On the p-length of some finite p-soluble groups*

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Abstract

The main aim of this paper is to give structural information of a finite group of minimal order belonging to a subgroup-closed class of finite groups and whose p-length is greater than 1, p a prime number. Alternative proofs and improvements of recent results about the influence of minimal p-subgroups on the p-nilpotence and p-length of a finite group arise as consequences of our study.

1 Introduction and statement of results

All groups considered are finite. In the following p will be a prime number. The motivation for this paper comes from [7], where some results about the influence of minimal p-subgroups on the p-nilpotence and p-length of groups were given. More precisely, the authors proved there that if p is odd and G is a group with a Sylow p-subgroup P such that the elements of order p of P are in $\mathrm{Z}_{p-1}(P)$, then G is p-nilpotent if and only if $\mathrm{N}_G(P)$ is p-nilpotent (Theorem D). In addition, if G is p-soluble and the elements of order p are

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actually in $\mathrm{Z}_{p-2}(P)$, then G is of p-length at most 1 (Theorem E). The p-nilpotence of G in the above theorem is deduced from [12, Main Theorem], whose proof depends on the interesting fact that every p-Schur-Frattini extension of certain groups of p-length 2 has a subgroup isomorphic to a certain p-group called $Y_p(m)$ provided that p is odd ([12, Section 3.4 and Proposition 3.5]).

Unfortunately, we have found some delicate points in the proof of the above statement. For instance, the image of the form defined in Equation (3.14) is not contained in general in $\mathrm{GF}(p^e)$ because we cannot assure in general that this image is fixed by the corresponding Frobenius-type automorphism. Moreover, in the construction of the subgroup isomorphic to $Y_p(m)$ in Case 1.B in the proof of [12, Proposition 3.5], it is not sufficient to ensure that the chosen element x is not fixed under the automorphism $x \mapsto x^{p^f}$, because x could be taken as an element of the maximal submodule of the regular module and hence x might generate a non-regular submodule.

We have been unable to overcome those difficulties, especially the second one, just following Weigel's proof and so we have tried to solve them by presenting an alternative proof of Proposition 3.5 of [12]. This is done in the paper [2].

The aim of this paper is to describe a completely different approach based on the classical theory of Hall and Higman (see Chapter IX of [10]). An improvement of Theorem E and Theorem D of [7] follow from our main result.

We prove the following general result.

Theorem A. *Let $\mathcal{P}$ be a subgroup-closed class of p-groups, and let $\mathfrak{Y}(\mathcal{P})$ denote the class of all p-soluble groups whose Sylow p-subgroups are in $\mathcal{P}$. Also, let $\mathfrak{L}_p$ be the class of all groups of p-length at most one. Suppose that $\mathfrak{Y}(\mathcal{P})$ is not contained in $\mathfrak{L}_p$, and let G be a p-soluble group of minimal order in $\mathfrak{Y}(\mathcal{P}) \setminus \mathfrak{L}_p$. If P is a Sylow p-subgroup of G, then $\Phi(G)$, the Frattini subgroup of G, is contained in P and one of the following holds.*

1. *If p is not a Fermat prime or the Hall p'-subgroups of G are abelian, then the nilpotence class of $P/\Phi(G)$ is greater than or equal to p.*

2. *If p is a Fermat prime, then the nilpotence class of $P/\Phi(G)$ is greater than or equal to $p-1$.*

We now come to our principal applications of Theorem A.

If P is a p-group and k is a natural number, we denote

$$\Omega_k(P) = \langle x \in P \colon x^{p^k} = 1 \rangle, \quad \text{and} \quad \Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p \text{ is odd,} \\ \Omega_2(P) & \text{if } p = 2. \end{cases}$$

Our first corollary is an improvement of Theorem E in [7].

Corollary 1. *Let p be a prime. Let G be a p-soluble group.*

1. *If $\Omega(P) \leq \mathrm{Z}_{p-2}(P)$, then G has p-length at most 1.*
2. *If p is not a Fermat prime or the Hall p'-subgroups of G are abelian and $\Omega(P) \leq \mathrm{Z}_{p-1}(P)$, then G has p-length at most 1.*

Our second corollary confirms Theorem D of [7] and gives some additional information.

Corollary 2 (See [7, Theorem D])**.** *Suppose that p is a prime. Let G be a group and P a Sylow p-subgroup of G. Assume that $\mathrm{N}_G(P)$ is p-nilpotent.*

1. *If $\Omega(P) \leq \mathrm{Z}_{p-1}(P)$, then G is p-nilpotent.*
2. *If $p = 2$, and either $\Omega(P) \leq \mathrm{Z}(P)$, or $\Omega_1(P) \leq \mathrm{Z}(P)$ and P is quaternion-free, then G is 2-nilpotent.*

We round the paper off with some examples showing that the lower bounds in Theorem A are attained (Examples 3 and 4). Example 5 shows that the hypotheses on the Sylow 2-subgroups in Corollary 2 are necessary.

2 Proof of Theorem A

By [5, A, 10.2; IV, 3.4(a) and 4.8(a); IX, 1.11 and 1.12], the class $\mathfrak{L}_p$ of all p-soluble groups of p-length at most 1 is a subgroup-closed saturated Fitting formation. Moreover, since $\mathcal{P}$ is subgroup-closed, the class $\mathfrak{Y}(\mathcal{P})$ is also subgroup-closed. These facts will be used repeatedly in what follows.

We proceed in a number of steps, the first of which consists of three closely related statements, all of which are consequences of the structure of the proper subgroups of G.

(1) *Every proper subgroup of G has p-length at most 1. In particular:*

1. $\mathrm{O}^{p'}(G) = G$ *and then* $\mathrm{O}^p(G)$ *is a proper normal subgroup of* G*;*

2. G *is a group which has only one maximal normal subgroup;*

3. $G/\mathrm{O}^p(G)$ *is cyclic.*

Note that every proper subgroup of G belongs to $\mathfrak{Y}(\mathcal{P})$. The minimality of G implies that all of them have p-length at most 1.

If $\mathrm{O}^{p'}(G)$ were a proper subgroup of G, then $\mathrm{O}^{p'}(G)$ would have p-length at most 1. Then G would be of p-length at most 1, contradicting our assumption. Hence $\mathrm{O}^{p'}(G) = G$.

Since G is p-soluble, $\mathrm{O}^p(G)$ is a proper normal subgroup of G. Let H be a maximal normal subgroup of G such that $\mathrm{O}^p(G) \leq H$. Then H has p-length at most 1. If there were two such maximal normal subgroups, then G would have p-length at most 1 since the class $\mathfrak{L}_p$ is a Fitting class. Therefore $G/\mathrm{O}^p(G)$ has exactly one maximal subgroup. This implies that $G/\mathrm{O}^p(G)$ is cyclic.

(2) $\mathrm{O}_{p'}(G) = 1$. *Therefore if* F *is the Fitting subgroup of* G*, then* $F = \mathrm{O}_p(G)$ *and* $\mathrm{C}_G(F) \leq F$.

Suppose that $\mathrm{O}_{p'}(G) \neq 1$ and let N be a minimal normal subgroup of G contained in $\mathrm{O}_{p'}(G)$. Then PN/N is a Sylow p-subgroup of G/N such that $PN/N \cong P$. Since $|G/N| < |G|$ the group G/N is of p-length at most 1 and so is G. This contradicts our assumption. Thus $\mathrm{O}_{p'}(G) = 1$.

In particular, $\mathrm{O}_q(G) = 1$ for all primes $q \neq p$. Then $\mathrm{F}(G) = \mathrm{O}_p(G)$ and, since G is p-soluble, $\mathrm{C}_G(F) \leq F$ by [9, VI, 6.5].

(3) $G/\Phi(G)$ *is primitive and so* $F/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$ *is a chief factor of* $G/\Phi(G)$.

Moreover, G/F *is* p*-nilpotent and* $\mathrm{O}_p(G/F) = 1$.

Since $\mathfrak{L}_p$ is a saturated formation, it follows that $G/\Phi(G) \notin \mathfrak{L}_p$. Let $R/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$. Assume that $R/\Phi(G)$ is a p'-group. By [9, VI, 1.7] R has Hall p'-subgroups and all of them are conjugate in G. Consequently $R = K\Phi(G)$ for some Hall p'-subgroup K of R and $G = \mathrm{N}_G(K)(K\Phi(G)) = \mathrm{N}_G(K)$. Then $K \leq \mathrm{O}_{p'}(G) = 1$. Thus $R/\Phi(G)$ is an elementary abelian p-group, $R \leq F$ and $R/\Phi(G)$ is complemented in G. Let M be a maximal subgroup of G such that $MR = G$ and $M \cap R = \Phi(G)$. Then $G/R \cong M/\Phi(G) \in \mathfrak{L}_p$. Now suppose that $R_1/\Phi(G)$ and $R_2/\Phi(G)$ are distinct minimal normal subgroups of $G/\Phi(G)$. Then $R_1 \cap R_2 = \Phi(G)$, so $G/R_i \in \mathfrak{L}_p$, $i = 1, 2$, implies $G/\Phi(G) \in \mathfrak{L}_p$, against supposition. We can then conclude that $\mathrm{Soc}(G/\Phi(G)) = F/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$ and $G/F \in \mathfrak{L}_p$. Since $\mathrm{O}^{p'}(G) = G$, we have that G/F is p-nilpotent. If K/F is a normal p-subgroup of G/F, then $K \leq \mathrm{O}_p(G) = F$. This means that $\mathrm{O}_p(G/F) = 1$. Step (3) is therefore justified.

(4) *If U is a maximal subgroup of G containing P, then $U = \mathrm{N}_G(P)$.*

Let U be a maximal subgroup of G containing P. Since $F \leq U$ we have $\mathrm{O}_{p'}(U) \leq \mathrm{C}_U(F) \leq F$, and therefore $\mathrm{O}_{p'}(U) = 1$. Since $U \in \mathfrak{L}_p$, P is normal in U. Then $U \leq \mathrm{N}_G(P)$. If P were normal in G, then G would be of p-length at most 1, contradicting our hypothesis. Hence $U = \mathrm{N}_G(P)$.

(5) *G is a $\{p, q\}$-group for some prime $q \neq p$ and $G = \mathrm{O}_{p,q,p}(G)$. In particular G is soluble. Write $A = \mathrm{O}_{p,q}(G)$.*

If $N/\mathrm{O}_p(G) = \Phi(A/\mathrm{O}_p(G))$, then A/N is the only minimal normal subgroup of G/N and $U = PN$.

Moreover $\mathrm{O}^p(G) \leq A$ and, in particular, G/A is a cyclic p-group.

By Step (4), G has precisely one maximal subgroup containing P. Hence we can appeal to [11, X, 9.9] and conclude that $G = \mathrm{O}_{p,q,p}(G)$, for some prime $q \neq p$, A/N is the only minimal normal subgroup of G/N and $U = PN$. Since G is a $\{p, q\}$-group, G is soluble by the well-known theorem of Burnside ([5, I, Section I]).

Since $G/A = \mathrm{O}_p(G/A)$, $\mathrm{O}^p(G) \leq A$ and so G/A is a cyclic p-group by Step (1).

(6) *Let M be a maximal subgroup of G complementing $F/\Phi(G)$. Write $B = P \cap M$ and let Q be a Sylow q-subgroup of G contained in M. We have:*

(i) *B is a Sylow p-subgroup of M and $M = QB$.*

(ii) *$B/\Phi(G)$ is a cyclic p-group.*

(iii) *$M = \mathrm{N}_G(Q)$ and $\mathrm{Z}(M/\Phi(G))$ is cyclic.*

(iv) $[\mathrm{O}^p(G), \Phi(G)] = 1$.

(v) $B \leq \mathrm{C}_G(\Phi(Q))$.

(vi) $\mathrm{Z}_\infty(G) = \Phi(G)$.

(i) Since $G = MF$ and $F = \mathrm{O}_p(G)$, $P = F(P \cap M)$ and $B = P \cap M$ is a Sylow p-subgroup of M. Note that every Sylow q-subgroup of G is contained in A. In fact if Q is a Sylow q-subgroup of G contained in M, then $A = QF$ and $A \cap M = Q\Phi(G)$. Therefore $Q\Phi(G)$ is normal in M and $M = QB$.

(ii) Since $G = AM$ and $M = QB$, we have

$$G/A \cong M/Q\Phi(G) \cong B/\Phi(G)$$

and then $B/\Phi(G)$ is cyclic by Step (5).

(iii-iv) We know that $G/F \cong M/\Phi(G)$. Hence

$$M/\Phi(G) = \mathrm{O}^{p'}(M/\Phi(G)) = \mathrm{O}^{p'}(M)\Phi(G)/\Phi(G).$$

This means that $M = \mathrm{O}^{p'}(M)\Phi(G)$ and so $M = \mathrm{O}^{p'}(M)$ because $\Phi(G)$ is a p-group. This forces M to be p-nilpotent as M has p-length at most 1. Then Q is normal in M and $M \leq \mathrm{N}_G(Q)$. Since Q is not normal in G and M is maximal in G, $M = \mathrm{N}_G(Q)$. Thus $[Q, \Phi(G)] = 1$. Consequently $Q \leq \mathrm{C}_G(\Phi(G))$. Since $\mathrm{O}^p(G)$ is generated by all conjugates of Q, we have that $\mathrm{O}^p(G) \leq \mathrm{C}_G(\Phi(G))$ and hence $[\mathrm{O}^p(G), \Phi(G)] = 1$.

Now $F/\Phi(G)$ can be regarded as an irreducible M-module over the finite field of p elements. Since $\mathrm{C}_M(F/\Phi(G)) = \Phi(G)$, it follows that $\mathrm{Z}(M/\Phi(G))$ is cyclic by [5, B, 9.4].

(v) Note that $Q \cong A/F$ and then $N/F = \Phi(A/F) \cong \Phi(Q)$. Thus $N = \Phi(Q)F$ and $U = PN = P\Phi(Q)$. Hence $U \cap M = \mathrm{N}_G(P) \cap \mathrm{N}_G(Q) = P\Phi(Q) \cap M = \Phi(Q)(P \cap M) = \Phi(Q)B$ is a system normaliser of G and therefore is nilpotent by [5, I, 5.4]. Thus $B \leq \mathrm{C}_G(\Phi(Q))$.

(vi) Applying [5, I, 5.9], we have that $\mathrm{Core}_G(U \cap M) = \mathrm{Z}_\infty(G)$. Therefore $\Phi(G) \leq \mathrm{Z}_\infty(G)$. Now, since $(U \cap M)/\Phi(G)$ is core-free in $G/\Phi(G)$, $\mathrm{Z}_\infty(G) = \mathrm{Core}_G(U \cap M) = \Phi(G)$.

(7) *Let H be the maximal normal subgroup of G. Then $H = FQ = A$ and $B/\Phi(G)$ is a cyclic group of order p.*

It is clear that G/H has order p and $A = QF \leq H$. Therefore $H = H \cap FM = F(H \cap M) = F(H \cap QB) = FQ(H \cap B)$ and $B_0 = H \cap B$ is maximal in B. Since $B/\Phi(G)$ is cyclic, $B_0/\Phi(G) = \Phi(B/\Phi(G))$. Moreover H has p-length at most 1 and $\mathrm{O}_{p'}(H) = 1$ by Step (1). Therefore the Sylow p-subgroup FB_0 of H is normal in H. Hence $FB_0 \leq \mathrm{O}_p(G) = F$ and so $B_0 \leq F \cap M = \Phi(G)$. This implies that $H = A$ and the Frattini subgroup of the cyclic p-group $B/\Phi(G)$ is trivial. This is to say that $B/\Phi(G)$ has order p.

Next we focus our attention on the quotient group $\overline{G} = G/\Phi(G)$. For any subgroup X of G we will write $\overline{X}$ to denote the image of X in $\overline{G}$: $\overline{X} = X\Phi(G)/\Phi(G)$.

(8) *$\overline{Q}$ is either elementary abelian or an extraspecial q-group.*

Let $x \in G$ be an element of G such that $\overline{B} = \langle \overline{x} \rangle$, $\overline{x} = x\Phi(G)$. Since $\overline{G}$ is not nilpotent, it follows that $\overline{B}$ is a p-group of automorphisms of the q-group $\overline{Q}$. Let T be a proper subgroup of Q containing $\Phi(G)$ such that $\overline{T}$ is a $\overline{B}$-invariant normal subgroup of $\overline{Q}$. Then $C = FT\langle x \rangle$ is a proper subgroup of G and so it is of p-length at most 1. In addition, $\mathrm{O}_{p'}(C) = 1$. Thus $F\langle x \rangle$ is normal in C and so $\overline{T}$ is centralised by $\overline{B}$. Applying [8, 5.3.7], we have:

- either $\overline{Q}$ is elementary abelian,
- or $\overline{Q}$ has class 2 and $\overline{Q}' = \mathrm{Z}(\overline{Q}) = \Phi(\overline{Q})$ is elementary abelian, and $\overline{x}$ acts trivially on $\mathrm{Z}(\overline{Q})$. This implies that $\mathrm{Z}(\overline{Q}) \leq \mathrm{Z}(\overline{M})$. Since $\mathrm{Z}(\overline{M})$ is cyclic, by part (iii) of Step (6), we have that $\overline{Q}$ is extraspecial.

Recall that $\overline{F}$ can be regarded as an irreducible and faithful $\overline{M}$-module over $K = \mathrm{GF}(p)$, the finite field of p elements. Let $\overline{F}_{\overline{B}}$ denote the subgroup $\overline{F}$ regarded as $\overline{B}$-module over K by restriction.

(9) *If $\overline{Q}$ is abelian, then $\overline{F}_{\overline{B}}$ is a direct sum of copies of the regular $K\overline{B}$-module.*

Since $\overline{F}$ is an irreducible $\overline{M}$-module over K, we can apply [1, 3.3.40] to conclude that $\overline{F}_{\overline{B}}$ is isomorphic to a direct sum of copies of the regular $K\overline{B}$-module.

(10) *Assume that $\overline{Q}$ is extraspecial.*

- *If p is not a Fermat prime, then regular $K\overline{B}$-module is a direct summand of $\overline{F}_{\overline{B}}$.*
- *If p is a Fermat prime then two possibilities arise:*
  - *either the regular $K\overline{B}$-module is a direct summand of $\overline{F}_{\overline{B}}$,*
  - *or $\overline{F}_{\overline{B}}$ is a direct sum of copies of the Jacobson radical, $\mathrm{J}(K\overline{B})$, of the regular $K\overline{B}$-module.*

First we claim that $\overline{x} \in \overline{B}$ induces a fixed-point-free automorphism on $\overline{Q}/\overline{Q}'$. Thus if $\overline{g} \in \overline{Q} \setminus \overline{Q}'$, then $\overline{g}^{\overline{x}}\overline{Q}' \neq \overline{g}\overline{Q}'$.

Write $Q^* = \overline{Q}/\overline{Q}'$ and $M^* = \overline{M}/\overline{Q}'$. Then Q^* is the only minimal normal subgroup of M^* by (5). If $\mathrm{C}_{Q^*}(\overline{B}) \neq 1$, then $\mathrm{C}_{Q^*}(\overline{B})$ is a non-trivial normal subgroup of M^* contained in Q^*. Then $\mathrm{C}_{Q^*}(\overline{B}) = Q^*$. This implies that $\overline{B}$ stabilises the chain $1 \leq \overline{Q}' \leq \overline{Q}$ and then $[\overline{B}, \overline{Q}] = 1$, by [5, A, 12.3]. Therefore $\overline{P}$ is normal in $\overline{G}$, against supposition. Hence $\mathrm{C}_{Q^*}(\overline{B}) = 1$. In particular, $\overline{x}$ acts fixed-point-freely on Q^*.

Since $[\overline{Q}', \overline{B}] = 1$ by Statement (v) of Step (6), $\overline{M} = \overline{QB}$ is critical in the sense of [10, IX, 2.1].

Let L be an algebraic closure of K and let $\overline{F}^L$ be the LM-module obtained from $\overline{F}$ by extending the field to L. Since $\overline{F}$ is a faithful $\overline{M}$-module over K, $\overline{F}^L$ is a faithful $\overline{M}$-module over L by [5, B, 5.2]. According to [5, B, 5.15], $\overline{F}^L = F_1 \oplus \cdots \oplus F_r$ is a direct sum of irreducible $L\overline{M}$-modules and all F_i are Galois-conjugate. In particular $\mathrm{C}_{\overline{M}}(F_1) = \mathrm{C}_{\overline{M}}(F_i)$ for all $i = 1, \ldots, r$ by [5, B, 5.12]. Then $\mathrm{C}_{\overline{M}}(F_1) = \mathrm{C}_{\overline{M}}(\overline{F}^L) = \mathrm{C}_{\overline{M}}(\overline{F}) = 1$. Therefore F_1 is an irreducible and faithful $\overline{M}$-module over L. In particular, $\overline{Q}$ is represented faithfully on F_1.

Write $|\overline{Q}| = q^{2m+1}$ ($m > 0$). Applying a theorem of Hall and Higman [10, IX, 2.6], we have:

1. $\dim_L(F_1) = q^m$, and
2. $(F_1)_{\overline{B}} = V \oplus Y$ where V is a free $L\overline{B}$-module and Y is indecomposable and $\dim_L Y = 1$ or $p - 1$.

Suppose that no direct summand of $(F_1)_{\overline{B}}$ is isomorphic to the regular $L\overline{B}$-module. Then $V = 0$ and $(F_1)_{\overline{B}} = Y$. In this case $q^m = \dim_L F_1 = \dim_L Y = 1$, and therefore $m = 0$. This contradicts our assumption. Hence $q^m = \dim_L Y = p-1$. By [10, IX, 2.7], we have $q = 2$, $m = 2^r$ and $p = 2^{2^r}+1$ is a Fermat prime. In this case, by [10, VII, 5.3], we have:

$$Y \cong L\overline{B}/\mathrm{J}(L\overline{B})^{p-1} \cong \mathrm{J}(L\overline{B}) = \mathrm{J}(K\overline{B} \otimes L) \cong \mathrm{J}(K\overline{B}) \otimes L$$

by [10, VII, 1.5].

By [10, VII, 1.21], the regular $K\overline{B}$-module is a direct summand of $\overline{F}_{\overline{B}}$ provided the regular $L\overline{B}$-module is a direct summand of $(\overline{F}_{\overline{B}})^L \cong (\overline{F}^L)_{\overline{B}}$. Hence we have:

- If p is not a Fermat prime, then the $\overline{B}$-module $\overline{F}_{\overline{B}}$ contains a direct summand isomorphic to $K\overline{B}$, the regular $K\overline{B}$-module.
- If p is a Fermat prime then two possibilities arise:
  (a) either $\overline{F}_{\overline{B}}$ contains a direct summand isomorphic to $K\overline{B}$, the regular $K\overline{B}$-module,
  (b) or $\overline{F}_{\overline{B}}$ is a direct sum of indecomposable modules isomorphic to $\mathrm{J}(K\overline{B})$.

(11) *Conclusion.*

Write $W = C_p \wr C_p$. Note that that $\mathrm{Z}(W)$ is of order p, W' is elementary abelian of order p^p and the nilpotence class of W is p. Hence the nilpotence class of $W/\mathrm{Z}(W)$ is $p-1$.

(a) Suppose that p is not a Fermat prime or $\overline{Q}$ is abelian. Then a direct summand of $\overline{F}_{\overline{B}}$ is isomorphic to the regular $K\overline{B}$-module. In this case $P/\Phi(G)$ contains a subgroup isomorphic to W by [5, B, 11.1]. Then the nilpotence class of $P/\Phi(G)$ is greater or equal than p by [5, A, 8.2].

(b) Suppose that p is a Fermat prime. Then it could occur that $\overline{F}_{\overline{B}}$ is a direct sum of indecomposable $K\overline{B}$-modules isomorphic to $\mathrm{J}(K\overline{B})$. In this case $P/\Phi(G)$ contains a subgroup isomorphic to $W/\mathrm{Z}(W)$ and so the nilpotence class of $P/\Phi(G)$ is greater or equal to $p-1$ by [5, A, 8.2].

3 Proofs of the Corollaries

Proof of Corollary 1. Consider the class $\mathcal{P}_k$ the class of all p-groups P such that $\Omega(P) \le \mathrm{Z}_k(P)$, for some integer k. Then $\mathcal{P}_k$ is a subgroup-closed class of p-groups. Let $\mathfrak{Y}(\mathcal{P}_k)$ denote the class of all p-soluble groups whose Sylow p-subgroups are in $\mathcal{P}_k$. Assume that $\mathfrak{Y}(\mathcal{P}_k)$ is not contained in $\mathfrak{L}_p$. If G is a group of minimal order in $\mathfrak{Y}(\mathcal{P}_k) \setminus \mathfrak{L}_p$, then G is a group described in Theorem A. We follow the same notation. Consider the normal subgroup A. Suppose that every element of order p of A is in $\Phi(G)$. By part (vi) of (6) this is to say that $\Omega(F) \le \mathrm{Z}_\infty(G) \cap A \le \mathrm{Z}_\infty(A)$. Then A is p-nilpotent, by [3, Corollary 4]. This implies that $Q \le \mathrm{C}_G(F) \le F$, and this is not true. Therefore there exists an element of order p, or order 2 or 4 if $p = 2$, say g, in $F \setminus \Phi(G)$.

Since $F/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$, then the normal closure of $\langle g\Phi(G)\rangle$ in $G/\Phi(G)$ is $F/\Phi(G)$. Hence $\langle g\rangle^G\Phi(G) = F$. In fact. since $g \in F$, then $\langle g\rangle^G \leq F$ and then $\langle g\rangle^G \leq \Omega(P)$. Hence $F = \langle g\rangle^G\Phi(G) \leq \Omega(P)\Phi(G)$.

Since $\Omega(P) \leq \mathrm{Z}_k(P)$, then

$$F/\Phi(G) \leq \Omega(P)\Phi(G)/\Phi(G) \leq \mathrm{Z}_k(P)\Phi(G)/\Phi(G) \leq \mathrm{Z}_k(P/\Phi(G)).$$

Since $\overline{P}/\overline{F} \cong \overline{B}$ is a cyclic group, we have that the nilpotence class of $P/\Phi(G)$ is less than or equal to k.

Consequently, the class $\mathfrak{Y}(\mathcal{P}_k)$ is contained in $\mathfrak{L}_p$ for all $k < p-1$. If $k = p-1$ and p is not a Fermat prime, $\mathfrak{Y}(\mathcal{P}_{p-1})$ is contained in $\mathfrak{L}_p$ either. Moreover, every group G in $\mathfrak{Y}(\mathcal{P}_{p-1})$ whose Hall p'-subgroups of G are abelian is of p-length at most 1. This proves Corollary 1. □

Remark. Note that Corollary 1 improves Theorem E of [7] for non-Fermat odd primes and groups with abelian Hall p'-subgroups.

Proof of Corollary 2. (1) We suppose that the statement is false and derive a contradiction. Let G be a non-p-nilpotent group of minimal order subject to having a Sylow p-subgroup P, p odd, such that $\mathrm{N}_G(P)$ is p-nilpotent and $\Omega(P) \leq \mathrm{Z}_{p-1}(P)$. Then G satisfies the following properties.

(a) $\mathrm{O}_{p'}(G) = 1$.

Assume $\mathrm{O}_{p'}(G) \neq 1$, and let N be a minimal normal subgroup of G such that $N \leq \mathrm{O}_{p'}(G)$. Note that PN/N is a Sylow p-subgroup of G/N such that $PN/N \cong P$. Hence $\Omega(PN/N) \leq \mathrm{Z}_{p-1}(PN/N)$. Moreover, $\mathrm{N}_{G/N}(PN/N) = \mathrm{N}_G(P)N/N$ is p-nilpotent. The minimality of G yields G/N is p-nilpotent and so G is p-nilpotent, giving a contradiction.

(b) *Write $F = \mathrm{F}(G) = \mathrm{O}_p(G)$. G is p-soluble and $\mathrm{C}_G(F) \leq F$.*

Let $\mathrm{Z}\,\mathrm{J}(P)$ denote the centre of the Thompson subgroup of P (see [8, 8, Section 2]). Clearly $P \leq \mathrm{N}_G(P) \leq \mathrm{N}_G(\mathrm{Z}\,\mathrm{J}(P))$. If $\mathrm{N}_G(\mathrm{Z}\,\mathrm{J}(P))$ is a proper subgroup of G, then the choice of G ensures that $\mathrm{N}_G(\mathrm{Z}\,\mathrm{J}(P))$ is p-nilpotent. Applying [8, 8.3.1], we deduce that G is p-nilpotent, contrary to supposition. Hence $\mathrm{N}_G(\mathrm{Z}\,\mathrm{J}(P)) = G$. Therefore, $1 \neq \mathrm{Z}\,\mathrm{J}(P) \leq \mathrm{O}_p(G) = F(G) = F$ since $\mathrm{O}_{p'}(G) = 1$.

Suppose that G/F is not p-nilpotent. Then F is a proper subgroup of P and thus $\mathrm{Z}\,\mathrm{J}(P/F)$ is a non-trivial subgroup of P/F which is not normal in G/F. Write $Z/F = \mathrm{Z}\,\mathrm{J}(P/F)$. Then $\mathrm{N}_G(Z)$ is a non-p-nilpotent proper subgroup of G containing P. This contradicts the minimal choice of G. Hence G/F is p-nilpotent. In particular, G is p-soluble and then $\mathrm{C}_G(F) \leq F$ by [9, VI, 6.5].

(c) *G is a soluble group whose p-length is at most* 1.

Let U be a maximal subgroup of G containing P. Since $F \leq U$ we have $\mathrm{O}_{p'}(U) \leq \mathrm{C}_U(F) \leq F$, and therefore $\mathrm{O}_{p'}(U) = 1$. By minimality of G, U is a p-nilpotent group and then $P = U$.

Applying [11, X, 9.9], we have that G is a $\{p, q\}$-group for some prime $q \neq p$. Then G is a $\{p, q\}$-group, and there exists a Sylow q-subgroup Q of G such that $G = PQ$.

Note that $F = \mathrm{Core}_G(P)$. Then G/F is a primitive group and P/F is a core-free maximal subgroup of G, which is complemented in G/F by a minimal normal subgroup of G/F (note that G is soluble by Burnside's theorem). Then $\mathrm{Soc}(G/F) = QF/F$. Since $Q \cong QF/F$, it follows that Q is an elementary abelian Sylow q-subgroup of G.

Therefore G is a p-soluble group with abelian Hall p'-subgroups and $\Omega(P) \leq \mathrm{Z}_{p-1}(P)$. By Corollary 1, G has p-length at most 1.

(d) *We have a contradiction.*

Since G has p-length at most 1 and $\mathrm{O}_{p'}(G) = 1$, then P is normal in G. This is to say that $G = \mathrm{N}_G(P)$. Then G is p-nilpotent by hypothesis. This final contradiction completes the proof.

(2) Suppose that $p = 2$ and $\Omega(P) \leq \mathrm{Z}(P)$. Since $\mathrm{N}_G(P)$ is 2-nilpotent, it follows that $\Omega(P \cap G')$ is contained in the centre of $\mathrm{N}_G(P)$. We can apply then [4, Theorem 1] to conclude that G is 2-nilpotent. Moreover, if $\Omega_1(P) \leq \mathrm{Z}(P)$ and P is quaternion-free, then G is 2-nilpotent by [4, Theorem 2]. □

4 Examples

The next two examples show that there exist groups in which the bounds of Theorem A are attained.

Example 3. The group of automorphisms of $Q \cong C_{11}$ has a subgroup isomorphic to $H = C_5$. Let $S = [Q]H$ be the corresponding semidirect product.

Let V be an irreducible and faithful module for S over the field of 5 elements. The dimension of V as a GF(5)-vector space is 5. Let $G = [V]S$ be the corresponding semidirect product.

The Sylow 5-subgroup of G is isomorphic to $[V]H$, which is isomorphic to the wreath product $C_5 \wr C_5$. The nilpotence class of P is exactly 5. Moreover, the maximal subgroups of G are isomorphic to S, to $[V]S$ or to $[V]Q$, all of them of 5-length one. Since the Frattini subgroup $\Phi(G)$ of G is trivial, the bound of Theorem A cannot be improved in general.

Example 4. Let $Q = \langle g_2, g_3, g_4, g_5, g_6 \mid g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6, g_6^2 = 1, [g_3, g_2] = [g_4, g_3] = [g_5, g_4] = [g_6, g_2] = [g_6, g_3] = [g_6, g_4] = [g_6, g_5] = 1, [g_4, g_2] = [g_5, g_2] = [g_5, g_3] = g_6 \rangle$ be an extraspecial group of order 32 which is the central product of a quaternion group $\langle g_2, g_4 \rangle$ and a dihedral group $\langle g_3, g_4g_5 \rangle$ of order 8. This group has an automorphism g_1 of order 5 given by $g_2^{g_1} = g_2g_3g_4g_5$, $g_3^{g_1} = g_2$, $g_4^{g_1} = g_3$, $g_5^{g_1} = g_4$, $g_6^{g_1} = g_6$. We can take the semidirect product $R = [Q]\langle g_1 \rangle$. Now consider the extraspecial group $E = \langle g_7, g_8, g_9, g_{10}, g_{11} \mid g_7^5 = g_8^5 = g_9^5 = g_{10}^5 = g_{11}^5 = 1, [g_7, g_8] = [g_9, g_{10}] = g_{11}, [g_7, g_9] = [g_7, g_{10}] = [g_7, g_{11}] = [g_8, g_9] = [g_8, g_{10}] = [g_8, g_{11}] = [g_9, g_{10}] = [g_9, g_{11}] = [g_{10}, g_{11}] = 1 \rangle$ of order 5^5 and exponent 5. The group R is a subgroup of automorphism group of E by means of the action given by $g_7^{g_1} = g_7^2g_8g_9^2g_{10}g_{11}^3$, $g_8^{g_1} = g_7g_8^2g_9g_{10}^2g_{11}^3$, $g_9^{g_1} = g_7^4g_8^3g_9g_{10}^2g_{11}^3$, $g_{10}^{g_{11}} = g_7^2g_8g_9^3g_{10}^4g_{11}^3$, $g_{11}^{g_1} = g_{11}$, $g_7^{g_2} = g_8^4$, $g_8^{g_2} = g_7$, $g_9^{g_2} = g_{10}$, $g_{10}^{g_2} = g_9^4$, $g_{11}^{g_2} = g_{11}$, $g_7^{g_3} = g_{10}^3$, $g_8^{g_3} = g_9^3$, $g_9^{g_3} = g_8^3$, $g_{10}^{g_3} = g_7^3$, $g_{11}^{g_3} = g_{11}$, $g_7^{g_4} = g_8^3$, $g_8^{g_4} = g_7^3$, $g_9^{g_4} = g_{10}^3$, $g_{10}^{g_4} = g_9^3$, $g_{11}^{g_4} = g_{11}$, $g_7^{g_5} = g_8^3$, $g_8^{g_5} = g_7^3$, $g_9^{g_5} = g_{10}^2$, $g_{10}^{g_5} = g_9^2$, $g_{11}^{g_5} = g_{11}$, $g_7^{g_6} = g_7^4$, $g_8^{g_6} = g_8^4$, $g_9^{g_6} = g_9^4$, $g_{10}^{g_6} = g_{10}^4$, $g_{11}^{g_6} = g_{11}$ (the details can be checked with GAP [6]). The corresponding semidirect product $G = [E]R$ is a group of order $2^5 \cdot 5^6 = 500{,}000$. This is a soluble group of 5-length 2 and every maximal subgroup of G is of 5-length 1. Its Sylow 5-subgroup $P = \langle g_1, g_7, g_8, g_9, g_{10}, g_{11} \rangle$ and its Frattini subgroup is $\Phi(G) = \langle g_{11} \rangle$. The nilpotency class of $P/\Phi(G)$ is exactly $4 = 5 - 1$. This shows that the bound of Theorem A cannot be improved for the Fermat prime $p = 5$.

The thesis of Corollary 2 (2) does not hold if $\Omega_1(P) \leq \mathrm{Z}(G)$ but G has sections isomorphic to the quaternion group Q_8 of order 8, as the following example shows.

Example 5. There are groups G with a Sylow 2-subgroup P such that $\Omega_1(P) \leq \mathrm{Z}(P)$ but G is not p-nilpotent. Let $G = \mathrm{SL}_2(3)$ be the special linear group of dimension 2 over GF(3). Then G is not 2-nilpotent. However, a Sylow 2-subgroup P of G is isomorphic to a quaternion group of order 8 and $\Omega_1(P) \leq \mathrm{Z}(P)$. Therefore the hypothesis of Corollary 2 (2) cannot be removed.

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