# ABSOLUTELY CONTINUOUS MULTILINEAR OPERATORS

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Abstract. We introduce the new class of the absolutely $(p; p_1, ..., p_m; \sigma)$-continuous multilinear operators, that is defined using a summability property that provides the multilinear version of the absolutely $(p, \sigma)$-continuous operators. We give an analogue of Pietsch's Domination Theorem and a multilinear version of the associated Factorization Theorem that holds for absolutely $(p, \sigma)$-continuous operators, obtaining in this way a rich factorization theory. We present also a tensor norm which represents this multi-ideal by trace duality. As an application, we show that absolutely $(p; p_1, ..., p_m; \sigma)$-continuous multilinear operators are compact under some requirements. Applications to factorization of linear maps on Banach function spaces through interpolation spaces are also given.

## 1. Introduction and Notation

In 1987 Matter defined the ideal of absolutely $(p, \sigma)$-continuous linear operators in order to analyze super-reflexive Banach spaces, establishing many of his fundamental properties in [16]. In the nineties, López Molina and Sánchez Pérez studied the factorization properties and the trace duality for these operators in a series of papers, introducing the class of tensor norms that represent these operator ideals (see [13, 14, 25]). Roughly speaking, the class of absolutely $(p, \sigma)$-continuous operators can be considered as an "interpolated" ideal between the $p$-summing operators and the continuous operators, preserving some of the characteristic properties of the first class. Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. A linear operator $T$ between Banach spaces $X$ and $Y$ is absolutely $(p, \sigma)$-continuous if there is a positive constant $C$ such that for all $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset X$, we have

$$\left( \sum_{i=1}^{n} \|T(x_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq C \sup_{\xi \in B_{X^*}} \left( \sum_{i=1}^{n} \left( |\langle x_i, \xi \rangle|^{1-\sigma} \|x_i\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}. \qquad (1)$$

The smallest constant $C$ such that the inequality (1) holds is called the absolutely $(p, \sigma)$-continuous norm of $T$, and is denoted by $\pi_{p,\sigma}(T)$. It is in fact a norm on the space $\mathcal{P}_{p,\sigma}$ of all absolutely $(p, \sigma)$-continuous linear operators from $X$ into $Y$, that becomes a Banach space. In particular, we have that $\mathcal{P}_{p,0}(X, Y)$ coincides with $\Pi_p(X, Y)$, the well known operator ideal of absolutely $p$-summing operators introduced by Pietsch in [19] (see also [9, 20]).

The aim of this paper is to study the multilinear version of this class of operators and its tensor product representation, and to provide some applications in the setting of the factorization theory of bilinear maps. Regarding compactness, we show that as in the case of the $p$-summing operators the absolutely $(p, \sigma)$-continuous operators are always completely continuous, allowing some direct applications for giving sufficient conditions for compactness of multilinear maps on reflexive Banach spaces under weaker summability requirements. Other application is given by proving that absolutely $(p; p_1, p_2; \sigma)$-continuous

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bilinear maps defined on products of Banach function spaces satisfy also a concavity type property. This allows to prove a factorization theorem for operators between Banach function spaces through interpolation spaces.

The paper is divided in five sections. After the introductory one, in Section 2 we extend to multilinear mappings the concept of absolutely $(p,\sigma)$-continuous linear operators, for which the resulting vector space $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}$ of the absolutely $(p;p_1,...,p_m;\sigma)$-continuous multilinear operators is a normed (Banach) multi-ideal. In the third section we establish a domination theorem for such operators and we give the Factorization Theorem for the absolutely $(p,\sigma)$-continuous linear operators and its multilinear version.

In Section 4, we present a reasonable crossnorm $\beta_{p,\sigma}$ on $X_1\otimes ...\otimes X_m\otimes Y$ that satisfies that the topological dual of the corresponding normed tensor product is isometric to the space of $Y^*$-valued absolutely $(p;p_1,...,p_m;\sigma)$-continuous multilinear operators on $X_1\times ...\times X_m$ . We generalize in this way the result for the linear case that can be found in [13].

Finally, Section 5 is devoted to show some applications. Under adequate requirements we show that the summability property for multilinear operators that is considered in the definition of $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}$ implies compactness, providing in this way sufficient conditions for assuring such property for multilinear maps. We finish the paper by showing the factorization theorem for linear operators between Banach function spaces mentioned above.

The notation used in the paper is in general standard. Let $m\in\mathbb{N}$ and $X_j,(j=1,...,m),Y$ be Banach spaces over $\mathbb{K}$, ( either $\mathbb{R}$ or $\mathbb{C}$) we will denote by $\mathcal{L}\left(X_1,...,X_m;Y\right)$ the Banach space of all continuous $m$-linear mappings from $X_1\times ...\times X_m$ into $Y$, under the norm $\|T\|=\sup_{x_m\in B_{X_j},1\leq j\leq m}\left\|T(x^1,...,x^m)\right\|$, where $B_{X_j}$ denotes the closed unit ball of $X_j(1\leq j\leq m)$. If $Y=\mathbb{K}$, we write $\mathcal{L}\left(X_1,...,X_m\right)$. In the case $X_1=...=X_m=X$, we will simply write $\mathcal{L}\left({}^mX;Y\right)$.

Let now $X$ be a Banach space and $1\leq p\leq\infty$. We write $p^*$ for the real number satisfying $1/p+1/p^*=1$. We denote by $\ell_p^n\left(X\right)$ the space of all sequences $\left(x_i\right)_{i=1}^n$ in $X$ with the norm $\|(x_i)_{i=1}^n\|_p=(\sum\limits_{i=1}^n\|x_i\|^p)^{\frac{1}{p}}$, and by $\ell_{p,\omega}^n\left(X\right)$ the space of all sequences $(x_i)_{i=1}^n$ in $X$ with the norm $\|(x_i)_{i=1}^n\|_{p,\omega}=\sup\limits_{\|\xi\|_{X^*}\leq 1}(\sum\limits_{i=1}^n|\langle x_i,\xi\rangle|^p)^{\frac{1}{p}}$, where $X^*$ denotes the topological dual of $X$.

Let $\ell_p\left(X\right)$ be the Banach space of all absolutely $p$-summable sequences $(x_i)_{i=1}^\infty$ in $X$ with the norm $\|(x_i)_{i=1}^\infty\|_p=(\sum\limits_{i=1}^\infty\|x_i\|^p)^{\frac{1}{p}}$. We denote by $\ell_p^\omega\left(X\right)$ the Banach space of all weakly $p$-summable sequences $(x_i)_{i=1}^\infty$ in $X$ with the norm $\|(x_i)_{i=1}^\infty\|_{p,\omega}=\sup\limits_{\|\xi\|_{X^*}\leq 1}\|(\xi(x_i))_{i=1}^\infty\|_p$. If $p=\infty$ we are restricted to the case of bounded sequences and in $\ell_\infty\left(X\right)$ we use the sup norm.

We denote by $\mathcal{L}_f(X_1,...,X_m;Y)$, the space of all $m$-linear mappings of finite type, which is generated by the mappings of the special form

$$x_1^*\otimes ...\otimes x_m^*y:\left(x^1,...,x^m\right)\to x_1^*\left(x^1\right)...x_m^*\left(x^m\right)y$$

for some non-zero $x_j^*\in X_j^*\left(1\leq j\leq m\right)$ and $y\in Y$.

**Definition 1.1.** *An ideal of multilinear mappings (or multi-ideal) is a subclass $\mathcal{M}$ of all continuous multilinear mappings between Banach spaces such that for all $m\in\mathbb{N}$ and Banach spaces $X_1,...,X_m$ and $Y$, the components $\mathcal{M}(X_1,...,X_m;Y):=\mathcal{L}(X_1,...,X_m;Y)\cap\mathcal{M}$ satisfy:*

(i) *$\mathcal{M}(X_1,...,X_m;Y)$ is a linear subspace of $\mathcal{L}(X_1,...,X_m;Y)$ which contains the $m$-linear mappings of finite type.*

(ii) *The ideal property: If $T\in\mathcal{M}(G_1,...,G_m;F)$, $u_j\in\mathcal{L}(X_j;G_j)$ for $j=1,...,m$ and $v\in\mathcal{L}(F;Y)$, then $v\circ T\circ(u_1,...,u_m)$ is in $\mathcal{M}(X_1,...,X_m;Y)$.*

*If $\|.\|_{\mathcal{M}}:\mathcal{M}\to\mathbb{R}^+$ satisfies*

(i') *$(\mathcal{M}(X_1,...,X_m;Y),\|.\|_{\mathcal{M}})$ is a normed ( Banach) space for all Banach spaces $X_1,...,X_m$ and $Y$ and all $m$,*

(i") *$\left\|T^m:\mathbb{K}^m\to\mathbb{K}:T^m\left(x^1,...,x^m\right)=x^1...x^m\right\|_{\mathcal{M}}=1$ for all $m$,*

(iii') *If $T\in\mathcal{M}(G_1,...,G_m;F)$, $u_j\in\mathcal{L}(X_j,G_j)$ for $j=1,...,m$ and $v\in\mathcal{L}(F,Y)$, then*

$$\|v\circ T\circ(u_1,...,u_m)\|_{\mathcal{M}}\leq\|v\|\,\|T\|_{\mathcal{M}}\,\|u_1\|\ldots\|u_m\|,$$

*then $(\mathcal{M},\|.\|_{\mathcal{M}})$ is called a normed (Banach) multi-ideal.*

**Definition 1.2.** *Let $\mathcal{M}$ be a multi-ideal and operator ideals $\mathcal{A}_1,...,\mathcal{A}_m$, an $m$-linear mapping $A\in\mathcal{L}(X_1,...,X_m;Y)$ is said to be of type $\mathcal{M}\circ(\mathcal{A}_1,...,\mathcal{A}_m)$, in symbols $A\in\mathcal{M}\circ(\mathcal{A}_1,...,\mathcal{A}_m)(X_1,...,X_m;Y)$, if there are Banach spaces $G_1,...,G_m$, linear operators $u_j\in\mathcal{A}_j(X_j;G_j), 1\leq j\leq m$, and a continuous $m$-linear mapping $M\in\mathcal{M}(G_1,...,G_m;Y)$ such that $A=M\circ(u_1,,...,u_m)$.* The proof that $\mathcal{M}\circ(\mathcal{A}_1,...,\mathcal{A}_m)$ is an ideal of $m$-linear mappings can be found in [10, Th. 2.2.2].

The definition of absolutely summing $m$-linear functional is due to Pietsch [21]. In [15], Matos presented a definition for vector-valued mappings.

**Definition 1.3.** *Let $1\leq p,p_1,...,p_m<\infty$, with $\frac{1}{p}\leq\frac{1}{p_1}+...+\frac{1}{p_m}$. An $m$-linear operator $T\in\mathcal{L}(X_1,...,X_m;Y)$ is said to be absolutely $(p;p_1,...,p_m)$-summing if there is a constant $C>0$ such that for any $x_1^j,...,x_n^j\in X_j$ we have*

$$\left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^n\right\|_p\leq C\prod_{j=1}^m\|(x_i)_{i=1}^n\|_{p_j,\omega},$$

*for every $n,m\in\mathbb{N}, j=1,...,m$ and $i=1,...,n$.* The *vector space of these mappings is indicated by $\mathcal{L}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$ and the smallest $C$ satisfying the inequality above, by $\|T\|_{\mathcal{L}_{as,(p;p_1,...,p_m)}}$. This defines a norm on $\mathcal{L}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$.*

This definition is equivalent to say that $\left(T(x_i^1,...,x_i^m)\right)_{i=1}^\infty$ belongs to $\ell_p(Y)$ for every $(x_i^j)_{i=1}^\infty\in\ell_{p_j}^\omega(X_j)$.

The next results can be found in [10] and [15], and will be used in the sequel.

**Proposition 1.4.** *Let $1\leq p,p_1,...,p_m<\infty$, with $\frac{1}{p}=\frac{1}{p_1}+...+\frac{1}{p_m}$ and $T\in\mathcal{L}(X_1,...,X_m;Y)$. The following statements are equivalent:*

(a) *$T$ is absolutely $(p;p_1,...,p_m)$-summing.*

(b) *There is a constant $C>0$ and regular Borel probability measures $\mu_j$ on $B_{X_j^*}$ (with the weak star topology) so that for all $\left(x^1,...,x^m\right)\in X_1\times....\times X_m$ the inequality*

$$\left\|T\left(x^1,...,x^m\right)\right\|\leq C\prod_{j=1}^m\left(\int_{B_{X_j^*}}\left|\phi_j(x^j)\right|^{p_j}d\mu(\phi_j)\right)^{\frac{1}{p_j}},\tag{1.2}$$

*is valid.*

(c) *(Factorization Theorem) There exist Banach spaces $Z_1,...,Z_m$, a map $S\in\mathcal{L}(Z_1,...,Z_m;Y)$ and for each $j=1,...,m$ and operators $u_j\in\Pi_{p_j}(X_j,Z_j)$ such that $T=S\circ(u_1,...,u_m)$.*

*Moreover, we have*

$$\begin{aligned}\|T\|_{\mathcal{L}_{as,(p;p_1,...,p_m)}} &= \inf\{C>0 : \textit{for all } C \textit{ verifying the inequality } (1.2)\} \\ &= \inf\left\{\|S\|\prod_{j=1}^{m}\pi_{p_j}(u_j) : T = So(u_1,...,u_m)\right\}.\end{aligned}$$

**Proposition 1.5.** *Let $1 \leq p \leq q < \infty$ and $1 \leq p_j \leq q_j < \infty, j = 1,...,m$ be such that $\sum_{j=1}^{m}\frac{1}{p_j}-\frac{1}{p} \leq \sum_{j=1}^{m}\frac{1}{q_j}-\frac{1}{q}$, then $\mathcal{L}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y) \subset \mathcal{L}_{as,(q;q_1,...,q_m)}(X_1,...,X_m;Y)$.*

## 2. Absolutely $(p;p_1,...,p_m;\sigma)$-Continuous Multilinear Operators

In this section we extend the definition of class of $(p,\sigma)$-absolutely continuous linear operators to the case of multilinear operators and we show that the inclusion between a couple of multi-ideals of the class with different parameters works as one would expect.

Let $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} \leq \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. For all $(x_i^j)_{i=1}^n \subset X_j, (1 \leq j \leq m)$ we put

$$\delta_{p_j\sigma}((x_i^j)_{i=1}^n) = \sup_{\phi_j \in B_{X_j^*}}\left(\sum_{i=1}^{n}\left(\left|\phi_j(x_i^j)\right|^{1-\sigma}\left\|x_i^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}}$$

It is clear that

$$\left\|(x_i^j)_{i=1}^n\right\|_{\frac{p_j}{1-\sigma},\omega} \leq \delta_{p_j\sigma}((x_i^j)_{i=1}^n), \text{ for all } (x_i^j)_{i=1}^n \subset X_j, (1 \leq j \leq m) \tag{2.1}$$

**Definition 2.1.** *A mapping $T \in \mathcal{L}(X_1,...,X_m;Y)$ is absolutely $(p;p_1,...,p_m;\sigma)$-continuous if there is a constant $C > 0$ such that for any $x_1^j,...,x_n^j \in X_j$, $(1 \leq j \leq m)$ we have*

$$\left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \leq C\prod_{j=1}^{m}\delta_{p_j\sigma}((x_i^j)_{i=1}^n) \tag{2.2}$$

*We denote this class of mappings by $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$ which is a Banach space with the norm*

$$\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} = \inf\{C>0 : \textit{for all } C \textit{ verifying the inequality } (2.2)\}$$

It is obvious that $\|T\| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$ for all $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$.

For $\sigma = 0$, we have $\mathcal{L}^{0}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y) = \mathcal{L}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$.

**Proposition 2.2.** *(Inclusion Theorem).*
*Let $p \leq q, p_j \leq q_j (1 \leq j \leq m)$. If $\frac{1}{p_1}+...+\frac{1}{p_m}-\frac{1}{p} \leq \frac{1}{q_1}+...+\frac{1}{q_m}-\frac{1}{q}$, then*

$$\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y) \subset \mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)}(X_1,...,X_m;Y).$$

*Proof.* By [5, Prop. 3.2] we may assume $\frac{1}{p_1}+...+\frac{1}{p_m}-\frac{1}{p} = \frac{1}{q_1}+...+\frac{1}{q_m}-\frac{1}{q}$.
Considering $1 \leq r, r_j < \infty$ with $\frac{1}{r}+\frac{1}{q} = \frac{1}{p}$, $\frac{1}{r_j}+\frac{1}{q_j} = \frac{1}{p_j}$ $(1 \leq j \leq m)$ it follows that $\frac{1}{r_1}+...+\frac{1}{r_m} = \frac{1}{r}$.
Now select a multilinear mapping $T$ in $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y)$ and $x_1^j,...,x_n^j \in X_j$, for $j = 1,..,m$. Then, with $\lambda_i^j = \left\|T\left(x_i^1,...,x_i^m\right)\right\|^{\frac{q}{r_j}}$, we have $\left\|T(\lambda_i^1x_i^1,...,\lambda_i^mx_i^m)\right\|^{\frac{p}{1-\sigma}} = \left\|T\left(x_i^1,...,x_i^m\right)\right\|^{\frac{q}{1-\sigma}}$. An application of Hölder's inequality reveals that

$$\left(\sum_{i=1}^{n}\left\|T\left(x_i^1,...,x_i^m\right)\right\|^{\frac{q}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{n} \left\|T(\lambda_i^1 x_i^1, ..., \lambda_i^m x_i^m)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{p_j\sigma}\left(\left(\lambda_i^j x_i^j\right)_{i=1}^{n}\right) \\
&= \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \sup_{\phi_j \in B_{X_j^*}} \left(\sum_{i=1}^{n} \left(\lambda_i^j \left|\phi_j(x_i^j)\right|^{1-\sigma} \left\|x_i^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}} \\
&\leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \left\|\left(\lambda_i^j\right)_{i=1}^{n}\right\|_{\frac{r_j}{1-\sigma}} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right) \\
&= \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \left(\sum_{i=1}^{n} \left\|T\left(x_i^1, ..., x_i^m\right)\right\|^{\frac{q}{1-\sigma}}\right)^{\frac{1-\sigma}{r}} \prod_{j=1}^{m} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right)
\end{aligned}
$$

Since $\frac{1-\sigma}{p} - \frac{1-\sigma}{r} = \frac{1-\sigma}{q}$, we end up with

$$
\left\|\left(T\left(x_i^1, ..., x_i^m\right)\right)_{i=1}^{n}\right\|_{\frac{q}{1-\sigma}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \delta_{q_j\sigma}\left(\left(x_i^j\right)_{i=1}^{n}\right).
$$

Hence $T \in \mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)}(X_1, ..., X_m; Y)$ and $\|T\|_{\mathcal{L}^{\sigma}_{as,(q;q_1,...,q_m)}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.

□

The proof of the following proposition is straightforward.

**Proposition 2.3.** *Let $T \in \mathcal{L}(X_1, ..., X_m; Y)$, $R \in \mathcal{L}(Y, Z)$ and $u_j \in \mathcal{L}(E_j, X_j)$, $1 \leq j \leq m$.*

(i) *If $T$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous, then $R \circ T$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous and $\|R \circ T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \|R\| \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.*

(ii) *If $T$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous, then $T \circ (u_1, ..., u_m)$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous and*

$$
\|T \circ (u_1, ..., u_m)\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^{m} \|u_j\|.
$$

We can establish the following comparison between the classes of absolutely $(p; p_1, ..., p_m; \sigma)$-continuous and absolutely $(p; p_1, ..., p_m)$-summing $m$-linear operators.

**Proposition 2.4.** *Let $1 \leq p_j, p < \infty, j = 1, ..., m$ such that $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. Then $\mathcal{L}_{as,(\frac{p}{1-\sigma}; \frac{p_1}{1-\sigma}, ..., \frac{p_m}{1-\sigma})}(X_1, ..., X_m; Y) \subset \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$ Consequently, $\mathcal{L}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y) \subset \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$*

*Proof.* It is immediate by the inequality (2.1) and Proposition 1.5.

□

## 3. Domination and Factorization Theorems

3.1. **Pietsch Domination Theorem.** In the case of absolutely $(p, \sigma)$-continuous linear maps it is possible to obtain a Domination Theorem as the one that holds for $p$-summing operators (see [16]). It can be also extended to the multilinear case. For the proof of this Domination Theorem we use the full general Pietsch Domination Theorem recently presented by Pellegrino et al in [18].

Let $X_1, ..., X_m, Y$ and $E_1, ..., E_k$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X_1 \times ... \times X_m$ to $Y$. Let also $K_1, .., K_t$ be compact Hausdorff topological spaces, $G_1, ..., G_t$ be Banach spaces and suppose that the maps

$$\left\{ \begin{array}{l} R_j : K_j \times E_1 \times ... \times E_k \times G_j \rightarrow [0, +\infty) \ , j = 1, ..., t \\ S : \mathcal{H} \times E_1 \times ... \times E_k \times G_1 \times ... \times G_t \rightarrow [0, +\infty) \end{array} \right.$$

satisfy

(1) For each $x^l \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, ..., t\} \times \{1, ..., k\}$ the mapping $(R_j)_{x^1,...,x^k,b} : K_j \rightarrow [0, +\infty)$ defined by $(R_j)_{x^1,...,x^k,b}(\phi) = R_j(\phi, x^1, ..., x^k, b)$ is continuous.

(2) The following inequalities hold:
$$\left\{ \begin{array}{l} R_j(\phi, x^1, ..., x^k, \eta_j b^j) \leq \eta_j R_j(\phi, x^1, ..., x^k, b^j) \\ S(f, x^1, ..., x^k, \alpha_1 b^1, ..., \alpha_t b^t) \geq \alpha_1 ... \alpha_t S(f, x^1, ..., x^k, b^1, ..., b^t), \end{array} \right.$$
for every $\phi_j \in K_j, x^l \in E_l$(with $l \in \{1, ..., k\}), 0 \leq \eta_j, \alpha_j \leq 1, b^j \in G_j$ with $j = 1, ..., t$ and $f \in \mathcal{H}$.

**Definition 3.1.** *If $0 < q_1, ..., q_t, q < \infty$, with $\frac{1}{q} = \frac{1}{q_1} + ... + \frac{1}{q_t}$, a mapping $f : X_1 \times ... \times X_m \longrightarrow Y$ in $\mathcal{H}$ is said to be $R_1, ..., R_t$-$S$-abstract $(q_1, ..., q_t)$-summing if there is a constant $C > 0$ so that*

$$\left( \sum_{i=1}^{n} S(f, x_i^1, ..., x_i^k, b_i^1, ..., b_i^t)^q \right)^{\frac{1}{q}} \leq C \prod_{j=1}^{t} \sup_{\phi \in K_j} \left( \sum_{i=1}^{n} R_j(\phi, x_i^1, ..., x_i^k, b_i^j)^{q_j} \right)^{\frac{1}{q_j}},$$

*for all $x_1^s, ..., x_n^s \in E_s, b_1^j, ..., b_n^j \in G_j, n \in N$ and $(s, j) \in \{1, ..., k\} \times \{1, ..., t\}$.*

**Theorem 3.2.** [18] *A map $f \in \mathcal{H}$ is $R_1, ..., R_t$-$S$-abstract $(q_1, ..., q_t)$-summing if and only if there is a constant $C > 0$ and Borel probability measures $\mu_j$ on $K_j$ such that*

$$S(f, x^1, ..., x^k, b^1, ..., b^t) \leq C \prod_{j=1}^{t} \left( \int_{K_j} R_j(\phi, x^1, ..., x^k, b^j)^{q_j} d\mu_j(\phi) \right)^{\frac{1}{q_j}},$$

*for all $x^l \in E_l$, $l \in \{1, ..., k\}$) and $b^j \in G_j$ with $j = 1, ..., t$.*

**Theorem 3.3.** *Let $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. An $m$-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous if and only if there is a constant $C > 0$ and Borel probability measures $\mu_j$ on $B_{X_j^*}, 1 \leq j \leq m$, (with the weak star topology) so that for all $\left(b^1, ..., b^m\right) \in X_1 \times .... \times X_m$ the inequality*

$$\left\| T(b^1, ..., b^m) \right\| \leq C \prod_{j=1}^{m} \left( \int_{B_{X_j^*}} (\left|\phi(b^j)\right|^{1-\sigma} \left\|b^j\right\|^{\sigma})^{\frac{p_j}{1-\sigma}} d\mu_j(\phi) \right)^{\frac{1-\sigma}{p_j}} \tag{3.1}$$

*is valid.*
*The infimum of all these possible $C$ is equal to $\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}$.*

*Proof.* Note that by choosing the parameters

$$\left\{ \begin{array}{l} t = m \\ E_j = \mathbb{K}, \ j = 1, ..., k \\ G_j = X_j, \ j = 1, ..., m \\ K_j = B_{X_j^*}, \ j = 1, ..., m \\ \mathcal{H} = \mathcal{L}(X_1, ..., X_m; Y) \\ q = \frac{p}{1-\sigma}, q_j = \frac{p_j}{1-\sigma}, \ j = 1, ..., m \\ S(T, x^1, ..., x^k, b^1, ..., b^m) = \left\|T\left(b^1, ..., b^m\right)\right\| \\ R_j(\phi, x^1, ..., x^k, b^j) = \left|\phi(b^j)\right|^{1-\sigma} \left\|b^j\right\|^{\sigma}, \ j = 1, ..., m \end{array} \right.$$

we can easily conclude that $T : X_1 \times ... \times X_m \longrightarrow Y$ is absolutely $(p; p_1, ..., p_m; \sigma)$-continuous if and only if $T$ is $R_1, ..., R_m$-$S$ abstract $(\frac{p_1}{1-\sigma}, ..., \frac{p_m}{1-\sigma})$-summing. Theorem 3.2 tells us that $T$ is $R_1, ..., R_m$-$S$ abstract $(\frac{p_1}{1-\sigma}, ..., \frac{p_m}{1-\sigma})$-summing if and only if there is a $C > 0$ and there are probability measures $\mu_j$ on $K_j$, $j = 1, ..., m$, such that

$$S(T, x^1, ..., x^k, b^1, ..., b^t) \leq C \prod_{j=1}^{t} \left( \int_{B_{X_j^*}} R_j(\phi, x^1, ..., x^k, b^j)^{\frac{p_j}{1-\sigma}} d\mu_j(\phi) \right)^{\frac{1-\sigma}{p_j}}$$

i.e; $\left\| T(b^1, ..., b^m) \right\| \leq C \prod_{j=1}^m \left( \int_{B_{X_j^*}} (\left| \phi(b^j) \right|^{1-\sigma} \left\| b^j \right\|^{\sigma})^{\frac{p_j}{1-\sigma}} d\mu_j(\phi) \right)^{\frac{1-\sigma}{p_j}}$
and we obtain the inequality in the statement of the theorem.

□

3.2. **Pietsch Factorization Theorem.** Now we give the Pietsch Factorization Theorem for the absolutely $(p, \sigma)$-continuous linear operators and his multilinear version for absolutely $(p; p_1, ..., p_m; \sigma)$-continuous multilinear operators.

Let $X$, $Y$ be Banach spaces, $p \geq 1$, $0 \leq \sigma < 1$ and a regular Borel probability measure $\eta$ on $B_{X^*}$, (with the weak star topology). We denote by $i_X$ the isometric embedding $X \to C(B_{X^*})$ given by $i_X(x) = \langle x, . \rangle$.

For $f \in i_X(X) \subset C(B_{X^*})$, we define the semi norm,

$$\|f\|_{p,\sigma} = \inf \left\{ \sum_{k=1}^{n} \|f_k\|_{i_X(X)}^{\sigma} \cdot \left( \int_{B_{X^*}} |f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}}, \; f = \sum_{k=1}^{n} f_k, \; f_k \in i_X(X), \forall k, \; 1 \leq k \leq n \right\}$$

Let $S$ the closed subspace of $i_X(X)$ given by $S = \left\{ f \in i_X(X), \|f\|_{p,\sigma} = 0 \right\}$, we write $L_{p,\sigma}(\eta)$ the quotient space $i_X(X)/S$ with the norm

$$\|[f]\|_{p,\sigma} = \inf \left\{ \|g\|_{p,\sigma}, g \in i_X(X), g \in [f] \right\}$$

where $[f]$ is the equivalence class of $f \in i_X(X)$. We need the following lemma.

**Lemma 3.4.** *The canonical map $J_{p,\sigma} : C(B_{X^*}) \to L_{p,\sigma}(\eta)$ is absolutely $(p, \sigma)$-continuous, and $\pi_{p,\sigma}(J_{p,\sigma}) = \|J_{p,\sigma}\| = 1$*

*Proof.* Let $\delta_\omega : C(B_{X^*}) \to \mathbb{K} : f \mapsto f(\omega)$ be the Dirac's delta associated with $\omega \in B_{X^*}$. As $\|\delta_\omega\| = 1$, we may write, for every $(f_k)_{k=1}^n \subset C(B_{X^*})$

$$\begin{aligned}
&\left( \sum_{k=1}^n \|J_{p,\sigma}(f_k)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq \left( \int_{B_{X^*}} \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} |f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}} \\
&\leq \sup_{\lambda \in C(B_{X^*})^*, \|\lambda\| \leq 1} \left| \left\langle \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} |f_k|^p, \lambda \right\rangle \right|^{\frac{1-\sigma}{p}} \\
&\leq \left\| \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} |f_k|^p \right\|_{C(B_{X^*})}^{\frac{1-\sigma}{p}} \\
&= \sup_{\omega \in B_{X^*}} \left| \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} . |f_k(\omega)|^p \right|^{\frac{1-\sigma}{p}} \\
&= \sup_{\delta_\omega} \left| \int_K \sum_{k=1}^n \|f_k\|^{\frac{\sigma p}{1-\sigma}} |f_k|^p \, d\delta_\omega \right|^{\frac{1-\sigma}{p}} \\
&= \sup_{\lambda \in C(B_{X^*})^*, \|\lambda\| \leq 1} \left| \sum_{k=1}^n \left( \|f_k\|^{\sigma} |\langle f_k, \lambda \rangle|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \right|^{\frac{1-\sigma}{p}}
\end{aligned}$$

Then $J_{p,\sigma} \in \mathcal{P}_{p,\sigma}(C(B_{X^*}), L_{p,\sigma}(\eta))$ and $\pi_{p,\sigma}(J_{p,\sigma}) \leq 1$. Together with

$$\pi_{p,\sigma}(J_{p,\sigma}) \geq \|J_{p,\sigma}\| = \sup_{f\in C(B_{X^*}),\|f\|\leq 1} \|J_{p,\sigma}(f)\| \geq 1,$$

we get $\pi_{p,\sigma}(J_{p,\sigma}) = \|J_{p,\sigma}\| = 1$.

□

**Theorem 3.5.** *For every operator $T : X \to Y$, the following are equivalent*

(i) *$T$ is absolutely $(p,\sigma)$-continuous*

(ii) *There exist a regular Borel probability measure $\mu$ on $B_{X^*}$, a (closed) subspace $X_{p,\sigma}$ of $L_{p,\sigma}(\mu)$ and an operator $\hat{T} : X_p \to Y$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i_X \downarrow & & \uparrow \widetilde{T} \\ i_X(X) & \xrightarrow{\overline{J}_{p,\sigma}} & X_{p,\sigma} \\ \cap & & \cap \\ C(B_{X^*}) & \xrightarrow{J_{p,\sigma}} & L_{p,\sigma}(\mu), \end{array}$$

*where $\overline{J}_{p,\sigma}$ is the map $i_X(X) \to X_{p,\sigma}$ induced by $J_{p,\sigma}$.*

*Proof.* (i)$\Longrightarrow$(ii). If $T$ absolutely $(p,\sigma)$-continuous, the Pietsch Domination (see [16, Th. 4.1]) provides a regular Borel probability measure $\mu$ on $B_{X^*}$ for which

$$\|Tx\| \leq \pi_{p,\sigma}(T). \|x\|^{\sigma} . \left(\int_{B_{X_j^*}} |\langle x, x^*\rangle|^p \, d\mu\right)^{\frac{1-\sigma}{p_j}} \text{ for all } x \in X.$$

This informs us that if we denote the range of $J_{p,\sigma} \circ i_X$ by $S$ and consider it to be a normed subspace of $L_{p,\sigma}(\mu)$, the map $S \to Y : J_{p,\sigma} \circ i_X(x) \mapsto Tx$ is a well-defined operator. It is continuous for the $L_{p,\sigma}(\mu)$-topology with norm $\leq \pi_{p,\sigma}(T)$, since $\|Tx\| \leq \pi_{p,\sigma}(T). \|\langle x, .\rangle\|_{p,\sigma}, \forall x \in X$. Let $X_{p,\sigma}$ be the closure of $S$ in $L_{p,\sigma}(\mu)$. Then the natural extension of our map to $X_{p,\sigma}$ is the operator $\widetilde{T}$ we are looking for.

(ii)$\Longrightarrow$(i) From $\widetilde{T} \circ \overline{J}_{p,\sigma} \circ i_X = T$ and the previous lemma we get $T$ is $(p,\sigma)$-absolutely continuous and $\pi_{p,\sigma}(T) \leq \left\|\widetilde{T}\right\| .\pi_{p,\sigma}(\overline{J}_{p,\sigma}). \|i_X\| = \left\|\widetilde{T}\right\|$, so that even $\left\|\widetilde{T}\right\| = \pi_{p,\sigma}(T)$ is true.

□

**Theorem 3.6.** *(Multilinear Version)*
*Let $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $0 \leq \sigma < 1$. Then*

$$T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$$

*if and only if there exist Banach spaces $G_1, ..., G_m$, absolutely $(p_j,\sigma)$-continuous linear operators $u_j \in \mathcal{L}(X_j, G_j)$ and $m$-linear mapping $S \in \mathcal{L}(G_1, ..., G_m; Y)$ so that $T = S \circ (u_1, ..., u_m)$. Moreover,*

$$\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} = \inf\left\{\|S\| \prod_{j=1}^m \pi_{p_j,\sigma}(u_j) : T = So(u_1, ..., u_m)\right\}$$

*i.e., $\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)} = \mathcal{L} \circ (\mathcal{P}_{p_1,\sigma}, ..., \mathcal{P}_{p_m,\sigma})$ holds isometrically.*

*Proof.* First we prove the converse. Let $(x^1, ..., x^m) \in X_1 \times .... \times X_m$. If $T$, has such a factorization, we have

$$\left\|T(x^1, ..., x^m)\right\| = \left\|S\left(u_1\left(x^1\right), ..., u_m\left(x^m\right)\right)\right\| \leq \|S\| \prod_{j=1}^{m} \|u_j\left(x^j\right)\| .$$

We know that (see [16, Th. 4.1]), for each $j = 1, ..., m$, there is $\mu_j \in C(B_{X_j^*})^*$, such that

$$\left\|u_j\left(x^j\right)\right\| \leq \pi_{p_j,\sigma}\left(u_j\right)\left(\int_{B_{X_j^*}}\left(\left|\left\langle x^j,\phi\right\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\right)^{\frac{1-\sigma}{p_j}}.$$

Now we have

$$\left\|T(x^1,...,x^m)\right\| \leq \|S\| \prod_{j=1}^m \pi_{p_j,\sigma}\left(u_j\right) \prod_{j=1}^m \left(\int_{B_{X_j^*}}\left(\left|\left\langle x^j,\phi\right\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\right)^{\frac{1-\sigma}{p_j}}.$$

Therefore $T$ is absolutely $(p;p_1,...,p_m;\sigma)$-continuous and

$$\|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \leq \|S\| \prod_{j=1}^m \pi_{p_j,\sigma}\left(u_j\right).$$

To prove the first implication, take $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}\left(X_1,....,X_m;Y\right)$. Then, there exist probability measures $\mu_j \in C(B_{X_j^*})^*, (1 \leq j \leq m)$ such that for all $\left(x^1,...,x^m\right) \in X_1 \times .... \times X_m$,

$$\left\|T(x^1,...,x^m)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^m \left(\int_{B_{X_j^*}}\left(\left|\left\langle x^j,\phi\right\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j(\phi)\right)^{\frac{1-\sigma}{p_j}}.$$

We now consider the operator $u_j^0 : X_j \to L_{p_j,\sigma}(\mu_j)$ which is given by $u_j^0(x^j) = \left[\left\langle x^j,.\right\rangle\right]$ and notice that we have

$$\left\|u_j^0(x^j)\right\| = \left\|x^j\right\|_{p_j,\sigma} \leq \left\|x^j\right\|_{X_j}^{\sigma} \cdot \left(\int_{B_{X^*}} \left|\left\langle x^j,\phi\right\rangle\right|^{p_j} d\mu_j\right)^{\frac{1-\sigma}{p_j}}, \text{ for all } x^j \in X_j \text{ and } 1 \leq j \leq m$$

with $\left\|x^j\right\|_{p_j,\sigma} = \inf\left\{\sum\limits_{k=1}^n \left\|x_k^j\right\|_{X_j}^{\sigma} \cdot \left(\int_{B_{X_j^*}}\left|\left\langle x_k^j,\phi\right\rangle\right|^{p_j} d\mu_j\right)^{\frac{1-\sigma}{p_j}}, \ x^j = \sum\limits_{k=1}^n x_k^j, \ (x_k^j)_{k=1}^n \subset X_j\right\}$

Let $G_j$ be the closure in $L_{p_j,\sigma}(\mu_j)$ of the range of $u_j^0$ and let $u_j : X_j \to G_j$ be the induced operator. $u_j$ is $(p_j,\sigma)$-absolutely continuous with $\pi_{p_j,\sigma}\left(u_j\right) \leq 1$.

Let $S_0$ be the operator defined on $u_1^0(X_1) \times ... \times u_m^0(X_m)$, by

$$S_0(u_1^0\left(x^1\right),...,u_m^0\left(x^m\right)) := T(x^1,...,x^m).$$

We prove that the mapping $S_0$ is well defined and continuous, so we have

$$\left\|S_0\left(u_1^0\left(x^1\right),...,u_m^0\left(x^m\right)\right)\right\| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}} \prod_{j=1}^m \left(\int_{B_{X_j^*}}\left(\left|\left\langle x^j,\phi\right\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}} d\mu_j\right)^{\frac{1-\sigma}{p_j}}.$$

Fix $j = 1$ and $\varepsilon > 0$. Then there exists $\left(x_k^1\right)_{k=1}^n \subset X_1$ such that $x^1 = \sum\limits_{i=1}^n x_k^1$ and

$$\sum_{k=1}^n \left\|i_1(x_k^1)\right\|_{i_1(X_1)}^{\sigma} \cdot \left(\int_{B_{X_j^*}}\left|\left\langle x_k^j,\phi\right\rangle\right|^{p_1} d\mu_j\right)^{\frac{1-\sigma}{p_1}} \leq \varepsilon + \left\|i_1(x^1)\right\|_{p_1,\sigma},$$

where $i_1$ the isometric embedding $X_1 \to C(B_{X_1^*})$ given by $i_1(x) = \langle x, .\rangle$ . So we have

$$\begin{aligned}
& \left\| S_0 \left( u_1^0 \left( x^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \\
= & \left\| S_0 \left( u_1^0 \left( \sum_{k=1}^n x_k^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \\
\leq & \sum_{i=1}^n \left\| S_0 \left( u_1^0 \left( x_k^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \\
\leq & \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \sum_{k=1}^n \left\| x_k^1 \right\|^\sigma \cdot \left( \int_{B_{X_1^*}} \left| \langle x_k^1, \phi \rangle \right|^{p_1} d\mu_1 \right)^{\frac{1-\sigma}{p_1}} \prod_{j=2}^m \left( \int_{B_{X_j^*}} \left( \left| \langle x^j, \phi \rangle \right|^{1-\sigma} \left\| x^j \right\|^\sigma \right)^{\frac{p_j}{1-\sigma}} d\mu_j \right)^{\frac{1-\sigma}{p_j}} \\
\leq & \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \left( \varepsilon + \left\| x^1 \right\|_{p_1,\sigma} \right) \prod_{j=2}^m \left( \int_{B_{X_j^*}} \left( \left| \langle x^j, \phi \rangle \right|^{1-\sigma} \left\| x^j \right\|^\sigma \right)^{\frac{p_j}{1-\sigma}} d\mu_j \right)^{\frac{1-\sigma}{p_j}}
\end{aligned}$$

We can write the same domination result for $j = 2$, with this new domination, to obtain

$$\begin{aligned}
& \left\| S_0 \left( u_1^0 \left( x^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \leq \\
& \leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \left( \varepsilon + \left\| x^1 \right\|_{p_1,\sigma} \right) \left( \varepsilon + \left\| x^2 \right\|_{p_2,\sigma} \right) \prod_{j=3}^m \left( \int_{B_{X_j^*}} \left( \left| \langle x^j, \phi \rangle \right|^{1-\sigma} \left\| x^j \right\|^\sigma \right)^{\frac{p_j}{1-\sigma}} d\mu_j \right)^{\frac{1-\sigma}{p_j}}.
\end{aligned}$$

By induction, we get

$$\left\| S_0 \left( u_1^0 \left( x^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \prod_{j=1}^m \left( \varepsilon + \left\| x^j \right\|_{p_j,\sigma} \right).$$

Since this is true for all $\varepsilon > 0$, we obtain

$$\left\| S_0 \left( u_1^0 \left( x^1 \right), ..., u_m^0 \left( x^m \right) \right) \right\| \leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \left\| x^1 \right\|_{p_1,\sigma} .... \left\| x^m \right\|_{p_m,\sigma}.$$

It follows that $S_0$ is continuous on $u_1^0(X_1) \times ... \times u_m^0(X_m)$ and has a unique extension $S$ to $\overline{u_1^0(X_1)} \times ... \times \overline{u_m^0(X_m)} = G_1 \times ... \times G_m$ with $\|S\| \leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}}$.
Finally, note that $T = S \circ (u_1, ..., u_m)$ where $u_j \in \mathcal{P}_{p_j,\sigma}(X_j, G_j)$, $(1 \leq j \leq m)$, $S \in \mathcal{L}(G_1, ..., G_m; Y)$ and

$$\|S\| \prod_{j=1}^m \pi_{p_j,\sigma}(u_j) \leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}}.$$

This completes the proof.

□

**Remark 3.7.** Since $\mathcal{P}_{p_j,\sigma}$ is a Banach operator ideal $(1 \leq j \leq m)$, (see [16]), the space $\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)} = \mathcal{L} \circ (\mathcal{P}_{p_1,\sigma}, ..., \mathcal{P}_{p_m,\sigma})$ is a Banach multi-ideal [9, Theorem 2.2.2].

**Example 3.8.** *Let $p > 1, 0 < \sigma < 1$ such that $p^* < \frac{p}{1-\sigma}$.* Let $S \in \mathcal{L}\left( \ell_{\frac{p}{1-\sigma}}, ..., \ell_{\frac{p}{1-\sigma}}; \ell_{\frac{p}{1-\sigma}} \right)$ *and $u \in \mathcal{L}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$ defined by $u(e_i) = (\frac{1}{i})^{\frac{1}{p}} e_i$ where $(e_i)_{i=1}^\infty$ be the vector unit basis of $l_{p^*}$. The $m$-linear operator $T \in \mathcal{L}\left( \ell_{p^*}, ..., \ell_{p^*}; \ell_{\frac{p}{1-\sigma}} \right)$ given by $T = S \circ (u, ..., u)$ is absolutely $(\frac{p}{m}; p, ..., p; \sigma)$-continuous* but it is not *absolutely $(\frac{p}{m}; p, ..., p)$-summing.* In order to see this, note that by [13, Ex. 1.9] we have $u \in \mathcal{P}_{p,\sigma}(\ell_{p^*}, \ell_{\frac{p}{1-\sigma}})$ and $u \notin \Pi_p \left( \ell_{p^*}, \ell_{\frac{p}{1-\sigma}} \right)$ then by the factorization theorems for the classes $\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}$ and $\mathcal{L}_{as,(p;p_1,...,p_m)}$ (Theorem 3.6 and Proposition 1.4) we get the result.

## 4. Connection with tensor products

In this section we introduce a reasonable crossnorm (see [24, p. 127]) on $X_1 \otimes ... \otimes X_m \otimes Y$ in such way that the topological dual of this normed space is isometric to

$\left(\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1,...,X_m;Y^*),\ \|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}}\right)$. Our aim is to show that the representation of our multi-ideal by as a dual of a topological tensor product holds exactly for this tensor norm. Let $u \in X_1 \otimes ... \otimes X_m \otimes Y$. For $1 \leq p, p_1, ..., p_m, r < \infty$, $0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we consider

$$\beta_{p,\sigma}(u) = \inf \textstyle\prod_{j=1}^{m} \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_r$$

where the infimum is taken over all representations of $u$ of the form

$$u = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$$

with $x_i^j \in X_j, y_i \in Y, i = 1, ..., n, j = 1, ..., m$ and $n, m \in \mathbb{N}$.

**Proposition 4.1.** *$\beta_{p,\sigma}$ is a reasonable crossnorm and $\epsilon \leq \beta_{p,\sigma}$, where $\epsilon$, denotes the injective norm on $X_1 \otimes ... \otimes X_m \otimes Y$.*

*Proof.* Let $u', u'' \in X_1 \otimes ... \otimes X_m \otimes Y$, and let $\varepsilon > 0$. Choose representations of $u'$ and $u''$ of the form

$$u' = \sum_{i=1}^{n'} x_i'^1 \otimes ... \otimes x_i'^m \otimes y_i', \quad u'' = \sum_{i=1}^{n''} x_i''^1 \otimes ... \otimes x_i''^m \otimes y_i''$$

such that

$$\beta_{p,\sigma}(u') + \varepsilon \geq \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i'^j)_{i=1}^{n'}) . \left\|(y_i')_{i=1}^{n'}\right\|_r \text{ and } \beta_{p,\sigma}(u'') + \varepsilon \geq \prod_{j=1}^{m} \delta_{p_j\sigma}((x_i''^j)_{i=1}^{n''}) . \left\|(y_i'')_{i=1}^{n''}\right\|_r$$

we can write $u', u''$ in the following way

$$u' = \sum_{i=1}^{n'} z_i'^1 \otimes ... \otimes z_i'^m \otimes t_i', \quad u'' = \sum_{i=1}^{n''} z_i''^1 \otimes ... \otimes z_i''^m \otimes t_i''$$

with

$$z_i'^j = \frac{(\beta_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p_j}}}{\delta_{p_j\sigma}\left((x_i'^j)_{i=1}^{n'}\right)} x_i'^j, j = 1, ..., m,\ i = 1, ..., n',$$

$$t_i' = \frac{\prod_{j=1}^{m} \delta_{p_j\sigma}\left((x_i'^j)_{i=1}^{n'}\right)}{(\beta_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p}}} y_i', i = 1, ..., n',$$

$$z_i''^j = \frac{(\beta_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p_j}}}{\delta_{p_j\sigma}\left((x_i''^j)_{i=1}^{n''}\right)} x_i''^j, j = 1, ..., m,\ i = 1, ..., n'',$$

$$t_i'' = \frac{\prod_{j=1}^{m} \delta_{p_j\sigma}\left((x_i''^j)_{i=1}^{n''}\right)}{(\beta_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p}}} y_i'', i = 1, ..., n''.$$

It follows that

$$\delta_{p_j\sigma}((z_i'^j)_{i=1}^{n'}) = (\beta_{p,\sigma}(u') + \varepsilon)^{\frac{1-\sigma}{p_j}} \text{ and } \left\|(t_i')_{i=1}^{n'}\right\|_r \leq (\beta_{p,\sigma}(u') + \varepsilon)^{\frac{1}{r}}, j = 1, ..., m,$$

$$\delta_{p_j\sigma}((z_i''^j)_{i=1}^{n''}) = (\beta_{p,\sigma}(u'') + \varepsilon)^{\frac{1-\sigma}{p_j}} \text{ and } \left\|(t_i'')_{i=1}^{n''}\right\|_r \leq (\beta_{p,\sigma}(u'') + \varepsilon)^{\frac{1}{r}}, j = 1, ..., m.$$

Thus

$$\textstyle\prod_{j=1}^{m} \delta_{p_j\sigma}((z_i'^j)_{i=1}^{n'}) . \left\|(t_i')_{i=1}^{n'}\right\|_r \leq \beta_{p,\sigma}(u') + \varepsilon \leq \beta_{p,\sigma}(u') + \beta_{p,\sigma}(u'') + \varepsilon,$$

$$\textstyle\prod_{j=1}^{m} \delta_{p_j\sigma}((z_i''^j)_{i=1}^{n''}) . \left\|(t_i'')_{i=1}^{n''}\right\|_r \leq \beta_{p,\sigma}(u'') + \varepsilon \leq \beta_{p,\sigma}(u') + \beta_{p,\sigma}(u'') + \varepsilon.$$

The two last inequalities implies that

$$\beta_{p,\sigma}(u' + u'') \leq \beta_{p,\sigma}(u') + \beta_{p,\sigma}(u'') + \varepsilon, \forall \varepsilon > 0.$$

Hence the triangular inequality is proved for $\beta_{p,\sigma}$. It is easy to see that $\beta_{p,\sigma}(\lambda u) = |\lambda| \, \beta_{p,\sigma}(u)$ for all $u \in X_1 \otimes ... \otimes X_m \otimes Y$ and $\lambda \in \mathbb{K}$.

Let $u=\sum\limits_{i=1}^{n}x_i^1\otimes...\otimes x_i^m\otimes y_i\in X_1\otimes...\otimes X_m\otimes Y$. By Hölder's inequality and (2.1) we get

$$\begin{aligned}\epsilon(u) &= \sup\left\{\left|\sum_{i=1}^{n}\phi_1(x_i^1)...\phi_m(x_i^m)\psi(y_i)\right|;\phi_j\in B_{X_j^*},\psi\in B_{Y^*}\right\}\\ &\leq \sup_{\phi_j\in B_{X_j^*}}\left\|\left(\phi_1(x_i^1)...\phi_m(x_i^m)\right)_{i\leq=1}^n\right\|_{\frac{p}{1-\sigma}}\|(y_i)_{i=1}^n\|_r\\ &\leq \textstyle\prod_{j=1}^m\left\|(x_i^j)_{i=1}^n\right\|_{\frac{p_j}{1-\sigma},\omega}\|(y_i)_{i=1}^n\|_r\\ &\leq \textstyle\prod_{j=1}^m\delta_{p_j\sigma}((x_i^j)_{i=1}^n)\,\|(y_i)_{i=1}^n\|_r\end{aligned}$$

Since is holds for every representation of $u$, consequently $\epsilon(u)\leq\beta_{p,\sigma}(u)$. Thus $\beta_{p,\sigma}(u)=0$ imply $u=0$. Hence $\beta_{p,\sigma}$ is a norm on $X_1\otimes...\otimes X_m\otimes Y$.
It is clear that $\beta_{p,\sigma}(x^1\otimes...\otimes x^m\otimes y)\leq\left\|x^1\right\|...\|x^m\|\,\|y\|$ for every $x^j\in X_j, j=1,...,m$ and $y\in Y$.
Let $\phi_j\in X_j^*$ with $\phi_j\neq0, j=1,...,m$, let $\psi\in Y^*$ and let $u=\sum\limits_{i=1}^{n}x_i^1\otimes...\otimes x_i^m\otimes y_i$.
Then an application of Hölder's inequality yields

$$|\phi_1\otimes...\otimes\phi_m\otimes\psi(u)|=\left|\phi_1\otimes...\otimes\phi_m\otimes\psi(\sum_{i=1}^{n}x_i^1\otimes...\otimes x_i^m\otimes y_i)\right|$$

$$\leq\sum_{i=1}^{n}\left|\phi_1(x_i^1)...\phi_m(x_i^m)\psi(y_i)\right|\leq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\phi_j(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}}\|(\psi(y_i))_{i=1}^n\|_r$$

$$\leq\|\phi_1\|...\|\phi_m\|\,\|\psi\|\prod_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\frac{\phi_j}{\|\phi_j\|}(x_i^j)\right|^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}}\|(y_i)_{i=1}^n\|_r$$

$$\leq\|\phi_1\|...\|\phi_m\|\,\|\psi\|\prod_{j=1}^{m}\delta_{p_j\sigma}((x_i^j)_{i=1}^n)\,\|(y_i)_{i=1}^n\|_r\,.$$

From which it follows that $|\phi_1\otimes...\otimes\phi_m\otimes\psi(u)|\leq\|\phi_1\|...\|\phi_m\|\,\|\psi\|\,\beta_{p,\sigma}(u)$. Therefore $\phi_1\otimes...\otimes\phi_m\otimes\psi$ is bounded and satisfies $\|\phi_1\otimes...\otimes\phi_m\otimes\psi\|\leq\|\phi_1\|...\|\phi_m\|\,\|\psi\|$ and we have shown that $\beta_{p,\sigma}$ is a reasonable crossnorm. □

In particular, note that when $m=1$, the norm $\beta_{p,\sigma}$ is reduced to the norm $d_{p,\sigma}$ on $X_1\otimes Y$ was introduced by López Molina and Sánchez Pérez in [13]. In what follows we consider the tensor product of linear operators in connection with the reasonable crossnorm $\beta_{p,\sigma}$. We show that the reasonable crossnorm $\beta_{p,\sigma}$ is actually a tensor norm [24, p. 127].

**Proposition 4.2.** *Let $X_j,Y_j,X,Y$ be Banach spaces, and $T\in\mathcal{L}(X,Y)$, $T_j\in\mathcal{L}(X_j,Y_j)$, $(j=1,...,m)$. Then there is a unique continuous linear operator $T_1\otimes_{\beta_{p,\sigma}}...\otimes_{\beta_{p,\sigma}}T_m\otimes_{\beta_{p,\sigma}}T$ from $\left(X_1\widehat{\otimes}...\widehat{\otimes}X_m\widehat{\otimes}X,\beta_{p,\sigma}\right)$ into $\left(Y_1\widehat{\otimes}...\widehat{\otimes}Y_m\widehat{\otimes}Y,\beta_{p,\sigma}\right)$ such that*

$$T_1\otimes_{\beta_{p,\sigma}}...\otimes_{\beta_{p,\sigma}}T_m\otimes_{\beta_{p,\sigma}}T(x^1\otimes...\otimes x^m\otimes x)=(T_1x^1)\otimes...\otimes(T_mx^m)\otimes(Tx)$$

*for every $x^j\in X_j,(j=1,...,m)$ and $x\in X$. Moreover*

$$\left\|T_1\otimes_{\beta_{p,\sigma}}...\otimes_{\beta_{p,\sigma}}T_m\otimes_{\beta_{p,\sigma}}T\right\|=\|T_1\otimes...\otimes T_m\otimes T\|=\|T\|\prod_{j=1}^{m}\|T_j\|\,.$$

*Proof.* By [24, p.7] there is a unique linear operator

$$T_1\otimes...\otimes T_m\otimes T:(X_1\otimes...\otimes X_m\otimes X)\to(Y_1\otimes...\otimes Y_m\otimes Y)$$

such that $T_1 \otimes ... \otimes T_m \otimes T(x^1 \otimes ... \otimes x^m \otimes x) = (T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx)$ for every $x^j \in X_j, j = 1, ..., m$ *and* $x \in X$. We may suppose $T_j \neq 0$, $j = 1, ..., m$ and $T \neq 0$. Let $u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes x_i \in X_1 \otimes ... \otimes X_m \otimes X$, hence the sum $\sum_{i=1}^n \left(T_1 x_i^1\right) \otimes ... \otimes (T_m x_i^m) \otimes (Tx_i)$ is a representation of $T_1 \otimes ... \otimes T_m \otimes T(u)$ in $Y_1 \otimes ... \otimes Y_m \otimes Y$. Then, for every $1 \leq p, p_1, ..., p_m, r < \infty, 0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we have

$$\beta_{p,\sigma}\left(T_1 \otimes ... \otimes T_m \otimes T(u)\right) \leq \prod_{j=1}^m \delta_{p_j\sigma}((T_j x_i^j)_{i=1}^n) \left\|(Tx_i)_{i=1}^n\right\|_r$$

$$\leq \|T\| \prod_{j=1}^m \|T_j\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left\|(x_i)_{i=1}^n\right\|_r .$$

Since this holds for every representation of $u$, we obtain

$$\beta_{p,\sigma}\left(T_1 \otimes ... \otimes T_m \otimes T(u)\right) \leq \|T\| \prod_{j=1}^m \|T_j\| \, \beta_{p,\sigma}(u).$$

So that the linear operator

$$T_1 \otimes ... \otimes T_m \otimes T : (X_1 \otimes ... \otimes X_m \otimes X, \beta_{p,\sigma}) \to (Y_1 \otimes ... \otimes Y_m \otimes Y, \beta_{p,\sigma})$$

is continuous and we have $\|T_1 \otimes ... \otimes T_m \otimes T\| \leq \|T\| \prod\limits_{j=1}^m \|T_j\|$.

On the other hand, as $\beta_{p,\sigma}$ is an reasonable crossnorm we get that

$$\begin{array}{rcl} \|Tx\| \prod_{j=1}^m \left\|T_j x^j\right\| & = & \beta_{p,\sigma}\left((T_1 x^1) \otimes ... \otimes (T_m x^m) \otimes (Tx)\right) \\ & \leq & \|T_1 \otimes ... \otimes T_m \otimes T\| \, \beta_{p,\sigma}\left(x^1 \otimes ... \otimes x^m \otimes x\right) \\ & \leq & \|T_1 \otimes ... \otimes T_m \otimes T\| \, \|x\| \prod_{j=1}^m \left\|x^j\right\| . \end{array}$$

Thus $\|T_1 \otimes ... \otimes T_m \otimes T\| \geq \|T\| \prod_{j=1}^m \|T_j\|$ and therefore $\|T_1 \otimes ... \otimes T_m \otimes T\| = \|T\| \prod\limits_{j=1}^m \|T_j\|$.

Now taking the unique continuous extension of the operator $T_1 \otimes ... \otimes T_m \otimes T$ to the completions of $X_1 \otimes ... \otimes X_m \otimes X$ and $Y_1 \otimes ... \otimes Y_m \otimes Y$ which we denote by $T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T$ we obtain a unique linear operator from $\left(X_1 \widehat{\otimes} ... \widehat{\otimes} X_m \widehat{\otimes} X, \beta_{p,\sigma}\right)$ into $\left(Y_1 \widehat{\otimes} ... \widehat{\otimes} Y_m \widehat{\otimes} Y, \beta_{p,\sigma}\right)$ with the norm $\left\|T_1 \otimes_{\beta_{p,\sigma}} ... \otimes_{\beta_{p,\sigma}} T_m \otimes_{\beta_{p,\sigma}} T\right\| = \|T\| \prod_{j=1}^m \|T_j\|$.

□

Following the idea of [15, Th. 3.7] we prove the following result.

**Theorem 4.3.** *The space* $(\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y^*), \|.\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}})$ *is isometrically isomorphic to* $(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*$ *through the mapping* $\Psi$ *defined by*

$$\Psi(T)(x^1 \otimes ... \otimes x^m \otimes y) = T(x^1, ..., x^m)(y),$$

*for every* $T \in \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y^*), x^j \in X_j, j = 1, ..., m$ *and* $y \in Y$.

*Proof.* It is easy to see that the correspondence $\Psi$ defined as above is linear. It rest to shows the surjectivity and

$$\|\Psi(T)\|_{(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*} = \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}}$$

for all $T$ in $(\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y^*))$.

Let $\phi \in (X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*$ define the $m$-linear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y^*)$, by

$$T(x^1, ..., x^m)(y) = \phi(x^1 \otimes ... \otimes x^m \otimes y).$$

Let $(x_i^1, ..., x_i^m)_{i=1}^n \subset X_1 \times ... \times X_m$. For each $\varepsilon > 0$, choose $(y_i)_{i=1}^n \subset Y, \|y_i\| = 1, j = 1, ..., m$ such that

$$\sum_{i=1}^n \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{p}{1-\sigma}} \leq \varepsilon + \sum_{i=1}^n \left|T(x_i^1, ..., x_i^m)(y_i)\right|^{\frac{p}{1-\sigma}} = (*)$$

For a convenient choice of $\lambda_i \in \mathbb{K}$, $|\lambda_i| = 1$, $i = 1, ..., n$ we can write

$$\begin{aligned}
(*) &= \varepsilon + \sum_{i=1}^n \left| \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\frac{p}{1-\sigma}-1} \phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i) \right| \\
&= \varepsilon + \left| \sum_{i=1}^n \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\frac{p}{1-\sigma}-1} \lambda_i \phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i) \right| \\
&= \varepsilon + \left| \phi \left( \sum_{i=1}^n \lambda_i \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\frac{p}{1-\sigma}-1} x_i^1 \otimes ... \otimes x_i^m \otimes y_i \right) \right| \\
&\leq \varepsilon + \|\phi\| \beta_{p,\sigma} \left( \sum_{i=1}^n \lambda_i \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\frac{p}{1-\sigma}-1} x_i^1 \otimes ... \otimes x_i^m \otimes y_i \right) \\
&\leq \varepsilon + \|\phi\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left\| \left( \lambda_i \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\frac{p}{1-\sigma}-1} y_i \right)_{i=1}^n \right\|_r \\
&= \varepsilon + \|\phi\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left( \sum_{i=1}^n \left|\phi(x_i^1 \otimes ... \otimes x_i^m \otimes y_i)\right|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}} \\
&\leq \varepsilon + \|\phi\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \left( \sum_{i=1}^n \left\|T(x_i^1, ..., x_i^m)\right\|^{\left(\frac{p}{1-\sigma}-1\right)r} \right)^{\frac{1}{r}}.
\end{aligned}$$

Since $\varepsilon$ is arbitrary and $(\frac{p}{1-\sigma} - 1)r = \frac{p}{1-\sigma}$, these inequalities imply

$$\left( \sum_{i=1}^n \left\|T(x_i^1, ..., x_i^m)\right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq \|\phi\| \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n)$$

showing that

$$T \in \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y^*)$$

and

$$\|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \leq \|\phi\| = \|\Psi(T)\|_{(X_1 \otimes ... \otimes X_m \otimes Y, \beta_{p,\sigma})^*}.$$

In order to establish the reverse inequality, we take $T \in \mathcal{L}^\sigma_{as,(p;p_1,...,p_m)} (X_1, ..., X_m; Y^*)$, and let

$$u = \sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i \in X_1 \otimes ... \otimes X_m \otimes Y,$$

where $m \in \mathbb{N}, (x_i^j)_{i=1}^n \subset X_j, (y_i)_{i=1}^n \subset Y, j = 1, ..., m$. Hence, by Hölder's inequality it follows that

$$\begin{aligned}
|\Psi(T)(u)| &= \left| \sum_{i=1}^n T(x_i^1, ..., x_i^m)(y_i) \right| \\
&\leq \left\| \left(T(x_i^1, ..., x_i^m)\right)_{i=1}^n \right\|_{\frac{p}{1-\sigma}} \|(y_i)_{i=1}^n\|_r \\
&\leq \|T\|_{\mathcal{L}^\sigma_{as,(p;p_1,...,p_m)}} \textstyle\prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \|(y_i)_{i=1}^n\|_r .
\end{aligned}$$

So $|\Psi(T)(u)| \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}} \cdot \beta_{p,\sigma}(u)$. Since $u$ is arbitrary it follows that

$$\|\Psi(T)\|_{(X_1\otimes\ldots\otimes X_m\otimes Y,\beta_{p,\sigma})^*} \leq \|T\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}}.$$

Now we are ready to introduce a new formula of the tensor norm $\beta_{p,\sigma}$ in such way that we characterize the space of absolutely $(p;p_1,...,p_m;\sigma)$-continuous multilinear forms. Let $u \in X_1 \otimes ... \otimes X_m \otimes Y$. For $1 \leq p, p_1, ..., p_m, r < \infty, 0 \leq \sigma < 1$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $\frac{1}{r} + \frac{1-\sigma}{p} = 1$, we consider

$$\widetilde{\nu}_{p,\sigma}(u) = \inf \|(\lambda_i)_{i=1}^n\|_r \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(y_i)_{i=1}^n\|_\infty$$

taking the infimum over all representations of $u$ of the form

$$u = \sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$$

with $(x_i^j)_{i=1}^n \subset X_j, (y_i)_{i=1}^n \subset Y, (\lambda_i)_{i=1}^n \subset \mathbb{K}, j = 1, ..., m$ and $n, m \in \mathbb{N}$.

□

**Proposition 4.4.** *We have $\widetilde{\nu}_{p,\sigma}(u) = \beta_{p,\sigma}(u)$ for all $u \in X_1 \otimes ... \otimes X_m \otimes Y$.*

*Proof.* We note first that every representation of $u$ of the form $\sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ can be written as $\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes (\lambda_i y_i)$ and hence

$$\begin{aligned} \beta_{p,\sigma}(u) &\leq \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda_i y_i)_{i=1}^n\|_r \\ &\leq \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n) \, \|(\lambda_i)_{i=1}^n\|_r \, \|(y_i)_{i=1}^n\|_\infty \end{aligned}$$

from which it follows that $\beta_{p,\sigma}(u) \leq \widetilde{\nu}_{p,\sigma}(u)$.
On the other hand, let $\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ be a representation of $u$. We can write $u$ as $\sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes z_i$, where $\lambda_i = \|y_i\|$ and $\|z_i\| \leq 1$ for every $i = 1, ..., n$. Then $\widetilde{\nu}_{p,\sigma}(u) \leq \|(y_i)_{i=1}^n\|_r \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n)$ and hence $\widetilde{\nu}_{p,\sigma}(u) \leq \beta_{p,\sigma}(u)$.

□

**Remark 4.5.** Making $F = \mathbb{K}$, in Theorem 4.3 we obtain that for every family of Banach spaces $X_1, ..., X_m$, the space of absolutely $(p;p_1,...,p_m;\sigma)$-continuous multilinear forms

$$(\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}(X_1, ..., X_m), \|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,\ldots,p_m)}})$$

is isometric to $(X_1 \otimes ... \otimes X_m \otimes \mathbb{K}, \widetilde{\nu}_{p,\sigma})^*$.

We recall that by the universal property of tensor products [11, Th. 1.6.2], there is an algebraic isomorphism between the $m$-linear mapping from $X_1 \times ... \times X_m$ into $Y$ and the linear mapping from $X_1 \otimes ... \otimes X_m$ into $Y$. To each $m$-linear mapping $T$ corresponds the linear mapping $\widetilde{T}$ such that

$$\widetilde{T}(x^1 \otimes ... \otimes x^m) = T(x^1, ..., x^m)$$

for every $x^j \in X_j$, $j = 1, ..., m$.

In Proposition 4.1 if we take $Y = \mathbb{K}$, then we identify $X_1 \otimes ... \otimes X_m \otimes \mathbb{K}$ with $X_1 \otimes ... \otimes X_m$, and in this case the corresponding tensor norm will be denoted by $\nu_{p,\sigma}$ and can be described as follows:

$$\nu_{p,\sigma}(u) = \inf \|(\lambda_i)_{i=1}^n\|_r \prod_{j=1}^m \delta_{p_j\sigma}((x_i^j)_{i=1}^n)$$

where the infimum is taken over all representations of $u \in X_1 \otimes ... \otimes X_m$ of the form $u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m$ with $(\lambda_i)_{i=1}^n \subset \mathbb{K}, (x_i^j)_{i=1}^n \subset X_j, j = 1, ..., m$.

The next theorem and its proof are similar to Theorem 4.3.

**Theorem 4.6.** $(\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m), \|.\|_{\mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}})$ *is isometrically isomorphic to* $(X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*$ *through the mapping* $T \mapsto \widetilde{T}$.

A consequence of Remark 4.5 and Theorem 4.6 we see that $(X_1 \otimes ... \otimes X_m \otimes \mathbb{K}, \beta_{p,\sigma})^*$ is isometric to $(X_1 \otimes ... \otimes X_m, \nu_{p,\sigma})^*$.

## 5. Some Applications

5.1. **Compactness and absolutely $(p; p_1, ..., p_m; \sigma)$-continuous multilinear operators on reflexive Banach spaces.** Compactness and weak compactness of multilinear maps is in general a property that is not easy to characterize, and it is nowadays not very well known. In what follows we prove that under certain summability conditions we can assure that the multilinear map is compact, obtaining in this way some sufficient automatic conditions for compactness of multilinear maps. We relax the requirements that are necessary for the case of $p$-summing multilinear maps by using Theorem 3.5 and the factorization theorem for the class of absolutely $(p; p_1, ..., p_m; \sigma)$-continuous multilinear operators that we have proved (Theorem 3.6).

**Proposition 5.1.** *Let* $0 \le \sigma < 1$, $1 \le p < \infty$ *and* $X$ *be a Banach space. The inclusion/quotient map* $i : X \to L_{p,\sigma}(\eta)$ *is completely continuous.*

*Proof.* Clearly, the map $i$ can be isometrically factorized through its identification with the subspace $M = i_X(X)$ of $C(B_{X^*})$, and so we have

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, hook, "i_0"'] & & L_{p,\sigma}(\eta) \\ & M \arrow[ur, "i_1"'] & \end{tikzcd}$$

Take a sequence $(x_n)$ in $X$ converging weakly to zero. Then for each $x^* \in X^*$ we have that $(\langle x_n, x^* \rangle)_n$ converges to 0. But this means that the sequence $(\langle x_n, \cdot \rangle)_n$ converges pointwise to 0. Consider the functions $|\langle x_n, \cdot \rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}}$. Clearly, they converge to 0 too, and its sequence is order bounded in $L^1(\eta)$ by the $\eta$-integrable function $\sup_n \|x_n\|^{\frac{p}{1-\sigma}} \chi_{B_{X^*}}$. The Dominated Convergence Theorem gives that $\lim_n \int_{B_{X^*}} |\langle x_n, \cdot \rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}} = 0$. Therefore, since

$$\| [\langle x_n, \cdot \rangle] \|_{L_{p,\sigma}}^{\frac{p}{1-\sigma}} \le \int_{B_{X^*}} |\langle x_n, \cdot \rangle|^p \|x_n\|^{\frac{p\sigma}{1-\sigma}}$$

we obtain that $\| [\langle x_n, \cdot \rangle] \|_{L_{p,\sigma}} \to_n 0$. The result is proved. □

**Corollary 5.2.** *Let* $Y$ *a Banach space,* $0 \le \sigma < 1$ *and* $1 \le p, p_1, ..., p_m < \infty$ *with* $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ *and let* $X_1, ..., X_m$ *be reflexive Banach spaces. If* $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$, *then* $T$ *is compact.*

*Proof.* It is a consequence of Theorem 3.6 and the previous proposition. □

As a consequence of the factorization properties of compact bilinear maps that can be found in [22] and [23], we obtain the following

**Corollary 5.3.** *Let $Z$ be a Banach space, $0 \leq \sigma < 1$ and $1 \leq p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and let $X, Y$ be reflexive Banach spaces. If $T \in \mathcal{L}^{\sigma}_{as,(p;q,r)}(X, Y; Z)$, then $T$ factorizes through a (closed) subspace of $c_0$ by means of a compact bilinear map and a compact linear map.*

For the proof, see Theorem 3, Theorem 5 and Corollary 6 in [23].

**Corollary 5.4.** *Let $X, Y$ be reflexive Banach spaces and $T \in \mathcal{L}^{\sigma}_{as,(p;q,r)}(X, Y; c_0)$. Then $T$ can be written as $T(x, y) = (b_n(x, y))_n$ for a norm null sequence $(b_n)_n$ of continuous bilinear forms.*

*Proof.* It is a consequence of Proposition 8 in [23]. □

The dependence of the fact that an $m$-linear map belongs to a multi-ideal if it factorizes through $m$-linear maps that belong to the corresponding linear ideal can be found in [6]. It is said that this happens for the case of compact bilinear maps and for weakly compact bilinear maps. So we can our results in the case of operators that are defined on reflexive spaces. As a consequence of the main Theorem in [4] and Proposition 5.1, we obtain the following result (see also [1, 3]).

**Corollary 5.5.** *Let $Y$ be a Banach space, $0 \leq \sigma < 1$, $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and let $X_1, ..., X_m$ be reflexive Banach spaces. If $T \in \mathcal{L}^{\sigma}_{as,(p;p_1,...,p_m)}(X_1, ..., X_m; Y)$, then $T$ is weak-to-norm continuous on bounded sets.*

5.2. **Absolutely continuous bilinear maps on Banach function spaces.** Further domination requirements for the transpose of the $p$-summing operators provide the well-known class of the $(p, q)$-dominated operators. In the interpolated case of the absolutely $(p, \sigma)$-continuous operators the same construction provides also the class of the $(p, \sigma, q, \nu)$-dominated operators, which is also well-known, specially regarding its domination and factorization properties [13] as well as their tensor product representation (see [25]).

Consider a couple of indexes $1 \leq p_1, p_2 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$. Define $r$ by $\frac{(1-\sigma)}{p_1} + \frac{(1-\sigma)}{p_2} = 1/r$. An operator $T : X \to Y$ is said to be $(p_1, \sigma, p_2, \sigma)$-dominated if $T$ can be dominated as

$$\langle T(x), y^* \rangle \leq C \|x\|^{\sigma} \|S_1(x)\|^{1-\sigma} \|y^*\|^{\sigma} \|S_2(y^*)\|^{1-\sigma}, \ C > 0$$

for every $x \in E$ and $y^* \in Y^*$, where $S_1 : X \to G_1$ and $S_2 : Y^* \to G_2$ are $p_1$-summing and $p_2$-summing operators on Banach spaces $G_1$ and $G_2$, respectively. It can be also be defined by means of the following condition: if there exist $C > 0$ such that for every finite sequence $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$

$$\|(\langle T(x_i), y_i^* \rangle)_{i=1}^n\|_r \leq C \delta_{p_1,\sigma}((x_i)_{i=1}^n) \cdot \delta_{p_2,\sigma}((y_i^*)_{i=1}^n)$$

This kind of domination is in fact the same thing that characterizes that $B_T$, the bilinear operator associated to $T$, is absolutely $(r(1-\sigma); p_1, p_2; \sigma)$-continuous. This provides the domination (see [13, Th. 2.4])

$$|\langle T(x), y^* \rangle| \leq C (\int_{B_{X^*}} (|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma})^{\frac{p_1}{1-\sigma}} d\eta_1)^{\frac{1-\sigma}{p_1}} \cdot (\int_{B_{Y^{**}}} (| \langle y^*, y^{**} \rangle^{1-\sigma} \|y^*\|^{\sigma} d\eta_2)^{\frac{p_2}{1-\sigma}})^{\frac{1-\sigma}{p_2}},$$

where $\eta_1$ and $\eta_2$ are Radon measures on the corresponding unit balls.

After Theorem 3.5, we can find the following factorization scheme for the $(p_1, \sigma, p_2, \sigma)$-dominated operators (we use the same notation that in Theorem 3.5). Consider a $(p_1, \sigma, p_2, \sigma)$-dominated operator $T : X \to Y$. Then there are regular Borel probability measures $\eta_1$

and $\eta_2$ on $B_{X^*}$ and $B_{Y^{**}}$, respectively, such that $T$ factorizes as

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ i\downarrow & & \uparrow \widetilde{T} \\ M_1 & \xrightarrow{i} & S_1 \end{array}$$

and $T^*$ factorizes as

$$\begin{array}{ccc} Y^* & \xrightarrow{T^*} & X^* \\ i\downarrow & & \uparrow \widetilde{T^*} \\ M_2 & \xrightarrow{i} & S_2 \end{array}$$

where $S_1 \subseteq L_{p_1,\sigma}(\eta_1)$ and $S_2 \subseteq L_{p_2,\sigma}(\eta_2)$ are the subspaces appearing in Theorem 3.5. In fact, our multilinear factorization result Theorem 3.6 gives that the bilinear form $B_T$ associated to $T$ factorizes as

$$X \times Y^* \to S_1 \times S_2 \to \mathbb{R}.$$

In the case of operators defined between Banach lattices, and as a consequence of our results, more can be said on the factorization schemes for $(p_1,\sigma,p_2,\sigma)$-dominated operators. In order to do this, let us introduce now some notions regarding Banach function spaces. Let $(\Omega,\Sigma,\mu)$ a $\sigma$-finite measure space. Let $L^0(\mu)$ be the space of classes of $\mu$-a.e. measurable functions. We consider a Banach function space $X(\mu) \subseteq L^0(\mu)$ in the sense of [12, p.28], i.e. a Banach ideal of locally integrable functions containing all the characteristic functions of sets of finite measure (see also this text or [17] for the definition of order continuity and the Fatou property and the main results regarding this class of Banach lattices). We write $X$ for short if the measure is clear in the context, and $X(\mu)'$ for the Köthe dual of $X$, i.e. the elements of the dual space that can be represented as integrals of measurable functions. Assume that the Banach function space $X(\mu)$ is also $p$-convex. In this case, it is well-known that the $p$-th power space of $X$ that is defined as

$$X_{[p]} := \{f \in L^0(\mu) : |f|^{1/p} \in X(\mu)\}$$

with the quasi-norm $\|f\|_{X_{[p]}} := \||f|^{1/p}\|^p_{X(\mu)}$, is a Banach function space with a norm that is equivalent to $\|f\|_{X_{[p]}}$ when $X$ is $p$-convex (see Proposition 2.23 in [17] and the same book for the definitions and main results on $p$-th powers). As in the case of the spaces $L_{p,\sigma}$ that we have defined in the previous sections, we can define the interpolation space $(X(\mu), L^p(\nu))_\sigma$, where $\nu$ is absolutely continuous with respect to $\mu$ and $X(\mu) \hookrightarrow L^p(\nu)$ is the corresponding inclusion quotient map, is well defined. Then the expression

$$\|f\|_{p,\sigma} := \inf \sum_{i=1}^n \|x\|^\sigma \Big(\int |f|^p d\nu\Big)^{\frac{1-\sigma}{p}}$$

for $f \in X$, where the infimum is defined over all decomposition in $X$ as $\sum_{i=1}^n f_i = f$, is a seminorm on $X$. We write $(X(\mu), L^p(\nu))_\sigma$ for the corresponding quotient space and $i : X(\mu) \to (X(\mu), L^p(\nu))_\sigma$ for the inclusion/quotient map.

Let $X(\mu)$ be a Banach function space, let $E$ be a Banach space and let $T : X(\mu) \to E$ be an operator. Let $1 \leq p < \infty$ and let $0 \leq \sigma < 1$. We say that $T$ is $p_\sigma$-concave (see [27, Def. 3.1]) if there is a constant $C > 0$ such that for every finite sequence of functions $f_1, ..., f_n \in X(\mu)$, it holds

$$\|(T(f_i))_{i=1}^n\|_{\frac{p}{1-\sigma}} \leq C \left\| \left( \sum_{i=1}^n (|f_i|^{1-\sigma} \|f_i\|^\sigma)^{\frac{p}{1-\sigma}} \right)^{\frac{1}{p}} \right\|^{1-\sigma}.$$

These operators are characterized as the ones that allow a domination by means of an interpolation formula as follows (see Theorem 3.4 in [27]). Suppose that $X(\mu)$ is order continuous. An operator $T : X(\mu) \to E$ is $p_\sigma$-concave if and only if there is an nonnegative element $\varphi \in (X(\mu)_{[p]})'$ such that for every $f \in X(\mu)$,

$$\|T(f)\| \leq (\int |f|^p \varphi d\mu)^{\frac{1-\sigma}{p}} \|f\|_X^\sigma.$$

A $(p,\sigma)$-absolutely continuous operator is always $p_\sigma$-concave. This can be proved easily using Proposition 1.d.9 in [12] (see Example 3.3 in [27]). This result can be extended to the case of multilinear maps using the same inequalities. Let $1/p = 1/p_1 + 1/p_2$ and $0 \leq \sigma < 1$ such that $\frac{p}{1-\sigma} \geq 1$. It can be easily shown that every $(p_1,\sigma,p_2,\sigma)$-dominated operator satisfies that there is a constant $C > 0$ such that for every $f_1, ..., f_n \in X(\mu)$ and $g_1^*, ..., g_n^* \in Y^*(\nu)$,

$$\begin{aligned} & \big\| (\langle T(f_i), g_i^* \rangle)_{i=1}^n \big\|_{\frac{p}{1-\sigma}} \\ \leq \quad & C. \left\| \left( \sum_{i=1}^n \left( |f_i|^{1-\sigma} \|f_i\|^\sigma \right)^{\frac{p_1}{1-\sigma}} \right)^{\frac{1}{p_1}} \right\|_X^{1-\sigma} \left\| \left( \sum_{i=1}^n \left( |g_i^*|^{1-\sigma} \|g_i^*\|^\sigma \right)^{\frac{p_2}{1-\sigma}} \right)^{\frac{1}{p_2}} \right\|_{Y^*}^{1-\sigma} \end{aligned}$$

We will say that such an operator satisfies a $(p_1,\sigma,p_2,\sigma)$-concave domination.

**Theorem 5.6.** *Let $T : X(\mu) \to Y(\nu)$ be an operator between the order continuous Banach function space $X(\mu)$ and the Banach function space with the Fatou property $Y(\nu)$ such that its Köthe dual is order continuous. Assume also that $X(\mu)$ is $p_1$-convex and $Y(\nu)$ is $p_2^*$-concave for $1 \leq p < \infty$. Let $0 \leq \sigma < 1$. The following statements are equivalent.*

(i) *The operator $T$ satisfies a $(p_1,\sigma,p_2,\sigma)$-concave domination.*

(ii) *There is a couple of functions $f' \in X(\mu)'$ and $g \in Y(\mu)$ such that for all $f \in X(\mu)$ and $g^* \in Y^*(\nu)$,*

$$|\langle T(f), g^* \rangle| \leq \left( \int |f|^{p_1} f' d\mu \right)^{\frac{1-\sigma}{p_1}} \|f\|_X^\sigma \left( \int |g^*|^{p_2} g d\nu \right)^{\frac{1-\sigma}{p_2}} \|g^*\|_{Y^*}^\sigma$$

(iii) *There is a factorization for $T$ as*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "i"'] & F \\ (X, L^{p_1}(f_0\mu))_\sigma \arrow[r, "\hat{T}"] & ((Y', L^{p_2}(g_0\nu))_\sigma)^* \arrow[u, "i'"'] \end{tikzcd}$$

*Consequently, each operator as above satisfying that the associated bilinear form is $(p;p_1,p_2;\sigma)$-absolutely continuous factorizes as in (iii).*

*Proof.* For the equivalence between (i) and (ii) it can be used the same argument based in Ky Fan's Lemma that proves Theorem 1 of [8]; for obtaining the right inequalities from the ones given in (i), see also the proof of Theorem 3.4 of [27] that leads to the linear version of our result. Notice that the assumptions of $X$ being $p_1$-convex and $Y$ being $p_2^*$-concave (and so $Y^*$ is $p_2$-convex) is necessary for proving it. Also the requirements on the order continuity and the Fatou property, that allows to assure that $X^* = Y'$, $Y'^* = Y''$ and $Y'' = Y$.

Let us prove (ii) $\Rightarrow$ (iii). Clearly, the assumptions on $T$ allows to extend the bilinear form $\Phi(f,g) := \langle T(f), g \rangle$ as

$$X(\mu) \times Y'(\nu) \to (X, L^{p_1}(f_0\mu))_\sigma \times (Y', L^{p_2}(g_0\nu))_\sigma \to \mathbb{R}.$$

with a continuous bilinear form $\hat{\Phi} : (X, L^{p_1}(f_0\mu))_\sigma \times (Y', L^{p_2}(g_0\nu))_\sigma \to \mathbb{R}$. Therefore we can define the map $T_{\hat{\Phi}} : (X, L^{p_1}(f_0\mu))_\sigma \to ((Y', L^{p_2}(g_0\nu))_\sigma)^*$ by $\langle T_{\hat{\Phi}}(x), y' \rangle := \hat{\Phi}(x, y')$. We have that $i : Y' \to (Y', L^{p_2}(g_0\nu))_\sigma$, and so $i' : (Y', L^{p_2}(g_0\nu))_\sigma^* \to (Y')^*$. Since $Y'$ is order continuous, $(Y')^* = Y''$ and the Fatou property of $Y$ gives $Y'' = Y$. Consequently, the factorization is obtained for $\hat{T} := T_{\hat{\Phi}}$. The converse implication is obvious. □

**Remark 5.7.** More applications in this setting can be obtained regarding the positive version of the absolutely $(p, \sigma)$-continuous operators and their multilinear extensions. For example, boundedness properties for the associated bilinear form of an operator as the ones provided by the absolutely $(p, \sigma)$-continuous operators for the integration map associated to a vector measure provide information about the containment of an interpolated space into the space of integrable functions with respect to $m$ (see [7]). The same technique that we have shown above should provide also the corresponding result for the multilinear case.

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