Document downloaded from:

http://hdl.handle.net/10251/48422

This paper must be cited as:

Frerick, L.; Jorda Mora, E.; Kalmes, T.; Wengenroth, J. (2014). Strongly continuous semigroups on some Fréchet spaces. Journal of Mathematical Analysis and Applications. 412(1):121-124. doi:10.1016/j.jmaa.2013.10.053.

The final publication is available at

http://dx.doi.org/10.1016/j.jmaa.2013.10.053

Copyright Elsevier

STRONGLY CONTINUOUS SEMIGROUPS ON SOME FRÉCHET SPACES

LEONHARD FRERICK, ENRIQUE JORDÁ, THOMAS KALMES,
AND JOCHEN WENGENROTH

Abstract. We prove that for a strongly continuous semigroup T on the Fréchet space ω of all scalar sequences, its generator is a continuous linear operator $A \in L(\omega)$ and that, for all $x \in \omega$ and $t \geq 0$, the series $\exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)$ converges to $T_t(x)$. This solves a problem posed by Conejero. Moreover, we improve recent results of Albanese, Bonet, and Ricker about semigroups on strict projective limits of Banach spaces.

1. Introduction

In [Con07] Conejero asked whether on $\omega = \mathbb{K}^{\mathbb{N}}$ every strongly continuous semigroup T is of the form $T_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)$ for a continuous linear operator A on ω. This question arose in the context of hypercyclicity: It is shown in [Con07, Theorem 2.7] that no such semigroup on ω can be hypercyclic. Although it has been proved by Shkarin in [Shk11] that there are not even supercyclic strongly continuous semigroups on ω, the question of Conejero remained open. Only a partial answer is contained in [ABR10].

The definitions and most basic results for semigroups on locally convex spaces X are the same as for Banach spaces, we refer to [Kom64, Kōm68, Ōuc73, Yos65]. A strongly continuous semigroup T on X is thus a morphism from the semigroup $([0,\infty),+)$ to that of continuous linear operators $(L(X),\circ)$ such that all orbits $t \mapsto T_t(x)$ are continuous. If the convergence $T_t \to id_X$ for $t \to 0$ is uniorm on bounded subsets of X, the semigroup is called uniformly continuous.

T is called locally equicontinuous if $\{T(s) : 0 \leq s \leq t\}$ is equicontinuous for every $t > 0$, i.e., for every continuous seminorm p on X there is another continuous seminorm q on X such that $p(T_s(x)) \leq q(x)$ for all $x \in X$ and $0 \leq s \leq t$. On barrelled spaces, every strongly continuous semigroup is already locally equicontinuous [Kōm68, Proposition 1.1]. The generator $(A, D(A))$ of a strongly continuous semigroup on a locally convex space is defined as in the Banach space setting as the derivative of the orbit at 0.

Although under rather weak assumptions (sequential completeness to have a vector valued intergal and barrelledness to apply the uniform boundedness principle) many results from the Banach space setting carry over to strongly continuous semigroups of locally convex spaces there are some crucial differences because the

exponential $\exp(A)(x) = \sum_{k=0}^{\infty} A^k(x)/k!$ need not converge for continuous linear operators. Therefore, one does not always have the familar representation $T_t = \exp(tA)$ for semigroups with continuous generators:

Example 1. Consider $\mathscr{C}^\infty(\mathbb{R})$ with its usual topology and the strongly continuous semigroup defined by $T_t(f)(x) = f(x+t)$. Then $D(A) = \mathscr{C}^\infty(\mathbb{R})$ and $Af = f'$. For any f which is flat at the origin but does not vanish on $(0,\infty)$ the series $\exp(tA)(f) = \sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)}$ cannot converge to $T_t(f)$ because for $t > 0$ with $f(t) \neq 0$ we have

$$f(t) = T_t(f)(0) \neq \sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)}(0) = 0.$$

Using E. Borel's theorem, that a smooth function can have any given sequence of derivatives at 0, one also gets $f \in \mathscr{C}^\infty(\mathbb{R})$ such that the exonential series diverges.

Answering Conjero's question, we will prove that such phenomena do not occur on the Fréchet space ω.

2. Semigroups on strict projective limits

Let X_n be a sequence of Banach spaces, $\pi_m^n : X_m \to X_n$ norm decreasing operators for $n \leq m$ with $\pi_m^n \circ \pi_k^m = \pi_k^n$ as well as $\pi_n^n = id_{X_n}$, and

$$X = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \pi_m^n(x_m) = x_n \text{ for all } n \leq m\}$$

its projective limit. Every Fréchet space has such a representation, and X is called a *quojection* if there is a representaion with surjective (hence open) π_m^n. Countable products of Banach spaces are of this form, in particular spaces like $L^p_{loc}(\Omega)$ of locally p-integrable functions or $\mathscr{C}^n(\Omega)$ for open sets $\Omega \subseteq \mathbb{R}^d$, but there are quojections which are not isomorphic to a product.

If X has a strict representation as above then, by a simple induction, $\pi_m : X \to X_n$, $(x_n)_{n\in\mathbb{N}} \mapsto x_m$ are also surjective. Applying this observation to the spaces $\ell^\infty_I(X_n)$ of bounded functions $I \to X_n$ one obtains that π_m lifts bounded sets, that is, there are bounded sets $D_n \subseteq X$ such that $\pi_n(D_n)$ contains the unit ball B_n of the Banach space X_n. This lifting property was first proved in [DZ84].

The following theorem improves results of Albanese, Bonet, and Ricker [ABR10] who showed its first part under restrictive additional assumptions.

Theorem 2. *Let X be a quojection.*

(1) If T is a uniformly continuous semigroup on X then its generator is continuous and everywhere defined, and for all $x \in X$ and $t \geq 0$ we have

$$T_t(x) = \exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x).$$

(2) $A \in L(X)$ generates a strongly continuous semigroup if and only if

$$\forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}\ \forall\, k \in \mathbb{N}_0 :\ \pi_m(x) = 0 \implies \pi_n(A^k(x)) = 0.$$

Then the generated semigroup is even uniformly continuous.

Proof. As X is barrelled the semigroup is locally equicontinuous, so that, for every $t_0 > 0$ and $n \in \mathbb{N}$, there are $m \geq n$ and $c > 0$ such that for all $x \in X$ and $t \in [0, t_0]$

$$\|\pi_n(T_t(x))\|_n \leq c\|\pi_m(x)\|_m.$$

In particular, we have

$$(*) \quad \pi_m(x) = 0 \implies \pi_n(T_t(x)) = 0 \text{ for all } t \leq t_0.$$

As in the case of semigroups on Banach spaces it is easily seen that the Cesaro means

$$C_t(x) = \frac{1}{t}\int_0^t T_s(y)ds$$

belong to $D(A)$, $A(C_t(y)) = \frac{1}{t}(T_t(y) - y)$, and $C_t(y) \to y$ holds uniformly on bounded sets since T is uniformly continuous. Moreover, C_t satisfy the same continuity estimates as T_t. Therefore, the following operators are well-defined and continuous for $t \in (0, t_0]$:

$$\tilde{C}_t : X_m \to X_n \ \pi_m(y) \mapsto \pi_n(C_t(y)).$$

Since π_m lifts bounded sets, every z in the unit ball B_m of X_m can be represented as $z = \pi_m(y)$ with $y \in D_m$ and as $C_t \to id$ uniformly on D_m we obtain that $\tilde{C}_t$ converges uniformly on B_m to π_m^n. Since the set of surjective operators is open in $L(X_m, X_n)$ we conclude that $\tilde{C}_t$ is surjective for some sufficently small $t > 0$.

If now $n_0 \in \mathbb{N}$ is given we take $n \geq n_0$ with $\pi_n(z) = 0 \Rightarrow \pi_{n_0}(T_t(z)) = 0$ and then $m \geq n$ again with $(*)$ for $t_0 = 1$, say. Given $x \in X$ we choose $y \in X$ with $\pi_n(x) = \tilde{C}_t(\pi_m(y)) = \pi_n(C_t(y))$ so that $\pi_{n_0}(T_h(x)) = \pi_{n_0}(T_h(C_t(y)))$ for small h. Therefore,

$$\pi_{n_0}\left(\frac{1}{h}(T_h(x) - x)\right) = \pi_{n_0}\left(\frac{1}{h}(T_h(C_t(y)) - C_t(y))\right)$$

converges to $\pi_{n_0}(\frac{1}{t}(T_t(y) - y))$.

This shows that the difference quotients satisfy the Cauchy condition so that the completeness of X implies that $A(x)$ is defined for every $x \in X$. Moreover, A is continuous either because of the closed graph theorem or because of $\pi_{n_0}(A(x)) = \pi_{n_0}(\frac{1}{t}(T_t(y) - y))$ together with the fact that y can be chosen with $\|\pi_m(y)\| \leq \tilde{c}\|\pi_n(x)\|_n$ using that $\tilde{C}_t$ is open.

Since $A^k(x)$ is the kth derivative of the orbit $t \mapsto T_t(x)$ at 0 we obtain from $(*)$ that A satisfies the condition in the second part of the theorem (this argument uses only local equicontinuity and is thus true for strongly continuous semigroups).

We will now show that under the condition of (2) the exponential series

$$\exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)$$

converges absolutely and uniformly on bounded sets. Since then $\exp(tA)$ is a uniformly continuous semigroup with generator A we have proved both parts of the theorem because a locally equicontinuous semigroup is uniquely determined by its generator.

We define linear maps $\tilde{A}_0 = \pi_m^n$ and

$$\tilde{A}_k : X_m \to X_n, \ \pi_m(x) \mapsto \pi_n(A^k(x))$$

so that $\tilde{A}_k \circ \pi_m = \pi_n \circ A^k$ and $\tilde{A}_{k+1} \circ \pi_m = \tilde{A}_k \circ \pi_m \circ A$.

As above we take a bounded set $D_m \subseteq X$ with $B_m \subseteq \pi_m(D_m)$. The continuity of A then implies $\pi_m \circ A(D_m) \subseteq \lambda B_m$ for some $\lambda > 0$. We now claim that

$$\|\tilde{A}_k\|_{L(X_m,X_n)} \leq \lambda^k$$

for all $k \in \mathbb{N}_0$. For $k = 0$ this holds because $\tilde{A}_0 = \pi_m^n$ is norm decreasing. If the claim is true for some $k \in \mathbb{N}_0$ and $y \in B_m$ is given we choose $x \in D_m$ with $\pi_m(x) = y$ and obtain

$$\|\tilde{A}_{k+1}(y)\|_n = \|\tilde{A}_{k+1} \circ \pi_m(x)\|_n = \|\tilde{A}_k(\pi_m(A(x)))\|_n \leq \lambda^k \|\pi_m(A(x))\|_m \leq \lambda^{k+1}.$$

Finally, we obtain $\|\pi_n(A^k(x))\|_n = \|\tilde{A}_k(\pi_m(x))\|_n \leq \lambda^k \|\pi_m(x)\|_m$ which clearly implies the absolute convergence of the exponential series. □

The very strong form of the convergence of the exponential series shown at the end of the proof need not hold in arbitrary Fréchet spaces. The shift $T_t(f)(z) = f(z + t)$ on the space of entire functions is generated by $A(f) = f'$ and is of the form $\exp(tA)$ because of the Taylor representation of entire functions. However, one cannot estimate $\|A^k(f)\|_n \leq \lambda^k \|f\|_m$.

Let us remark that part of the implication in (2), namely that a strongly contiuous semigroup whose generator is everywhere defined is uniformly continuous, holds for all barrelled spaces. This follows easily from the Banach-Steinhaus theorem for $\{(T_h - id)/h : h \in [0, 1]\}$. Moreover, the equicontinuity of this family characterizes semigroups with a continuous generator.

The countability of the projective spectrum was only used to obtain barrelledness and that $\pi_m : X \to X_m$ lifts bounded sets. The theorem is thus true for non-countable strict projective limits of Banach spaces which are barrelled and satisfy this lifting property. In particular, it holds for arbitrary products of Banach spaces. Moreover, as in [ABR10, proposition 3.4] it is easy to extend the results to so-called prequojections X where only the bidual X'' has a strict representation.

Since strongly continuous, locally equicontinuous semigroups on Montel spaces are easily seen to be uniformly continuous we immediately obtain the answer to Conejero's question:

Corollary 3. *Every strongly continuous semigroup on $\mathbb{K}^I$ has a continuous generator A and is of the form* $\exp(tA)$.

Example 4. Consider the backward shift $A : \omega \to \omega$, $x \mapsto (x_2, x_3, \ldots)$. Theorem 2 implies that A does not generate a strongly continuous semigroup on ω. On the other hand, the operator $B(x) = (x_2, 0, x_4, 0, x_6, 0, \ldots)$ satisfies $B^2 = 0$ as well as $(A - B)^2 = 0$ so that both operators do generate strongly continuous semigroups on ω, given by $T(t) = id + tB$ and $S(t) = id + t(A - B)$, respectively. It is thus not only the "shape" of the matrix determined by the operator which decides whether it generates a semigroup.

Let us finally remark that the second part of theorem 2 implies that, if a continuous operator A on a quojection generates a uniformly continuous semigroup, then the same is true for A^2. This does not hold for the Fréchet space $\mathscr{C}^\infty(\mathbb{R})$: $A(f) = f'$ generates the shift semigroup but $B(f) = A^2(f) = f''$ does not generate a strongly continuous semigroup on $\mathscr{C}^\infty(\mathbb{R})$:

The Gauß-Weierstraß semigroup $\tilde{T}$ on $\mathscr{S}(\mathbb{R})$ given by the convolution with the Gauß kernel $k(t, x) = (\pi t)^{-1/2} \exp(-x^2/t)$ is uniformly continuous (which one can

check by Fourier transformation) and has the generator $\tilde{B} = B|_{\mathscr{S}(\mathbb{R})}$. If B would generate a semigroup T on $\mathscr{C}^\infty(\mathbb{R})$ we obtain for fixed $f \in \mathscr{S}(\mathbb{R})$ and $t > 0$ that $\varphi(s) = T_s \circ \tilde{T}_{t-s}(f)$ has vanishing derivative so that $T_t(f) = \varphi(t) = \varphi(0) = \tilde{T}_t(f)$. But this means that the convolution with the Gauß kernel can be continuously extended from $\mathscr{S}(\mathbb{R})$ to $\mathscr{C}^\infty(\mathbb{R})$ which is not true.

References

[ABR10] Angela A. Albanese, José Bonet, and Werner J. Ricker, *C_0-semigroups and mean ergodic operators in a class of Fréchet spaces*, J. Math. Anal. Appl. **365** (2010), no. 1, 142–157. MR 2585085 (2011g:47089)

[Con07] José A. Conejero, *On the existence of transitive and topologically mixing semigroups*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 3, 463–471. MR 2387043 (2009k:47027)

[DZ84] Susanne Dierolf and David N. Zarnadze, *A note on strictly regular Fréchet spaces*, Arch. Math. (Basel) **42** (1984), no. 6, 549–556. MR 756897 (86c:46002)

[Kom64] Hikosaburo Komatsu, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan **16** (1964), 230–262. MR 0182886 (32 #368)

[Kōm68] Takako Kōmura, *Semigroups of operators in locally convex spaces*, J. Functional Analysis **2** (1968), 258–296. MR 0234317 (38 #2634)

[Ōuc73] Sunao Ōuchi, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan **25** (1973), 265–276. MR 0365227 (51 #1480)

[Shk11] Stanislav Shkarin, *Hypercyclic and mixing operator semigroups*, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 3, 761–782. MR 2837479 (2012h:47021)

[Yos65] Kôsaku Yosida, *Functional analysis*, Die Grundlehren der Mathematischen Wissenschaften, Band 123, Academic Press Inc., New York, 1965. MR 0180824 (31 #5054)

Fachbereich IV – Mathematik, Universität Trier, D-54286 Trier, Germany
E-mail address: frerick@uni-trier.de

Departamento de Matemática Aplicada, E. Politécnica Superior de Alcoy, Universidad Politécnica de Valencia, Plaza Ferrándiz y Carbonell 2, E-03801 Alcoy (Alicante), Spain
E-mail address: ejorda@mat.upv.es

Fachbereich IV – Mathematik, Universität Trier, D-54286 Trier, Germany
E-mail address: kalmesth@uni-trier.de

Fachbereich IV – Mathematik, Universität Trier, D-54286 Trier, Germany
E-mail address: wengenroth@uni-trier.de