE FITTING FORMATIONS OF
FINITE SOLUBLE GROUPS

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Dedicated to Professor K. Doerk on his 60th Birthday

In this paper the subnormal subgroup closed saturated formations of finite soluble groups containing nilpotent groups are fully characterised by means of extensions of well-known properties enjoyed by the formation of all nilpotent groups.

1. INTRODUCTION AND NOTATION

All groups considered in the paper will be finite and soluble.

The class of groups whose members are a direct product of Hall subgroups corresponding to a given partition of the set of all primes is a subgroup-closed saturated formation enjoying many interesting properties of nilpotent type. In fact, this family of formations was characterised in [2] as the subgroup-closed saturated formations $\mathcal{F}$ for which the set of all $\mathcal{F}$-subnormal subgroups of each group G is a sublattice of the subgroup lattice of G. Hence the members of this family are called lattice formations.

The purpose of this paper is to prove that the majority of properties of nilpotent type enjoyed by the lattice formations actually characterise them (Theorem 1).

On the other hand, it is known that the Fitting subgroup of a group which is the product of two nilpotent subgroups inherits the factorisation [1, Lemma 2.5.7]. The natural question is now for which subgroup-closed Fitting formations $\mathcal{F}$ the $\mathcal{F}$-radical of a group G which is the product of two $\mathcal{F}$-groups inherits the factorisation. As application of Theorem 1, we see that this property also characterises the aforesaid subgroup-closed Fitting formations. This is Theorem 2 of the paper.

We shall adhere to the notation and terminology employed in [5]. This book is our main reference for results concerning formations and Fitting classes. For the sake of completeness, we gather some notation and recall some definitions and results. We

denote by $\mathbb{P}$ the set of all prime numbers and, for a prime p, C_p is the cyclic group of order p. For a group G, $\pi(G)$ denotes the set of all prime divisors of $|G|$. If $\mathcal{X}$ is a class of groups, the characteristic of $\mathcal{X}$ is $\mathrm{char}\mathcal{X} = \{p \in \mathbb{P} \mid C_p \in \mathcal{X}\}$. The boundary $b(\mathcal{X})$ of a class of groups $\mathcal{X}$ consists of all groups G satisfying $G \notin \mathcal{X}$ and $G/N \in \mathcal{X}$ for all $1 \neq N \lhd G$. If π is a set of prime numbers, let $\mathcal{S}$ and $\mathcal{S}_\pi$ denote the classes of soluble and soluble π-groups, respectively.

The well-known Gaschütz-Lubeseder-Schmid Theorem states that in the general finite universe, saturated formations are exactly local formations, that is, formations $\mathcal{F} = LF(f)$ defined by a formation function f: $LF(f) = \big(G \in \mathcal{E} \mid$ if H/K is a chief factor of G and $p \in \pi(H/K)$, then $G/C_G(H/K) \in f(p)\big)$, where $\mathcal{E}$ is the class of all finite groups. In this case, f is said to be a local definition of $\mathcal{F}$. Among all possible local definitions of a local formation $\mathcal{F}$ there exists exactly one, denoted by F, such that F is integrated (that is, $F(p) \subseteq \mathcal{F}$ for all $p \in \mathbb{P}$) and full (that is, $S_p F(p) = F(p)$ for all $p \in \mathbb{P}$). F is called the canonical local definition of $\mathcal{F}$.

If $\mathcal{F}$ is a Fitting class, a subgroup V of a group G is called an $\mathcal{F}$-injector of G if $V \cap N$ is an $\mathcal{F}$-maximal subgroup of N for every subnormal subgroup N of G. By a well-known result due to Fischer, Gaschütz and Hartley [5, IX, Theorem 1.4], every soluble group G has a unique conjugacy class of $\mathcal{F}$-injectors.

It is also known that a subgroup-closed Fitting class of finite soluble groups is a saturated formation (Bryce and Cossey, [5, XI, Theorem 1.1]).

2. THE RESULTS

LEMMA 1. *Let $\mathcal{F}$ be a subgroup-closed Fitting class containing $\mathcal{N}$, the class of nilpotent groups. Then the following statements are equivalent:*

(i) *Let F be the full and integrated local formation function defining $\mathcal{F}$. Then there exists a partition $\{\pi_i\}_{i \in I}$ of $\mathbb{P}$ such that $F(p) = \mathcal{S}_{\pi_i}$ for every prime number $p \in \pi_i$ and for every $i \in I$.*

(ii) *For each prime number $p \in \mathbb{P}$, every primitive group $G \in \mathcal{F} \cap b\big(F(p)\big)$ is cyclic.*

PROOF: (i) implies (ii). By applying [2, Lemma 3.2], a group G is an $\mathcal{F}$-group if and only G has a normal Hall π_i-subgroup, for every $i \in I$. Therefore, it is clear that (ii) holds.

(ii) implies (i). Arguing as in (2) and (3) of [2, Theorem 3.3], we obtain that if p and q are two prime numbers such that $p \in \mathrm{char}F(q)$, then $\mathrm{char}F(p) = \mathrm{char}F(q)$. Now, by [5, IV,Lemma 3.16], $F(p)$ is a subgroup-closed Fitting class for every prime p. Hence, if $p, q \in \mathbb{P}$ and $p \in \mathrm{char}F(q)$, then $\mathcal{S}_p \subseteq F(q)$.

We see that if $p \in \mathbb{P}$ and $\pi(p) = \mathrm{char}F(p)$, then $F(p) = \mathcal{S}_{\pi(p)} \cap \mathcal{F}$. Since $F(p)$ is a subgroup-closed Fitting class and F is integrated, it follows that $F(p) \subseteq \mathcal{S}_{\pi(p)} \cap \mathcal{F}$,

for every prime p. Assume that there exists a prime p such that $F(p) \neq \mathcal{S}_{\pi(p)} \cap \mathcal{F}$ and consider a group $G \in (\mathcal{S}_{\pi(p)} \cap \mathcal{F}) \backslash F(p)$ of minimal order. We have that G is a primitive group, because $F(p)$ is a saturated formation. Since $G \in \mathcal{F} \cap b(F(p))$, then G is cyclic by (ii). Therefore G is a q-group with $q \in \pi(p)$ and so $G \in F(p)$ because $F(p)$ is a Fitting class, a contradiction.

Consequently, if p and q are two prime numbers such that $p \in \mathrm{char} F(q)$, then $F(p) = F(q)$.

Finally, we claim that for every prime p if $\pi(p) = \mathrm{char} F(p)$, then $F(p) = \mathcal{S}_{\pi(p)}$. It is clear that $F(p) \subseteq \mathcal{S}_{\pi(p)}$. If $F(p) \neq \mathcal{S}_{\pi(p)}$, we choose a group G of minimal order in $\mathcal{S}_{\pi(p)} \backslash F(p)$. Then G is again a primitive group, $Soc(G)$ is a q-group, with $q \in \pi(p)$, and $G/Soc(G) \in F(p)$. Consequently, $G \in \mathcal{S}_q F(p) = \mathcal{S}_q F(q) = F(q) = F(p)$, a contradiction. $\qquad\square$

Recall that if $\mathcal{F}$ is a saturated formation and G is a group, a maximal subgroup M of G is said to be $\mathcal{F}$-normal in G if $G/\mathrm{Core}_G(M) \in \mathcal{F}$. A subgroup H of G is called $\mathcal{F}$-subnormal in G if either $H = G$ or there exists a chain $H = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_n = G$ such that H_i is an $\mathcal{F}$-normal maximal subgroup of H_{i+1} for $0 \leqslant i < n$. It is clear that if $\mathcal{F} = \mathcal{N}$, the saturated formation of nilpotent groups, the $\mathcal{F}$-subnormal subgroups of G are exactly the subnormal subgroups of G.

THEOREM 1. *Let $\mathcal{F}$ be an s_n-closed saturated formation containing $\mathcal{N}$. Then the following statements are pairwise equivalent:*

(1) *Let F be the full and integrated local formation function defining $\mathcal{F}$. Then there exists a partition $\{\pi_i\}_{i \in I}$ of $\mathbb{P}$ such that $F(p) = \mathcal{S}_{\pi_i}$ for every prime number $p \in \pi_i$ and for every $i \in I$.*

(2) *If H and K are two $\mathcal{F}$-subnormal subgroups of a group G, then $\langle H, K \rangle^{\mathcal{F}} = \langle H^{\mathcal{F}}, K^{\mathcal{F}} \rangle$.*

(3) *$\mathcal{F}$ is a Fitting class satisfying that if G is a group, V is an $\mathcal{F}$-injector of G and H is an $\mathcal{F}$-subnormal subgroup of G, then $V \cap H$ is an $\mathcal{F}$-injector of H.*

(4) *$\mathcal{F}$ is a Fitting class such that the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of a group G is:*
$$G_{\mathcal{F}} = \langle X \in \mathcal{F} \mid X \text{ is } \mathcal{F}\text{-subnormal in } G \rangle.$$

(5) *If H and K are two $\mathcal{F}$-subnormal $\mathcal{F}$-subgroups of a group G, then $\langle H, K \rangle \in \mathcal{F}$.*

(6) *$\mathcal{F}$ is a Fitting class and if H is an $\mathcal{F}$-subnormal $\mathcal{F}$-subgroup of a group G, then $\langle H, H^g \rangle \in \mathcal{F}$ for every $g \in G$.*

PROOF: First we claim that if $\mathcal{F}$ is a saturated Fitting formation containing $\mathcal{N}$ and satisfying (6), then $\mathcal{F}$ is subgroup-closed.

Denote $\mathcal{F}_S = (X \mid S(X) \subseteq \mathcal{F})$, where $S(X) = (H \in \mathcal{S} \mid H \leqslant X)$. We prove that $\mathcal{F} = \mathcal{F}_S$. Suppose not and let G be a group of minimal order in $\mathcal{F} \setminus \mathcal{F}_S$. Then there

exists a subgroup H of G such that $H \notin \mathcal{F}$. We choose H of minimal order. Then every proper subgroup of H is in $\mathcal{F}$.

By the minimal choice of G, it is clear that $HK/K \in \mathcal{F}$ for every proper normal subgroup K of G. In particular, G has a unique minimal normal subgroup, N say. Let p be the prime dividing $|N|$. Since $H/H \cap N \in \mathcal{F}$, it is clear that $H^{\mathcal{F}}$ is a p-group.

We claim that $H/H^{\mathcal{F}}$ has a unique maximal subgroup, and so $H/H^{\mathcal{F}}$ is a cyclic q-group for some prime $q \neq p$. First we shall show that $H/H^{\mathcal{F}}$ is a nilpotent group. Suppose not and let $M/H^{\mathcal{F}}$ be a maximal subgroup of $H/H^{\mathcal{F}}$ such that $M/H^{\mathcal{F}}$ is not a normal subgroup. Then there exists $g \in H$ such that $M/H^{\mathcal{F}} \neq M^g/H^{\mathcal{F}}$. Notice that $M \in \mathcal{F}$ by the choice of H. Moreover, M is an $\mathcal{F}$-subnormal subgroup of H since $H^{\mathcal{F}} \leqslant M$. By applying (6), we have that $H = \langle M, M^g \rangle \in \mathcal{F}$, a contradiction. Hence any maximal subgroup of $H/H^{\mathcal{F}}$ is a normal subgroup and so $H/H^{\mathcal{F}} \in \mathcal{N}$. Assume now that there exist two maximal subgroups of $H/H^{\mathcal{F}}$, $M_1/H^{\mathcal{F}}$ and $M_2/H^{\mathcal{F}}$, such that $M_1 \neq M_2$. Then M_i is a normal subgroup of H for $i = 1, 2$ and $M_i \in \mathcal{F}$ for $i = 1, 2$ by the choice of H. Since $\mathcal{F}$ is a Fitting class it follows that $H = M_1 M_2 \in \mathcal{F}$, a contradiction.

Therefore we have $H = H_q H^{\mathcal{F}}$ where H_q is a Sylow q-subgroup of H. Notice that $G/C_G(N) \in F(p)$ since $G \in \mathcal{F}$. If $H_q \leqslant C_G(N)$, we have $H = H_q \times H^{\mathcal{F}} \in \mathcal{N} \subseteq \mathcal{F}$, a contradiction. Hence $q \in \pi(G/C_G(N))$. This is to say that $q \in \operatorname{char} F(p)$ and so $\mathcal{S}_q \subseteq F(p)$ because $F(p)$ is a Fitting class by [5, IV,Lemma 3.16]. Therefore $H = H^{\mathcal{F}} H_q \in \mathcal{S}_p F(p) = F(p) \subseteq \mathcal{F}$ since F is full and integrated, a contradiction.

Notice now that if $\mathcal{F}$ is a saturated Fitting formation satisfying either (3) or (4), we can apply similar arguments to those used above to conclude that $\mathcal{F}$ is also subgroup-closed.

(1) equivalent to (2). First observe that if $\mathcal{F}$ is an s_n-closed saturated formation satisfying (2), then it is a Fitting class since every subnormal subgroup is $\mathcal{F}$-subnormal. Hence $\mathcal{F}$ satisfies (6) and it is subgroup-closed because of the previous proof. Consequently, (1) equivalent to (2) is [4, Theorem 2].

(1) implies (3). This is [2, Theorem 4.5].

(3) implies (4). Let X be an $\mathcal{F}$-subnormal $\mathcal{F}$-subgroup of G. If V is an $\mathcal{F}$-injector of G, then $V \cap X = X$ by (3). Hence X is contained in every $\mathcal{F}$-injector V of G and then $X \leqslant \bigcap_{g \in G} V^g = G_{\mathcal{F}}$.

It is clear that (4) implies (5) and (5) implies (6).

(6) implies (1). By Lemma 1 it is enough to show that every primitive group $G \in \mathcal{F} \cap b(F(p))$ is cyclic.

It is clear that G has a unique minimal normal subgroup N, and N is a q-group, where q is a prime number, $q \neq p$. Then there exists an irreducible and faithful G-module V_p over $GF(p)$. We claim that G has a unique maximal subgroup M such that $\operatorname{Core}_G(M) = 1$. Assume that M_1 and M_2 are maximal subgroups of G, $M_1 \neq M_2$ and $\operatorname{Core}_G(M_i) = 1$,

$i = 1, 2$. Then $M_i \in F(p)$. Consider now the semidirect product $H = [V_p]G$ with respect to the action of G on V_p. Clearly $H \notin \mathcal{F}$, so $H^{\mathcal{F}} = V_p$. On the other hand, for $i = 1, 2$, $V_p M_i$ is an $\mathcal{F}$-normal maximal subgroup of H and $V_p M_i \in \mathcal{S}_p F(p) = F(p) \subseteq \mathcal{F}$. Now $H = \langle V_p M_1, V_p M_2 \rangle = \langle V_p M_1, (V_p M_1)^g \rangle$ for some $g \in H$. By applying (6), we have $H \in \mathcal{F}$, a contradiction.

Consequently, G is cyclic and the circle of implications is complete. □

We say that a saturated Fitting formation containing $\mathcal{N}$ is a *lattice-formation* if it satisfies one of the conditions of Theorem 1.

3. SOME APPLICATIONS

In this section we give some applications of our main result in the framework of factorised groups. We recall here some definitions and elementary facts about these groups (see [1]). If a group G is the product of two subgroups A and B, we say that a subgroup S is *factorised* if $S = (A \cap S)(B \cap S)$ and $A \cap B \leqslant S$. If N is a normal subgroup of a factorised group $G = AB$, then there exists a subgroup $X(N)$, called the *factoriser* of N in $G = AB$, which is the smallest factorised subgroup of G containing N and it has the form $X(N) = AN \cap BN = (A \cap BN)N = (B \cap AN)N = (A \cap BN)(B \cap AN)$. This fact motivates the study of trifactorised groups $G = AB = AK = BK$, where K is a normal subgroup of G.

It is well-known that a triply factorised group $G = AB = AK = BK$ of two subgroups A and B and a normal subgroup K is nilpotent provided that A, B and K are nilpotent (see [1, Corollary 1.3.5]). The next result shows that the same is true for lattice-formations.

PROPOSITION 1. *Let $\mathcal{F}$ be a lattice-formation. Let G be a group of the form $G = AB = AK = BK$, where A and B are subgroups of G and K is a normal subgroup of G. If A, B and K are $\mathcal{F}$-groups, then G is also an $\mathcal{F}$-group.*

PROOF: First notice that, although G is assumed soluble, the solubility of G follows automatically from the conditions on the subgroups A, B and K.

Let $\{\pi_i\}_{i \in I}$ be a partition of the set $\mathbb{P}$ of all primes and let $\mathcal{F}$ be the formation locally defined by the integrated and full formation function F given by $F(p) = \mathcal{S}_{\pi_i}$ for every prime number $p \in \pi_i$ and for every $i \in I$. Assume that the result is false and choose for G a counterexample of smallest order.

It is clear that G is in the boundary of $\mathcal{F}$. Consequently G must be a primitive group with a unique minimal normal subgroup N. Let p be the prime dividing the order of N and let $i \in I$ such that $p \in \pi_i$. Now, since K is a normal $\mathcal{F}$-subgroup of G, if $j \in I$ and $j \neq i$, we have that $O_{\pi_j}(K) \leqslant C_G(N) \leqslant N$. So $O_{\pi_j}(K) = 1$ and K is a π_i-group.

Let $A_{\pi_i'}$ and $B_{\pi_i'}$ be the Hall π_i'-subgroups of A and B. Since $G \notin \mathcal{F}$, it follows that G is not a π_i-group. So we can assume that $A_{\pi_i'} \neq 1$. On the other hand it is clear that

$A_{\pi_i'} B_{\pi_i'}^g = B_{\pi_i'}^g A_{\pi_i'}$ for every element $g \in G$. So $[A_{\pi_i'}^G, B_{\pi_i'}^G] = 1$ by [1, Lemma 2.5.1]. Since $A_{\pi_i'}^G \neq 1$, we have that $N \leqslant A_{\pi_i'}^G$ and so $B_{\pi_i'} \leqslant C_G(N) \leqslant N$. This implies that B is a π_i-group and so is G, a contradiction.   □

The previous result is a particular case of a more general one which will appear in [**3**].

THEOREM 2. *Let $\mathcal{F}$ be a subgroup-closed Fitting formation containing $\mathcal{N}$. Then the following statements are equivalent:*

(1)   *$\mathcal{F}$ is a lattice-formation.*

(2)   *$\mathcal{F}$ satisfies the property:*

(*)   *If $G = AB$ and A and B are $\mathcal{F}$-groups, then the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of G is a factorised subgroup, that is, $G_{\mathcal{F}} = (G_{\mathcal{F}} \cap A)(G_{\mathcal{F}} \cap B)$ and $A \cap B$ is contained in $G_{\mathcal{F}}$.*

PROOF: (1) implies (2). Let $\{\pi_i\}_{i \in I}$ be a partition of the set $\mathbb{P}$ of all primes and assume that $\mathcal{F}$ is the saturated Fitting formation locally defined by the integrated and full formation function F given by $F(p) = \mathcal{S}_{\pi_i}$ for every prime number $p \in \pi_i$ and for every $i \in I$.

By induction on the order of G, we may suppose that the $\mathcal{F}$-radical $L/G_{\mathcal{F}}$ of the factor group $G/G_{\mathcal{F}} = (AG_{\mathcal{F}}/G_{\mathcal{F}})(BG_{\mathcal{F}}/G_{\mathcal{F}})$ is factorised. Therefore $G_{\mathcal{F}} \leqslant X(G_{\mathcal{F}}) = AG_{\mathcal{F}} \cap BG_{\mathcal{F}} \leqslant L$ and hence $X(G_{\mathcal{F}})$ is an $\mathcal{F}$-subnormal subgroup of G since $X(G_{\mathcal{F}})/G_{\mathcal{F}}$ is $\mathcal{F}$-subnormal in $L/G_{\mathcal{F}} \in \mathcal{F}$. Moreover $X(G_{\mathcal{F}}) = (A \cap BG_{\mathcal{F}})G_{\mathcal{F}} = (B \cap AG_{\mathcal{F}})G_{\mathcal{F}} = (A \cap BG_{\mathcal{F}})(B \cap AG_{\mathcal{F}})$ is an $\mathcal{F}$-group by Proposition 1. Hence by Theorem 1, we have that $X(G_{\mathcal{F}}) = G_{\mathcal{F}}$ and $G_{\mathcal{F}}$ is factorised.

(2) implies (1). We prove that every primitive group $G \in \mathcal{F} \cap b(F(p))$ is cyclic and then we apply Lemma 1 to obtain the result. Let N be the unique minimal normal subgroup of the primitive group $G \in \mathcal{F} \cap b(F(p))$. Then N is a q-group, where q is a prime number, $q \neq p$. If we assume that G is not a cyclic group, there exists $M \neq 1$, $M \in F(p)$ such that G is the semidirect product $G = [N]M$. There exists an irreducible and faithful G-module V_p over $GF(p)$. Consider now the semidirect product $H = [V_p]G$ with respect to the action of G on V_p. Then $H = (V_p M)(NM)$ where $V_p M \in \mathcal{S}_p F(p) = F(p) \subseteq \mathcal{F}$ and $G = NM \in \mathcal{F}$. By (2), $H_{\mathcal{F}}$ is a factorised subgroup, so $M = V_p M \cap NM \leqslant H_{\mathcal{F}}$. On the other hand, since $H_{\mathcal{F}} \cap G$ is a non-trivial normal subgroup of G, we have $N \leqslant H_{\mathcal{F}} \cap G \leqslant H_{\mathcal{F}}$. Thus $G = [N]M \leqslant H_{\mathcal{F}}$. But then $H = [V_p]G \in \mathcal{F}$ and $G \in F(p)$, a contradiction.   □

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