Generation of Bivariate Nakagami-m Fading Envelopes with Arbitrary not Necessary Identical Fading Parameters

J. Reig, M. A. Martinez-Amoraga and L. Rubio[1]

I. ABSTRACT

In this letter, a generation procedure of two correlated Nakagami-m random variables for arbitrary fading parameters values (not necessary identical) is described. For the generation of two correlated Nakagami-m samples, the proposed method uses the generalized Rice distribution which appears in the conditional distribution of two correlated Nakagami-m variables. This procedure can be applied to simulate diversity systems such as selection combiners, equal-gain combiners and maximal-ratio combiners as well as multiple-input multiple-output (MIMO) receiver systems, in Nakagami-m channels.

Keywords: fading, Nakagami-m channels, correlation, diversity systems.

[1]Dr. Reig, Mr. Martínez-Amoraga and Dr. Rubio are with the Departamento de Comunicaciones, Universidad Politécnica de Valencia, 46022 Valencia (Spain), Ph.: +34 96 387 97 62 Fax: +34 96 387 73 09, e-mail jreigp@dcom.upv.es.

II. INTRODUCTION

The effect of correlated Nakagami-m fading has been extensively studied on evaluating performances of diversity receivers in handheld phones or compact base stations.

The Nakagami-m distribution is characterized by two parameters, m and Ω; m is the fading parameter related to the depth of fading and Ω is the average power. In previous works, procedures for correlated Nakagami-m random variables generation have been described restricted to m identical [1]-[3] or m integer values [2], [3].

A method of generation of multivariate Nakagami-m variables with arbitrary covariance matrix was addressed by Q.T. Zhang in [1]. C. Tellambura and A.D.S. Jayalath in [2] have also investigated the generation of two correlated Nakagami-m variables using a modified inverse transform method for m integers. In [3], a multivariate Nakagami-m generation procedure is described restricted to identical integer m. However, note that in urban wireless environments, fading parameters reported in literature [4] oscillate between 1 and 2.5, thus there should not be any restriction to m values. Furthermore, fading parameters of each finger signal in Rake receivers could be different. Recently, in [5] K. Zhang *et al.* described a method of generating multivariate gamma variables using Cholesky decomposition of the covariance matrix, where fading parameters and correlation coefficients between variables are arbitrary. However, the marginal gamma distributions obtained in [5] are only mathematically exact if the ratio between the average power and fading parameter of each gamma distribution is identical.

In this letter, the generation of dual correlated Nakagami-m variables is addressed assuming arbitrary (not necessary identical) parameters: both fading and average power. Nevertheless, no Doppler effect is included in the procedure of generation, that is, a number of independent samples following a Nakagami-m distribution is generated correlated with another set of independent samples described by a Nakagami-m distribution. The distribution of the maximum of two correlated Nakagami-m variates at the output of a selection combiner (SC) is calculated. The cumulative distribution function (CDF) at the output of the SC calculated from the proposed method is compared to that obtained using [5] and the analytical CDF expression of [6].

III. PROCEDURE

Let r_1 and r_2 be Nakagami-m distributed where m_1, m_2, Ω_1 and Ω_2 are the fading parameters and the average powers of r_1 and r_2, respectively.

If r_1 and r_2 are dependent with correlation coefficient ρ in power terms defined as

$$\rho = \frac{\operatorname{cov}\left(r_1^2, r_2^2\right)}{\sqrt{\operatorname{var}\left(r_1^2\right) \cdot \operatorname{var}\left(r_2^2\right)}} \tag{1}$$

where cov(·,·) and var(·) denote covariance and variance, respectively, the joint distribution can be symbolized as

$$(r_1, r_2) \sim \mathcal{M}(m_1, \Omega_1; m_2, \Omega_2 | \rho) \tag{2}$$

Note that the correlation coefficient between amplitudes r_1 and r_2 can be numerically approximated by ρ [7, eq. (119)].

For identical fading parameters $m_1 = m_2 = m$, the procedure of generation r_1 and r_2 is summarized as follows: r_1 is generated as a square root of a gamma distribution applying a transformation. Once r_1 is generated, r_2 is obtained using the conditional distribution function $p(r_2 / r_1)$ derived from [7].

The Nakagami-m bivariate conditional probability can be expressed as

$$p(r_2 / r_1) = \frac{p(r_1, r_2)}{p(r_1)} \tag{3}$$

where $p(r_1)$ is the univariate Nakagami-m probability density function (PDF) given by

$$p(r_1) = \frac{2m^m r_1^{2m-1}}{\Gamma(m)\Omega_1^m} \exp\left(-\frac{m}{\Omega_1} r_1^2\right), \; r_1 \geq 0 \tag{4}$$

and $\Gamma(\cdot)$ is the gamma function. The conditional probability density function is obtained as

$$\begin{aligned} p(r_2 / r_1) = {} & \frac{2m r_2^m \Omega_1^{m-1}}{\Omega_2 (1-\rho)(\sqrt{\Omega_1 \Omega_2 \rho})^{m-1} r_1^{m-1}} \exp\left(-m \frac{\Omega_1 r_2^2 + \Omega_2 r_1^2 \rho}{\Omega_1 \Omega_2 (1-\rho)}\right) \\ & \times I_{m-1}\left(\frac{2m\sqrt{\rho} r_1 r_2}{\sqrt{\Omega_1 \Omega_2}(1-\rho)}\right) \end{aligned} \tag{5}$$

where $I_n(\cdot)$ is the modified Bessel function of order n.

Substituting $\sigma = \sqrt{\frac{\Omega_2(1-\rho)}{2m}}$ and $r_0 = r_1\sqrt{\frac{\Omega_2}{\Omega_1}\rho}$ into (5), it yields

$$p(r_2 / r_1) = \frac{r_2^m}{\sigma^2 r_0^{m-1}} \exp\left(-\frac{r_2^2 + r_0^2}{2\sigma^2}\right) I_{m-1}\left(\frac{r_2 r_0}{\sigma^2}\right) \tag{6}$$

which is the probability density function of n-generalized distribution or generalized Rice distribution $\mathcal{N}_m\,(r_0, \sigma)$ given by [7, eq.(66)].

Firstly, the method of generating two correlated Nakagami-m random variables for $m_1 = m_2 = m$ will be shown later on and then it will be extended for different values of m_1 and m_2.

A. Identical fading parameters

Step 1: Generate $s_1 \sim \mathcal{G}\,(m, 1)$

where $\mathcal{G}\,(\alpha, \beta)$ represents the gamma distribution, whose PDF is given by

$$p(s) = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} s^{\alpha-1} \exp\left(-\frac{s}{\beta}\right), \quad s \geq 0 \tag{7}$$

This gamma random variable is generated as a sum of $2m$ squared Gaussian variates for integer $2m$ values. For $m < 1$ values, it is generated as a product of beta and exponential random variables following Jonhk's algorithm [8]. For any other m values, a recursive method is used: a gamma distribution for the integer part is generated $s_{11} \sim \mathcal{G}\,(\lfloor m \rfloor, 1)$ and a gamma distribution for the decimal part is also generated $s_{12} \sim \mathcal{G}\,(m - \lfloor m \rfloor, 1)$. The sum of both random variables follows the desired gamma distribution $s_1 = s_{11} + s_{12} \sim \mathcal{G}\,(m, 1)$ [8].

Step 2: Obtain $r_1 = \sqrt{\frac{\Omega_1}{m} s_1}$.

From elemental transformation variable theory, one can derive that the PDF of the Nakagami-m distribution of r_1 is given by (4).

Step 3: Generate $r_2 \sim \mathcal{N}_m(r_0, \sigma)$, (8)

where r_2 is a conditional variable generated as a generalized Rice distribution whose r_0 parameter is related to r_1 as

$$r_0 = r_1 \sqrt{\frac{\Omega_2}{\Omega_1}\rho} \tag{9}$$

and σ is given by

$$\sigma = \sqrt{\frac{(1-\rho)}{2m}} . \tag{10}$$

Using the three properties of the n-distribution or generalized Rice distribution compiled in the appendix, the generation of $r_2 \sim \mathcal{N}_m(r_0, \sigma)$ in (8) is as follows:

3.a.- *m integer*

Using property 1 (16), m Rician variables x_i should be generated as

$$x_i \sim \mathcal{N}\left(\frac{r_0}{\sqrt{m}}, \sigma\right), \tag{11}$$

where $\mathcal{N}(C, \sigma)$ denotes the Rice distribution whose PDF is given by

$$p(x_i) = \frac{x_i}{\sigma^2} \exp\left(-\frac{x_i^2 + C^2}{2\sigma^2}\right) I_0\left(\frac{Cx_i}{\sigma^2}\right), \quad x_i \geq 0 \tag{12}$$

If $r_{22} = \sqrt{\sum_{i=1}^{m} x_i^2}$, then $r_{22} \sim \mathcal{N}_m\left(r_0,\sigma\right)$.

3.b.- *m non-integer*

r_2 is generated using property 3 (19) as the sum of two generalized Rice distributions

$$r_2 = \sqrt{r_{21}^2 + r_{22}^2}\ , \tag{13}$$

where $r_{21} \sim \mathcal{N}_{m_a}\left(0,\sigma\right)$, $r_{22} \sim \mathcal{N}_{m_b}\left(r_0,\sigma\right)$, $m_a = m - \lfloor m \rfloor$ is the decimal part and $m_b = \lfloor m \rfloor$ the integer part of m. For r_{21} generation, property 2 of appendix (17) is used

$$r_{21} \sim \mathcal{M}\left(m_a, \sigma m_a\right). \tag{14}$$

For r_{22} generation, we can use step 3.a for m integer (11). In spite of the fact that the Nakagami-m distribution is not defined for fading parameters lower than 0.5, as Nakagami-m distribution is generated as the root square of the gamma function, and the gamma function is defined for $m_a > 0$ in [8], the distribution obtained for $m_a < 0.5$ is consistent. Figure 1 shows the flowchart corresponding to the generation of two Nakagami-m variables with the same fading parameter m, average powers Ω_1 and Ω_2, respectively, and correlated ρ.

Therefore, the distribution obtained of r_1 and r_2 follows the bivariate Nakagami–m distribution given by (2).

B. Non-identical fading parameters

If m_1 is not equal to m_2 the procedure is similar, but with some differences

Step 1: Generate $(r_1', r_{21}) \sim \mathcal{M}\left(m_1, m_1\Omega_1; m_1, m_1\Omega_2 \mid \rho\sqrt{\frac{m_2}{m_1}}\right)$ following A method, where $m_1 < m_2$.

Step 2: Generate $r_{22} \sim \mathcal{M}\left(m_2 - m_1, (m_2 - m_1)\Omega_2\right)$.

Step 3: Calculate $r_2' = \sqrt{r_{21}^2 + r_{22}^2}$.

Step 4: Obtain $r_1 = \frac{r_1'}{\sqrt{m_1}}$ and $r_2 = \frac{r_2'}{\sqrt{m_2}}$.

This procedure of obtaining r_2' is based on the property of the sum of independent gamma variates, i.e. if $s_1,\ldots, s_n$ are gamma distributed $s_i \sim \mathcal{G}(\alpha_i, \beta)$, $i = 1,.., n$, then the sum $s = \sum_{i=1}^{n} s_i$ is gamma distributed $s \sim \mathcal{G}\left(\sqrt{\sum_{i=1}^{n}\alpha_i}, \beta\right)$ [7, eq.(78)], [8]. Since $r_i = \sqrt{s_i}$ is Nakagami-m distributed $r_i \sim \mathcal{M}(\alpha_i, \beta\alpha_i), i = 1,..,n$, r defined as $r = \sqrt{s}$ follows a Nakagami-m distribution $r \sim \mathcal{M}\left(\sum_{i=1}^{n}\alpha_i, \beta\sum_{i=1}^{n}\alpha_i\right)$. Therefore, r_2' obtained in step 3 of sub-section B is Nakagami-m distributed with fading parameter m_2 and average power $m_2\Omega_2$. Applying the definition of the correlation coefficient given by (1), we can demonstrate that the correlation coefficient obtained between r_1^2 and r_2^2 is ρ [9].

Some restrictions for this method are to be exposed. Firstly, m_2 - m_1 should be theoretically higher than 0.5 for r_{22} generation. Since r_{22} is generated from a gamma distribution, it can be verified that the distribution obtained is Nakagami-m for m_2 - $m_1 > 0$ as commented in case A. On the other hand, m_2 should be the higher fading parameter of both distributions. Finally, the correlation coefficient $\rho\sqrt{\frac{m_2}{m_1}}$ in the generation of $r_1{}'$ and r_{21} could be higher than 1 as $m_2 > m_1$. Therefore, this method is limited to $0 \le \rho \le \sqrt{\frac{m_1}{m_2}}$ [9].

IV. VERIFICATION

Figure 2 shows a comparison between the analytical Nakagami-m PDF and the Nakagami-m PDF generated with the method exposed in this letter for $\rho = 0.3$. For this aim, $m_1 = 1.2$, $m_2 = 1.5$ and $\Omega_1 = \Omega_2 = 1$ are used. The difference between both theoretical and experimental distributions is minimal.

The outage probability for dual correlated Nakagami-m SC with arbitrary fading parameters has been evaluated using the described procedure and compared with the analytical result.

Assuming identical channels, the probability of outage for dual selection combiners can be written as [6]

$$P_{out}(q) = \frac{(1-\alpha)^{m_2}}{\Gamma(m_1)} \sum_{k=0}^{\infty} \frac{\gamma\left(m_1 + k, \frac{m_1 q}{\overline{SNR}(1-\alpha)}\right)}{k!} \sum_{l=0}^{\infty} \frac{\Gamma(m_2 - m_1 + l)}{\Gamma(m_2 - m_1)\Gamma(m_2 + k + l) l!} \alpha^{l}, \quad \gamma\left(m_2 + k + l, \frac{m_2 q}{\overline{SNR}(1-\alpha)}\right), \quad 0 \le \alpha < 1 \tag{15}$$

where $\overline{SNR}$ is the average signal-to-noise ratio at both inputs of the combiner, $\gamma(\cdot)$ is the incomplete gamma function, $\alpha = \rho\sqrt{\frac{m_2}{m_1}}$ and q is the protection ratio of the instantaneous

SNR at the output of the combiner related to the required threshold probability of error for an specified modulation [10].

Outage probabilities for non-coherent frequency shift keying, NCFSK, with dual SC are drawn in figure 3 by using analytical expression (15) and Monte Carlo method proposed in this letter for $\rho = 0$, 0.3 and 0.7. One million of samples were generated for each Nakagami-m marginal distribution following both the method proposed in this letter and the Cholesky decomposition procedure described by K. Zhang *et al.* in [5]. The differences between the results obtained by the simulation proposed and the analytical expression are minimal. Nevertheless, a substantial deviation between the analytical result and the Cholesky method is found for $\rho = 0.3$ and 0.7 due to the approximation for the sum of independent gamma variates with arbitrary parameters as a gamma distribution in [5, eq.(7)]. For instance, the deviation for $\rho = 0.3$ and outage probability of 10^{-3} is around 2.3 dB in the curves of figure 3.

V. CONCLUSIONS

A method for the generation of two Nakagami-m correlated variates with arbitrary fading parameters and average powers is proposed in this letter. The simulated distributions based on the generation of a generalized Rice distribution as a conditional variable are equivalent to the analytical distributions. This generation procedure can be used for simulating diversity systems to obtain the performance with the accuracy due to the Monte Carlo techniques (number of samples and accurateness of basis distributions: Gaussian and uniform generation)

APPENDIX

The generation of the generalized form of the n-distribution or generalized Rice distribution $\mathcal{N}_m\ (r_0,\ \sigma)$ can be achieved by using the properties derived in [7, eqs. (66), (70) and (72)].

Properties of generalized Rice distribution

1) Let be $x_i \sim \mathcal{N}\ (r_{0i},\ \sigma)$ Rician variables for i=1,2,...n,. If $x = \sqrt{\sum_{i=1}^{n} x_i^2}$, then x is distributed as

$$x \sim \mathcal{N}_n\left(\sqrt{\sum_{i=1}^{n} r_{0i}^2}, \sigma\right) . \tag{16}$$

For the generation of a Rician variable $x_i \sim \mathcal{N}\ (r_{0i},\ \sigma)$, let x_{i1} and x_{i2} be Gaussian variables with zero mean and standard deviation σ. If $x_i = \sqrt{\left(x_{i1} + r_{oi}\right)^2 + x_{i2}^2}$,then x_i is Rician-distributed.

2) If r_0 tends to zero in (8), then the generalized Rice distribution follows a Nakagami distribution

$$x_i \sim \mathcal{N}_{\substack{m \\ r_0 \to 0}}(r_0, \sigma) \approx \mathcal{M}(m, m\sigma) \ , \tag{17}$$

where m and $m\sigma$ are the fading parameter and the average power of the Nakagami-m distribution, respectively.

3) Let be $x_i \sim \mathcal{N}_{m_i}(r_{0_i}, \sigma)$ generalized Rice distributed. If $x = \sqrt{\sum_{i=1}^{n} x_i^2}$ then

$$x \sim \mathcal{N}_m(r_0, \sigma), \tag{18}$$

where

$$m = \sum_{i=1}^{n} m_i, \quad r_0 = \sqrt{\sum_{i=1}^{n} r_{0i}^2}\ . \tag{19}$$

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers and the Editor for their helpful comments which considerably improved the quality of this letter.

REFERENCES

[1] Zhang QT. Efficient generation of correlated Nakagami fading channels with arbitrary fading parameter. Proc. *IEEE International Conference on Communications* 2002; **3**: 1358-1362. DOI: 10.1109/icc.2002.997072.

[2] Tellambura C, Jayalath ADS. Generation of bivariate Rayleigh and Nakagami-m fading envelopes. *IEEE Communications Letters* 2000; **4**(5): 170-172. DOI: 10.1109/4234.846501.

[3] Luo J, Zeidler JR. A statistical simulation model for correlated Nakagami fading channels. Proc. *International Conference on Communication Technology* 2000; **2**: 1680-168. DOI: 10.1109/icct.2000.890981.

[4] Rubio L, Cardona N, Juan-Llacer L, Reig J. A semi-deterministic propagation model based on a digital map information for the prediction of the coverage area and fading statistics in urban environments. Proc. *Wireless Personal Multimedia Communications* 1998; 380-384.

[5] Zhang K, Song Z, Guan YL. Simulation of Nakagami fading channels with arbitrary cross-correlation and fading parameters. *IEEE Transactions on Wireless Communications* 2004; **3**(5): 1462-1468. DOI: 10.1109/twc.2004.833469.

[6] Reig J, Rubio L, Cardona N. Outage probabilities of dual selection combiners in a correlated Nakagami fading with arbitrary fading parameters. Proc. *IEEE Vehicular Technology Conference Spring* 2004; **1**: 367-370.

[7] Nakagami M. The *m*-distribution: A general formula of intensity distribution of rapid fading. *Statistical Methods in Radio Wave Propagation*. W. G. Hoffman, Oxford, England: Pergamon, 1960.

[8] Debroye L. *Non-Uniform Random Variate Generation*. Springer Verlag: New York, 1986.

[9] Reig J, Rubio L, Cardona N. Bivariate Nakagami-*m* distribution with arbitrary fading parameters. *Electronics Letters* 2002; **38**(25): 1715-1717. DOI: 10.1049/el:20021124.

[10] Schwartz M, Bennet WR, Stein S. *Communication Systems and Techniques.* McGraw-Hill: New York, 1966.

LIST OF CAPTIONS

Figure 1. Flowchart of generation of two Nakagami-m variates, r_1 and r_2 with the same fading parameter m, average powers Ω_1 and Ω_2, respectively and correlated ρ.

Figure 2. Univariate probability density functions generated by Monte Carlo simulation from the bivariate Nakagami-m distribution for $m_1 = 1.2$, $m_2 = 1.5$, $\Omega_1 = \Omega_2 = 1$ and $\rho = 0.3$.

Figure 3. Outage probability in a dual selection combiner for NCFSK for $m_1 = 1.2$, $m_2 = 1.5$ and threshold probability of error of 10^{-3}.

Figure 1

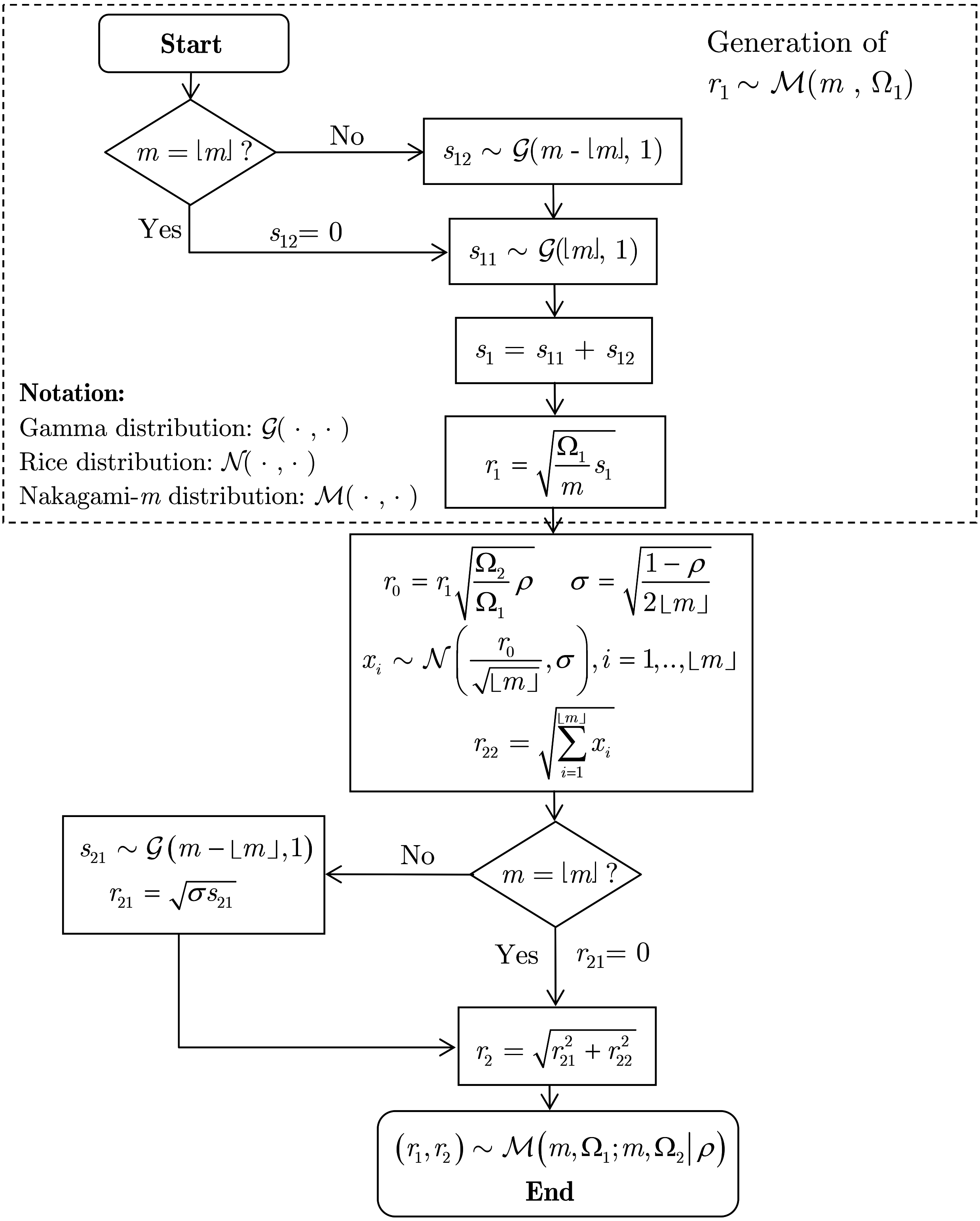

Start
Generation of
$r_1 \sim \mathcal{M}(m\ ,\ \Omega_1)$
$m = \lfloor m \rfloor$?
No
$s_{12} \sim \mathcal{G}(m - \lfloor m \rfloor,\ 1)$
Yes
$s_{12} = 0$
$s_{11} \sim \mathcal{G}(\lfloor m \rfloor,\ 1)$
$s_1 = s_{11} + s_{12}$
Notation:
Gamma distribution: $\mathcal{G}(\ \cdot\ ,\ \cdot\ )$
Rice distribution: $\mathcal{N}(\ \cdot\ ,\ \cdot\ )$
Nakagami-m distribution: $\mathcal{M}(\ \cdot\ ,\ \cdot\ )$
$r_1 = \sqrt{\frac{\Omega_1}{m} s_1}$
$r_0 = r_1\sqrt{\frac{\Omega_2}{\Omega_1}\rho}$ $\sigma = \sqrt{\frac{1-\rho}{2\lfloor m \rfloor}}$
$x_i \sim \mathcal{N}\left(\frac{r_0}{\sqrt{\lfloor m \rfloor}}, \sigma\right), i = 1,..,\lfloor m \rfloor$
$r_{22} = \sqrt{\sum_{i=1}^{\lfloor m \rfloor} x_i}$
$s_{21} \sim \mathcal{G}(m - \lfloor m \rfloor, 1)$
$r_{21} = \sqrt{\sigma s_{21}}$
No
$m = \lfloor m \rfloor$?
Yes
$r_{21} = 0$
$r_2 = \sqrt{r_{21}^2 + r_{22}^2}$
$(r_1, r_2) \sim \mathcal{M}(m, \Omega_1; m, \Omega_2 | \rho)$
End

Figure 2

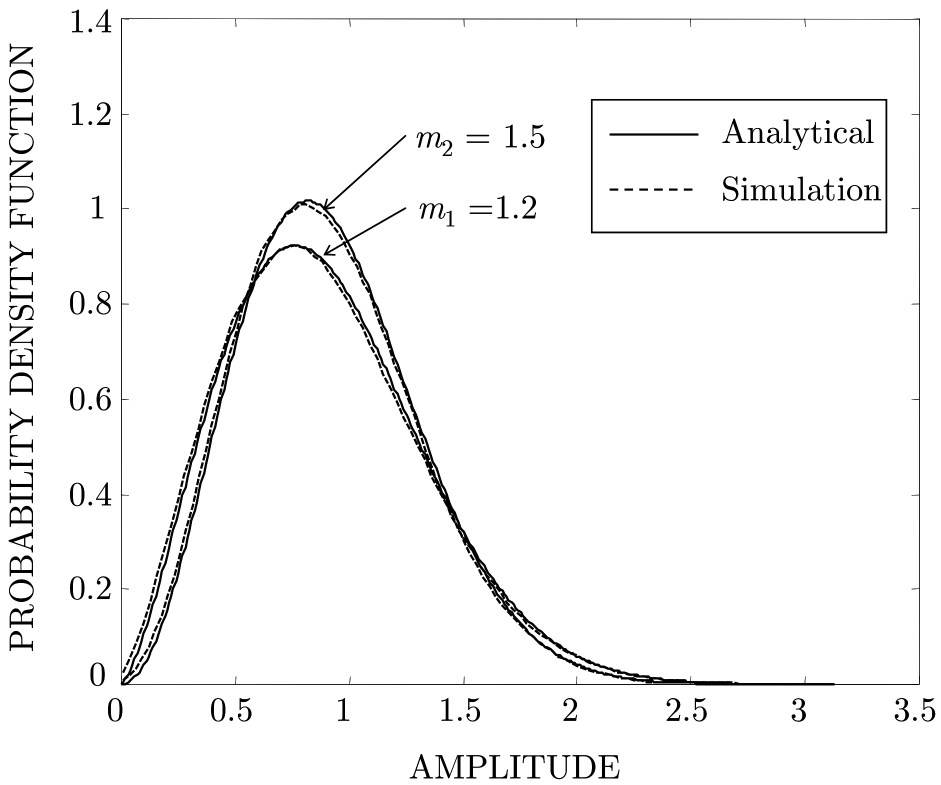
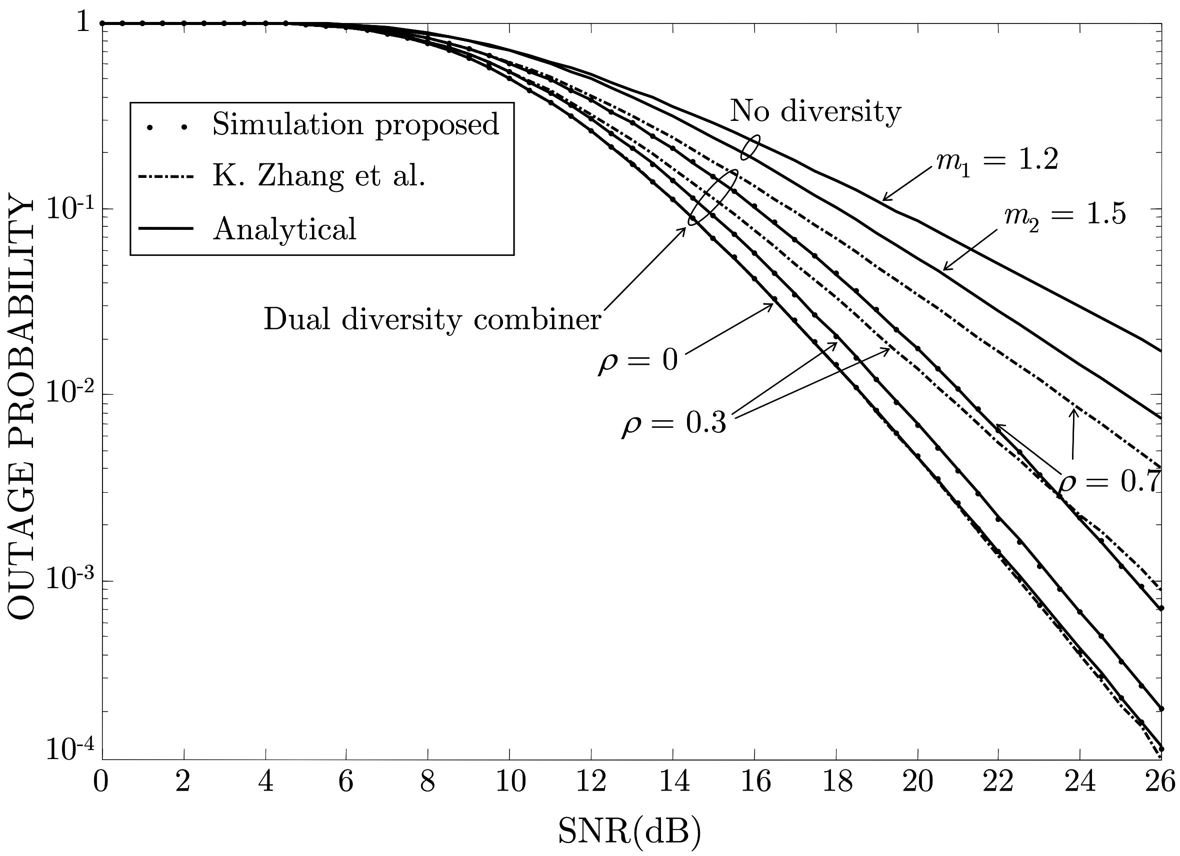

1.4
1.2
1
0.8
0.6
0.4
0.2
0
PROBABILITY DENSITY FUNCTION
0
0.5
1
1.5
2
2.5
3
3.5
AMPLITUDE
$m_2 = 1.5$
m_1 =1.2
Analytical
Simulation

Figure 3



Simulation proposed
K. Zhang et al.
Analytical
No diversity
$m_1 = 1.2$
$m_2 = 1.5$
Dual diversity combiner
$\rho = 0$
$\rho = 0.3$
$\rho = 0.7$
OUTAGE PROBABILITY
SNR(dB)