Complexity spaces as quantitative domains of computation

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Abstract

We study domain theoretic properties of complexity spaces. Although the so-called complexity space is not a domain for the usual pointwise order, we show that, however, each pointed complexity space is an ω-continuous domain for which the complexity quasi-metric induces the Scott topology, and the supremum metric induces the Lawson topology. Hence, each pointed complexity space is both a quantifiable domain in the sense of M. Schellekens and a quantitative domain in the sense of P. Waszkiewicz, via the partial metric induced by the complexity quasi-metric.

Key words: Complexity space, pointed, continuous domain, Scott topology, quantitative domain.

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1 Introduction

Quantitative Domain Theory is concerned with models of computation that, in addition to *qualitative information* - such as specifying the meaning of a computation in an order context - allow also for the extraction of *quantitative information* - such as determining the complexity of a program. Quantitative Domain Theory also plays a role in models for real-number computation where quantitative aspects arise directly due to the numeric nature of the processes under consideration.

On the other hand, addressing the long standing open problem to combine Semantics and Complexity has generated models which target the extraction of quantitative information of programs based on traditional semantics techniques. This has led to the theory of complexity spaces among other approaches (see [2, 11, 12, 16, 14, etc])

Complexity spaces enabled elegant semantics style proofs (unique fixed point arguments) for the complexity of Divide and Conquer style algorithms [3, 11, 13, 16].

Since quantitative domains are partially metrizable and complexity spaces are partial metric spaces and enable the extraction of quantitative information, it is natural to ask to what extent the complexity spaces can be incorporated as a Quantitative Domain. This is the topic of the present paper.

We recall some relevant results from Quantitative Domain Theory before stating our main results.

A central result in Quantitative Domain Theory states that all ω-continuous domains, are "quantifiable", i.e., they can be equipped with a partial metric that induces the Scott topology and the partial metric order coincides with the domain order. This quantification theorem was independently obtained by Schellekens [18] and Waszkiewicz [20], by using different techniques. They also deduced a quantification theorem for the ω-algebraic case which was previously obtained by O'Neill [10] in terms of generalized valuation spaces. The results, in view of the countable base requirement, regard models for traditional programming languages. More recently, Waszkiewicz proved in [21] that every ω-continuous domain can be equipped with a partial metric whose induced topology is weaker than the Scott topology but the supremum metric induces the Lawson topology.

In this paper we will rely on the notion of a quantifiable domain as discussed in [18] and of a quantitative domain as discussed in [21] (see also [20, Section 7]).

To this end we consider two classes of complexity spaces: pointed complexity spaces, i.e., complexity spaces with a minimum element on the complexity functions, and the general complexity space.

Pointed complexity spaces are interesting in their own right, as motivated below. We remark that the weighting function (and hence the self-distance of the associated partial metric) of the complexity space is not bounded. However, as discussed in [11], complexity functions of programs computing a given problem frequently can be shown to have a complexity lower bound.

A case in point is the collection of comparison based sorting algorithms that satisfy the well-known $\Omega(nlogn)$ lower bound.

A theoretical justification for the existence of lower bounds has been given in [12] based on Levin's Theorem (e.g. [6]). It is remarked in [6] that for an important class of problems that occur in practice an optimal algorithm does exist, and hence one does obtain a least element for these classes.

So it is reasonable to study the restriction of the complexity space to complexity functions respecting a given least element, i.e., consider pointed complexity spaces.

It is easy to verify that the complexity quasi-metric is bounded on such restricted spaces and that, as a corollary, these spaces are weightable. For more information on complexity spaces with a lower bound we refer the reader to [12, 11].

In this work, pointed complexity spaces are shown to be ω-continuous domains and hence quantitative domains. In fact, we will show that they are both quantifiable in the sense of [18] and quantitative in the sense of [21], via the partial metric induced by the complexity quasi-metric. The general complexity space is shown not to be a continuous domain. However, we will observe that the space of formal balls associated with the complexity space is both a quantifiable and quantitative domain.

2 Preliminaries

Our basic reference for Domain Theory is [4].

Let us recall that a partially ordered set, or poset for short, is a set L equipped with a partial order $\leq$. It will be denoted in the sequel by $(L, \leq)$.

A subset D of a poset $(L, \leq)$ is directed provided that it is nonempty and every finite subset of D has an upper bound in D (equivalently, if for each $a, b \in D$ there is $c \in D$ such that $a \leq c$ and $b \leq c$).

A poset $(L, \leq)$ is said to be directed complete, and is called a dcpo, if every directed subset of L has a least upper bound.

The least upper bound of a subset D of $(L, \leq)$ will be denoted by $\sup D$ if it exists.

An element x of L is called maximal if condition $x \leq y$ implies $x = y$. The set of all maximal elements of L is denoted by $\mathrm{Max}((L, \leq))$, or simply by $\mathrm{Max}(L)$ if no confusion arises.

Given a poset $(L, \leq)$, we say that x is way-below y, in symbols $x \ll y$, if for each directed subset D of L for which $\sup D$ exists, the relation $y \leq \sup D$ implies the existence of some $z \in D$ with $x \leq z$.

A poset $(L, \leq)$ is called continuous if it satisfies the axiom of approximation, i.e. for all $x \in L$, the set $\Downarrow x = \{u \in L : u \ll x\}$ is directed and $x = \sup(\Downarrow x)$.

A continuous poset which is also a dcpo is called a domain.

A subset B of a poset $(L, \leq)$ is a basis for L if for each $x \in L$, the set $\Downarrow x_B = \{u \in B : u \ll x\}$ is directed and $x = \sup(\Downarrow x_B)$.

Recall that a poset has a basis if and only if it is continuous. Therefore, a dcpo has a basis if and only if it is a domain.

A dcpo having a countable basis is said to be an ω-continuous domain [4]. In order to simplify the terminology, ω-continuous domains will be simply called ω-domains in the sequel.

The Scott topology $\sigma(L)$ of a dcpo $(L, \leq)$ is constructed as follows (Chapter II in [4]): A subset U of L is open in the Scott topology provided that: (i) $U = \uparrow U$, where $\uparrow U = \{y \in X : x \leq y$ for some $x \in U\}$; and (ii) for each directed subset D of L such that $\sup D \in U$, it follows that $D \cap U \neq \emptyset$.

The lower (or weak) topology of a dcpo $(L, \leq)$ is the one that has as a subbase the collection of sets of the form $L \setminus \uparrow x$, where $x \in L$, and denote it by $\omega(L)$. Let us recall that the supremum topology of $\sigma(L)$ and $\omega(L)$ is the Lawson topology of $(L, \leq)$, which is denoted by $\lambda(L)$.

According to Smyth ([19]), by $\mathrm{CMax}(L)$ we denote the set of the constructively maximal points of L, i.e., $x \in \mathrm{CMax}(L)$ provided that every $\lambda(L)$-neighborhood of x contains a $\sigma(L)$-neighborhood of x.

We conclude this section with some pertinent concepts and results on quasi-metric spaces and partial metric spaces.

Following the modern terminology, by a quasi-metric on a set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Each quasi-metric d on X induces a T_0 topology $\mathcal{T}(d)$ on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that if (X, d) is a quasi-metric space, then the binary relation $\leq_d$ defined on X by $x \leq_d y \Leftrightarrow d(x, y) = 0$, is a partial order on X, called the specialization order. Hence $(X, \leq_d)$ is a poset.

Given a quasi-metric d on X, then the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the conjugate of d, and the function d^s defined by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$, is a metric on X.

The notion of a partial metric space, and its equivalent weightable quasi-metric space, was introduced by Matthews in [9] as a part of the study of denotational semantics of dataflow networks.

Let us recall that a partial metric ([9]) on a set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$; (ii) $p(x, x) \leq p(x, y)$; (iii) $p(x, y) = p(y, x)$; (iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a set and p is a partial metric on X.

Each partial metric p on X induces a T_0-topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-metric space (X, d) is called weightable if there exists a function $w : X \to \mathbb{R}^+$ such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. The function w is said to be a weighting function for (X, d) and the quasi-metric d is weightable by the function w.

The precise relationship between partial metric spaces and weightable quasi-metric spaces is provided in the next result.

Theorem A [9]. (a) *Let (X, p) be a partial metric space. Then, the function $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$ is a weightable quasi-metric on X with weighting function w given by $w(x) = p(x, x)$ for all $x \in X$. Furthermore $\mathcal{T}(p) = \mathcal{T}(d_p)$.*

(b) *Conversely, if (X, d) is a weightable quasi-metric space with weighting function w, then the function $p_d : X \times X \to \mathbb{R}^+$ defined by $p_d(x, y) = d(x, y) + w(x)$ for all $x, y \in X$, is a partial metric on X. Furthermore $\mathcal{T}(d) = \mathcal{T}(p_d)$.*

If (X, p) is a partial metric space, then the binary relation $\leq_p$ on X given by $x \leq_p y \Leftrightarrow p(x,y) = p(x,x)$, is a partial order on X, which is called the partial order induced by p. Hence $(X, \leq_p)$ is a poset. Note that in this case one has $\leq_p = \leq_{d_p}$.

In Definition 5.3 of [9], Matthews introduced the notion of a complete partial metric space. For our purposes here it suffices to recall that a partial metric space (X,p) is complete if and only if the metric space $(X,(d_p)^s)$ is complete.

3 Pointed complexity spaces are quantitative domains

Let us recall that the complexity (quasi-metric) space ([16]) is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \omega \to (0,\infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and d_C is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f,g) = \sum_{n=0}^{\infty} 2^{-n} \left((\frac{1}{f(n)} - \frac{1}{g(n)}) \vee 0 \right)$$

for all $f, g \in \mathcal{C}$.

Schellekens proved in [16] that the complexity space is weightable with weighting function $w_{\mathcal{C}}$ given by $w_{\mathcal{C}}(f) = \sum_{n=0}^{\infty} 2^{-n}(1/f(n))$, for all $f \in \mathcal{C}$. Later on, it was proved in [11] that $(d_{\mathcal{C}})^s$ is a complete metric on $\mathcal{C}$.

Note that the partial metric $p_{d_{\mathcal{C}}}$, induced by $d_{\mathcal{C}}$ (see Theorem A), is given by

$$p_{d_{\mathcal{C}}}(f,g) = \sum_{n=0}^{\infty} 2^{-n} (\frac{1}{f(n)} \vee \frac{1}{g(n)}),$$

for all $f, g \in \mathcal{C}$.

In the following, the partial metric $p_{d_{\mathcal{C}}}$ will be simply denoted by $p_{\mathcal{C}}$.

Furthermore, if we define a binary relation $\leq$ on $\mathcal{C}$ by

$$f \leq g \Longleftrightarrow f(n) \leq g(n) \text{ for all } n \in \omega,$$

then, it is well known, and easy to see, that $\leq$ is a partial order on $\mathcal{C}$; in fact $\leq$ is the pointwise order. Hence $(\mathcal{C}, \leq)$ is a poset.

Note that $\leq = \leq_{d_\mathcal{C}}$ and that $\mathrm{Max}(\mathcal{C}) = \{f_\infty\}$, where f_∞ is defined by $f_\infty(n) = \infty$ for all $n \in \omega$.

The proof of the following well-known fact is straightforward, so it is omitted.

Proposition 1. *For each non-empty subset D of $\mathcal{C}$, let $F : \omega \to (0, \infty]$ given by*

$$F(n) = \sup_{f \in D} f(n),$$

for all $n \in \omega$. Then F is the least upper bound of D in $(\mathcal{C}, \leq)$.

Corollary 1. *$(\mathcal{C}, \leq)$ is a dcpo.*

Next we show that, unfortunately, the dcpo $(\mathcal{C}, \leq)$ is not continuous. Actually, this fact is an obvious consequence of the following more general result.

Proposition 2. *For each $f \in \mathcal{C}$, $\Downarrow f = \emptyset$.*

Proof. Let $f \in \mathcal{C}$. Suppose that there is $g \in \mathcal{C}$ such that $g \ll f$. Define a sequence $(f_k)_k$ in $\mathcal{C}$ as follows: For each $k \in \mathbb{N}$, put

$$f_k(n) = \begin{cases} g(n)/2, & n > k \\ f(n), & n \leq k \end{cases}$$

whenever $n \in \omega$. It is clear that $f_k \leq f_{k+1}$ for all $k \in \mathbb{N}$, so $D = \{f_k : k \in \mathbb{N}\}$ is a directed set. Moreover, and according to Proposition 1, the function F given by $F(n) = \sup_k f_k(n)$ for all $n \in \omega$, is the least upper bound of D in $(\mathcal{C}, \leq)$. However $g(n) > f_{n-1}(n)$ for all $n \in \mathbb{N}$, which contradicts that $g \ll f$. We conclude that $\Downarrow f = \emptyset$.$\blacksquare$

Motivated by the computational interest of those subspaces of the complexity space $(\mathcal{C}, d_\mathcal{C})$ having a lower bound, and by the fact that $(\mathcal{C}, \leq)$ is not a domain, we shall focus our attention on the study of the domain-theoretic properties of the so-called pointed complexity spaces, a class of subspaces of the complexity space that are defined as follows.

Definition 1. A pointed complexity space is a pair $(\mathcal{C}_{f_0}, d_{\mathcal{C}_{f_0}})$ such that $f_0 \in \mathcal{C}$, $\mathcal{C}_{f_0} := \{f \in \mathcal{C} : f_0 \leq f\}$, and $d_{\mathcal{C}_{f_0}}$ is the restriction of the complexity quasi-metric $d_{\mathcal{C}}$ to $\mathcal{C}_{f_0}$.

Obviously $(\mathcal{C}_{f_0}, d_{\mathcal{C}_{f_0}})$ is weightable with weighting function the restriction of $w_{\mathcal{C}}$ to $\mathcal{C}_{f_0}$. Moreover $(d_{\mathcal{C}_{f_0}})^s$ is a complete metric on $\mathcal{C}_{f_0}$ by [11, Theorem 9].

On the other hand $(\mathcal{C}_{f_0}, \leq)$ is a dcpo by Proposition 1, with $\mathrm{Max}(\mathcal{C}_{f_0}) = \{f_\infty\}$ and f_0 its least element.

In fact, it is straightforward to see that $(\mathcal{C}_{f_0}, \leq)$ is a complete lattice. In the next theorem we prove that it is also an ω-domain and consequently it will be an ω-continuous lattice [4, Definition I-1.6 (iii), p. 54].

Theorem 1. *$(\mathcal{C}_{f_0}, \leq)$ is an ω-domain.*

Proof. Since $(\mathcal{C}_{f_0}, \leq)$ is a dcpo, it will be sufficient to prove that it has a countable basis. To this end, we shall show that the countable subset of $\mathcal{C}_{f_0}$,

$\mathcal{B} := \{f_0\} \cup \{f \in \mathcal{C}_{f_0}$: there is a finite subset ω_f of ω such that $f(n) \in \mathbb{Q}$ for all $n \in \omega_f$ and $f(n) = f_0(n)$ otherwise$\}$,

is a basis for $(\mathcal{C}_{f_0}, \leq)$. Indeed, fix $f \in \mathcal{C}_{f_0}$.

Claim 1. $\Downarrow f_{\mathcal{B}}$ is directed: In fact, $\Downarrow f_{\mathcal{B}} \neq \emptyset$ because $f_0 \ll f$. Moreover, if $f_1, f_2 \in \Downarrow f_{\mathcal{B}}$, then $f_1 \vee f_2 \in \Downarrow f_{\mathcal{B}}$ because, obviously, $f_1 \vee f_2 \in \mathcal{B}$, and if D is a directed subset of $(\mathcal{C}_{f_0}, \leq)$ such that $f \leq \sup D$, then there exist $g_1, g_2 \in D$ with $f_i \leq g_i$, $i = 1, 2$, so by directedness of D there exists $h \in D$ such that $g_1 \vee g_2 \leq h$, and hence $f_1 \vee f_2 \leq h$.

Claim 2. $f = \sup(\Downarrow f_{\mathcal{B}})$: Obviously $\sup(\Downarrow f_{\mathcal{B}}) \leq f$. Now let $h \in \mathcal{C}_{f_0}$ such that $g \leq h$ for all $g \in \Downarrow f_{\mathcal{B}}$. Suppose that $h(m) < f(m)$ for some $m \in \omega$. Let $h(m) < q < f(m)$, with $q \in \mathbb{Q}$, and consider the function $g \in \mathcal{B}$ defined by $g(m) = q$ and $g(n) = f_0(n)$ for all $n \in \omega \backslash \{m\}$. It is easily seen that $g \ll f$, but $g \nleq h$, which provides a contradiction. Therefore $f = \sup(\Downarrow f_{\mathcal{B}})$.

We conclude that $(\mathcal{C}_{f_0}, \leq)$ is an ω-domain.$\blacksquare$

Next we prove that $(\mathcal{C}_{f_0}, \leq)$ is a quantifiable domain in the sense of [18] and a quantitative domain in the sense of [21], by means of the partial metric $p_{\mathcal{C}_{f_0}}$ in both cases, where by $p_{\mathcal{C}_{f_0}}$ we denote the restriction to $\mathcal{C}_{f_0}$ of the partial metric $p_{\mathcal{C}}$.

Definition 2 ([18]). A quantifiable domain is a domain $(L, \leq)$ such that there is a partial metric p on L satisfying the following conditions:

(Sch1) $\mathcal{T}(p) = \sigma(L)$.
(Sch2) $\leq_p = \leq$.

Definition 3 ([21]). A quantitative domain is a domain $(L, \leq)$ such that there is a partial metric p on L satisfying the following conditions:
(Was1) $\mathcal{T}(p) \subseteq \sigma(L)$.
(Was2) The function $\mu_p : L \to \mathbb{R}^+$ given by $\mu_p(x) = p(x, x)$, is a measurement in the sense of Martin [8] (see also [20, 21]).
(Was3) $\ker \mu_p = \mathrm{CMax}(L)$.
(Was4) The metric $(d_p)^s$ induces the Lawson topology on L.

In the following, quantifiable domains and quantitative domains will be called S-quantitative domains and W-quantitative domains, respectively.

Since a domain can be simultaneously S-quantitative and W-quantitative via different partial metrics (see Remark 4 below), we propose the following notion.

Definition 4. A SW-quantitative domain is a domain $(L, \leq)$ such that there is a partial metric p on L for which $(L, \leq)$ is both a S-quantitative domain and a W-quantitative domain.

Remark 1. Note that condition (Sch1) implies (Sch2) because $\sigma(L)$ is an order-consistent topology in the sense of [4, Definition II-1.30]). (Sch1) also implies (Was2) by [20, Theorem 8]. Moreover (see, for instance, [21, p. 369]) one has $\mathrm{CMax}(L) = \mathrm{Max}(L)$ whenever the the Scott and Lawson topologies agree on $\mathrm{Max}(L)$.

We deduce from Remark 1 that a domain $(L, \leq)$ is SW-quantitative if and only if there is a partial metric p on L satisfying conditions (Sch1), (Was3) and (Was4).

Remark 2. (a) As we indicated in Section 1, Schellekens and Waszkiewicz ([18, 20]) independently proved, among other results, that every ω-domain is S-quantitative, and Waszkiewicz proved in [21, Theorem 6.5] that every ω-domain with a least element is W-quantitative.

(b) Notice that, actually, one has that each ω-domain is W-quantitative, as it is observed in the last comment of [21]: Indeed, if $(L, \leq)$ is an ω-domain, then its lifting $L \cup \{\perp\}$ is also an ω-domain with least element $\perp$, so by [21, Theorem 6.5], there is a partial metric p on $L \cup \{\bot\}$ for which conditions (Was1)-(Was4) of Definition 3 hold. Then, it is straightforward to verify that $(L, \leq)$ is a W-quantitative domain via the restriction of p to L.

From Theorem 1 and Remark 2 (a), it follows that $(\mathcal{C}_{f_0}, \leq)$ is both a S-quantitative domain and a W-quantitative domain. We shall prove that actually the partial metric induced by the quasi-metric $d_{\mathcal{C}_{f_0}}$ endows to $(\mathcal{C}_{f_0}, \leq)$ with the structure of a SW-quantitative domain. To this end, we need the next auxiliary two lemmas.

Lemma 1 ([5, Theorem 2.18]). *Let (X,d) be a weightable quasi-metric space. If D is a directed subset of $(X, \leq_d)$, then there exists an ascending sequence in D which has the same upper bounds as D.*

Although Lemma 2 below can be deduced from some statements in [**?**, p. 189] and [7, Remark 1], we give a direct proof of it in order to help the reader.

Lemma 2. *Let $(f_k)_k$ be an ascending sequence in $(\mathcal{C}, \leq)$ and let $F = \sup_k f_k$. Then $(d_{\mathcal{C}})^s(F, f_k) \to 0$ as $k \to \infty$.*

Proof. For each $k \in \mathbb{N}$ we have $d_{\mathcal{C}}(f_k, F) = 0$. So, it remains to show that $d_{\mathcal{C}}(F, f_k) \to 0$ *as* $k \to \infty$. To this end choose an arbitrary $\varepsilon > 0$. Then, there exists n_ε such that $\sum_{n=n_\varepsilon+1}^{\infty} 2^{-n}(1/f_1(n)) < \varepsilon$. Since $F = \sup_k f_k$ and $(f_k)_k$ is ascending, there is k_ε such that for each $k \geq k_\varepsilon$ and each $n \in \{0, 1, ..., n_\varepsilon\}$,

$$2^{-n}(\frac{1}{f_k(n)} - \frac{1}{F(n)}) < \varepsilon.$$

Hence, for each $k \geq k_\varepsilon$ we obtain

$$\begin{aligned} d_{\mathcal{C}}(F, f_k) &= \sum_{n=0}^{\infty} 2^{-n}((\frac{1}{f_k(n)} - \frac{1}{F(n)}) \vee 0) \\ &\leq \sum_{n=0}^{n_\varepsilon} 2^{-n}((\frac{1}{f_k(n)} - \frac{1}{F(n)}) \vee 0) + \sum_{n=n_\varepsilon+1}^{\infty} 2^{-n}\frac{1}{f_k(n)} \\ &< 2\varepsilon + \sum_{n=n_\varepsilon+1}^{\infty} 2^{-n}\frac{1}{f_1(n)} < 3\varepsilon. \end{aligned}$$

Consequently $d_{\mathcal{C}}(F, f_k) \to 0$ *as* $k \to \infty$. This concludes the proof.■

Theorem 2. *For each $f_0 \in \mathcal{C}$, the following hold:*
(1) $\mathcal{T}(d_{\mathcal{C}_{f_0}}) = \sigma(\mathcal{C}_{f_0})$;
(2) $\mathcal{T}((d_{\mathcal{C}_{f_0}})^{-1}) = \omega(\mathcal{C}_{f_0})$;
(3) $\mathcal{T}((d_{\mathcal{C}_{f_0}})^s) = \lambda(\mathcal{C}_{f_0})$.

Proof. (1) Since the inclusion $\sigma(\mathcal{C}_{f_0}) \subseteq \mathcal{T}(d_{\mathcal{C}_{f_0}})$ follows from [20, Lemma 20], we only show that $\mathcal{T}(d_{\mathcal{C}_{f_0}}) \subseteq \sigma(\mathcal{C}_{f_0})$. Indeed, let $f \in \mathcal{C}_{f_0}$ and $\varepsilon > 0$. Obviously, $B_{d_{\mathcal{C}_{f_0}}}(f, \varepsilon) = \uparrow B_{d_{\mathcal{C}_{f_0}}}(f, \varepsilon)$. Moreover, if D is a directed set in $(\mathcal{C}_{f_0}, \leq)$ such that $\sup D \in B_{d_{\mathcal{C}_{f_0}}}(f, \varepsilon)$, then, by Lemma 1**,** there exists an ascending sequence $(f_k)_k$ in D such that $\sup D$ is the least upper bound of $(f_k)_k$. Therefore, by Lemma 2**,** the sequence $(f_k)_k$ converges to $\sup D$ with respect to the topology $\mathcal{T}((d_{\mathcal{C}_{f_0}})^s)$. Hence $f_k \in B_{d_{\mathcal{C}_{f_0}}}(f, \varepsilon)$ for some k, by the triangle inequality. We conclude that $\mathcal{T}(d_{\mathcal{C}_{f_0}}) \subseteq \sigma(\mathcal{C}_{f_0})$.

Next we show that $\sigma(\mathcal{C}_{f_0}) \subseteq \mathcal{T}(d_{\mathcal{C}_{f_0}})$. Indeed, suppose that there exists $U \in \sigma(\mathcal{C}_{f_0}) \backslash \mathcal{T}(d_{\mathcal{C}_{f_0}})$. Then, there exist $f \in U$ and a sequence $(f_k)_k$ in $\mathcal{C}_{f_0} \backslash U$ such that $d_{\mathcal{C}}(f, f_k) < 2^{-k}$ for all k. Put $g_k = \inf_{n \geq k} f_n$ for all k. Then $(g_k)_k$ is an ascending sequence in $(\mathcal{C}_{f_0}, \leq)$. Set $g = \sup_k g_k$. It is not hard to check that $f \leq g$, so $g \in U$. Since $U \in \sigma(\mathcal{C}_{f_0})$, then $g_k \in U$ for some k; so $f_k \in U$ because $g_k \leq f_k$, which yields a contradiction. We conclude that $\sigma(\mathcal{C}_{f_0}) \subseteq \mathcal{T}(d_{\mathcal{C}_{f_0}})$.

(2) Since the inclusion $\omega(\mathcal{C}_{f_0}) \subseteq \mathcal{T}((d_{\mathcal{C}_{f_0}})^{-1})$ follows from [20, Lemma 20], we only show that $\mathcal{T}((d_{\mathcal{C}_{f_0}})^{-1}) \subseteq \omega(\mathcal{C}_{f_0})$. Indeed, let $f \in \mathcal{C}_{f_0}$ and $\varepsilon > 0$. Then, there exists n_ε such that $\sum_{n=n_\varepsilon+1}^{\infty} 2^{-n}(1/f(n)) < \varepsilon$.

Suppose that $f(n) = \infty$ for all $n \in \{0, 1, ..., n_\varepsilon\}$. Then $\mathcal{C}_{f_0} = B_{(d_{\mathcal{C}_{f_0}})^{-1}}(f, \varepsilon)$ because for each $g \in \mathcal{C}_{f_0}$ we have

$$\begin{aligned} d_{\mathcal{C}_{f_0}}(g, f) &= \sum_{n=0}^{\infty} 2^{-n} \left((\frac{1}{f(n)} - \frac{1}{g(n)}) \vee 0 \right) \\ &= \sum_{n=n_\varepsilon+1}^{\infty} 2^{-n} \left((\frac{1}{f(n)} - \frac{1}{g(n)}) \vee 0 \right) < \varepsilon. \end{aligned}$$

Finally, suppose that there exist $n \in \{0, 1, ..., n_\varepsilon\}$ for which $f(n) < \infty$. Then, for each $n \in \{0, 1, ..., n_\varepsilon\}$ with $f(n) < \infty$, we define a function $h_n \in \mathcal{C}_{f_0}$ by $h_n(n) = f(n) + \delta_n$, where $\delta_n = (f(n))^2 \varepsilon$ and $h_n(m) = f_0(m)$ whenever $m \neq n$. Put

$$U = \bigcap \{\mathcal{C}_{f_0} \backslash \uparrow h_n : n \in \{0, 1, ..., n_\varepsilon\} \text{ and } f(n) < \infty\}.$$

Then $f \in U \in \omega(\mathcal{C}_{f_0})$. Moreover for each $g \in U$ and each $n \in \{0,1,...,n_\varepsilon\}$ with $f(n) < \infty$, we have that $g(n) < f(n) + \delta_n$. Then, it is easily checked that

$$\frac{1}{f(n)} - \frac{1}{g(n)} < \varepsilon.$$

Hence

$$\begin{aligned} d_{\mathcal{C}_{f_0}}(g,f) &\leq \sum_{n=0}^{n_\varepsilon} 2^{-n}((\frac{1}{f(n)} - \frac{1}{g(n)}) \vee 0) + \sum_{n=n_\varepsilon+1}^{\infty} 2^{-n}\frac{1}{f(n)} \\ &< 2\varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

We have shown that $U \subseteq B_{(d_{\mathcal{C}_{f_o}})^{-1}}(f, 3\varepsilon)$. Consequently $\mathcal{T}((d_{\mathcal{C}_{f_0}})^{-1}) \subseteq \omega(\mathcal{C}_{f_0})$.

(3) Is an immediate consequence of (1) and (2).■

Remark 3. Note that $(\mathcal{C}_{f_0}, (d_{\mathcal{C}_{f_o}})^s)$ is a compact metric space ([11]), so Proposition 24 of [20] yields the equality obtained in the statement (3) of Theorem 2. Nevertheless, this equality is deduced here as a natural factorization of statements (1) and (2) of the aforementioned theorem.

Theorem 3. *For each $f_0 \in \mathcal{C}$, $(\mathcal{C}_{f_0}, \leq)$ is a SW- quantitative domain via the partial metric $p_{\mathcal{C}_{f_0}}$.*

Proof. By Theorems 1 and 2, it only remains to show that $\ker\mu_{p_{\mathcal{C}_{f_0}}} = \mathrm{CMax}(\mathcal{C}_{f_0})$. Indeed, it is clear that $\ker\mu_{p_{\mathcal{C}_{f_0}}} = \{f_\infty\}$, and that $f_\infty \in \mathrm{CMax}(\mathcal{C}_{f_0})$ because $\mathcal{C}_{f_0}$ is the only neighborhood of f_∞ in $\omega(\mathcal{C}_{f_0})$. Finally, if $f \in \mathcal{C}_{f_0} \backslash \{f_\infty\}$, then $f \in \mathcal{C}_{f_0} \backslash \uparrow f_\infty$, but $f_\infty \in B_{d_{\mathcal{C}_{f_0}}}(f, \varepsilon)$ for all $\varepsilon > 0$, so that $f \notin \mathrm{CMax}(\mathcal{C}_{f_0})$. We conclude that $\ker\mu_{p_{\mathcal{C}_{f_0}}} = \mathrm{CMax}(\mathcal{C}_{f_0}) = \{f_\infty\}$. The proof is complete.■

Remark 4. In [21, Example 6.1] an ω-domain $(L, \leq)$ is constructed for which there does not exist any partial metric satisfying at the same time conditions (Sch1) and (Was3). In fact, the Scott and Lawson topologies agree on $\mathrm{Max}(L)$ and thus $\mathrm{Max}(L) = \mathrm{CMax}(L)$. Hence, this ω-domain provides an example of a S-quantitative and W-quantitative domain which is not SW-quantitative.

We finish the paper with some comments on the poset of formal balls of the complexity space $(\mathcal{C}, d_{\mathcal{C}})$.

In order to help the reader, we first recall some notions and facts on formal balls for partial metric spaces (see [15]) which extend to our context well-known results by Edalat and Heckmann ([1]) for metric spaces.

Given a partial metric space (X, p), the associated set of formal balls is the poset $(\mathbf{B}X, \sqsubseteq_{d_p})$, where $\mathbf{B}X = \{(x, r) : x \in X,\ r \in \mathbb{R}^+\}$ and the order relation $\sqsubseteq_{d_p}$ is given by

$$(x, r) \sqsubseteq_{d_p} (y, s) \Leftrightarrow d_p(x, y) \leq r - s.$$

Among others results, the following theorem was proved in [15].

Theorem 4. *Let (X, p) be a partial metric space. Then the metric space $(X, (d_p)^s)$ is separable and complete if and only if $(\mathbf{B}X, \sqsubseteq_{d_p})$ is an ω-domain.*

In Proposition 2 we have shown that the poset $(\mathcal{C}, \leq)$ is a dcpo that is not continuous. Consequently the complexity space is not a domain and hence neither is a S-quantitative domain nor is a W-quantitative domain. However, since the partial metric space $(\mathcal{C}, p_{\mathcal{C}})$ verifies that $(\mathcal{C}, (d_{\mathcal{C}})^s)$ is a separable complete metric space (recall that $d_{\mathcal{C}} = d_{p_{\mathcal{C}}}$), it follows from the preceding theorem that the poset of formal balls $(\mathbf{B}\mathcal{C}, \sqsubseteq_{d_{\mathcal{C}}})$ is an ω-domain. Therefore it is both a S-quantitative and W-quantitative domain. So that, although the poset $(\mathcal{C}, \leq)$ is not a domain, we can obtain from it "computational models" which are quantitative domains. Despite these facts, the following natural question remains open: Is $(\mathbf{B}\mathcal{C}, \sqsubseteq_{d_{\mathcal{C}}})$ a SW-quantitative domain?

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