Cordero Barbero, A.; Hueso Pagoaga, JL.; Martínez Molada, E.; Torregrosa Sánchez, JR. (2011). Efficient high-order methods based on golden ratio for nonlinear system. Applied Mathematics and Computation. 217(9):4548-4556. doi:10.1016/j.amc.2010.11.006.

Efficient high-order methods based on golden ratio for nonlinear systems ⋆

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Abstract

We present new iterative methods of order of convergence four or higher, for solving nonlinear systems, by composing iteratively golden ratio methods with a modified Newton's method. In addition, we define a new efficiency index involving the computational effort as well as the functional evaluations per iteration. We use this new index, joint with the usual efficiency index, in order to compare the new methods with other known methods and present several numerical tests.

Key words: Nonlinear systems, Newton's method, fixed point iteration, convergence order, computational efficiency index

1 Introduction

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a real solution $\bar{x}$ of the nonlinear system $F(x) = 0$, with n equations and n unknowns. This solution can be obtained as a fixed point

⋆ This research was supported by Ministerio de Ciencia y Tecnología MTM2007-64477

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of some function G by means of the fixed point iteration method. The best known fixed point method is the classical Newton's method (N), given by

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \quad k = 0, 1, \ldots,$$

where $x^{(0)}$ is the initial estimation and $F'(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration. It is known that this method has order of convergence two under certain conditions.

The construction of iterative methods for the approximation of the solution of $F(x) = 0$ is an interesting task in numerical analysis and applied scientific branches. During the last years, numerous papers devoted to iterative methods for solving nonlinear systems have appeared in several journals. Some methods existing in the literature are based on the use of interpolation quadrature formulas (see [1–4]), or include the second partial derivative of the function F or different estimations of it (see [5–8]), or are Steffensen's type methods (see [9]), etc. We also pay attention to the known Jarratt's method (J) (see [10]) whose efficiency is widely recognized. As the order of convergence of this method is four we will compare it with the new methods in the numerical section.

On the other hand, a known acceleration technique consists of the composition of two iterative methods of orders p_1 and p_2, respectively, to obtain a method of order $p_1 p_2$ ([11]). Usually, new evaluations of the Jacobian matrix and the nonlinear function are needed in order to increase the order of convergence. However, the existence of an extensive literature on higher order methods reveals that they are only limited by the nature of the problem to be solved: in particular, the numerical solution of quadratic equations and nonlinear integral equations are needed in the study of dynamical models of chemical reactors [12], or in radioactive transfer [13]. Moreover, many of these numerical applications use high precision in their computations; the results of these numerical experiments show that the high order methods associated with a multiprecision arithmetic floating point are very useful, because it yields a clear reduction in iterations.

Nevertheless, some modifications on Newton's method can be made in order to limit the number of functional evaluations and increase the convergence order (see for example [14,15]).

In [16], the authors presented the following family of multi-point iterative methods for nonlinear systems

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} \left(\sum_{i=1}^{m} A_i F(\eta_i(x^{(k)})) \right),$$

with $\eta_i(x^{(k)}) = x^{(k)} - \tau_i [F'(x^{(k)})]^{-1} F(x^{(k)})$, where τ_i and A_i are parameters

to be chosen in $\mathbb{R}$ and m is a positive integer. The value of these parameters plays an important role in the order of convergence of the method.

In this paper, we will work with two elements of this family whose iterative expressions are

$$\begin{aligned} y^{(k)} &= x^{(k)} - \tau[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= x^{(k)} - A[F'(x^{(k)})]^{-1}F(y^{(k)}), \end{aligned} \tag{1}$$

where $\tau = \dfrac{-1 \pm \sqrt{5}}{2}$ and $A = \dfrac{3 \pm \sqrt{5}}{2}$.

In the following, we remember some known notions and results that we need in order to analyze the convergence of the new methods.

Definition 1.1 *Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$ convergent to $\bar{x}$. Then, convergence is called*

(a) linear, if there exists M, $0 < M < 1$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|, \ \forall k \geq k_0.$$

(b) of order p, $p > 1$, if there exists M, $M > 0$, and k_0 such that

$$\left\|x^{(k+1)} - \bar{x}\right\| \leq M \left\|x^{(k)} - \bar{x}\right\|^p, \ \forall k \geq k_0.$$

Weerakoon and Fernando introduced in [17] the concept of *Computational Order of Convergence (COC)*. One of the main drawback of the COC is that it involves the exact root, which in a real situation it is not known a priori. To avoid this, the authors introduced in [1] the concept of *Approximated Computational Order of Convergence* as follows:

Definition 1.2 *Let $\bar{x}$ be a zero of function F and suppose that $x^{(k-1)}$, $x^{(k)}$ and $x^{(k+1)}$ are three consecutive iterations close to $\bar{x}$. Then, the computational order of convergence p can be approximated using the formula*

$$p \approx \rho = \frac{\ln\left(\left\|x^{(k+1)} - x^{(k)}\right\| / \left\|x^{(k)} - x^{(k-1)}\right\|\right)}{\ln\left(\left\|x^{(k)} - x^{(k-1)}\right\| / \left\|x^{(k-1)} - x^{(k-2)}\right\|\right)}. \tag{2}$$

We call this number the approximated computational order of convergence (ACOC).

In order to compare different methods, we use the efficiency index, $I = p^{1/d}$ (see [18]), where p is the order of convergence and d is the number of functional evaluations per iteration required by the method. This is the most used index. However, in the n-dimensional case, it is also important to take into account the number of operations performed, since for each iteration a number of linear

systems must be solved. We recall that the number of products/quotiens that we need for solving m linear systems with the same matrix, by using LU factorization, is

$$\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n,$$

where n is the size of the linear systems.

For this reason we define the *Computational Efficiency Index* as

$$I_C = p^{1/(d+op)}, \tag{3}$$

where *op* is the number of products/quotients per iteration. We use this new index to compare the different methods.

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable in D. The qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \ldots, v_q) \in \mathbb{R}^n$. It is easy to observe that

1. $F^{(q)}(u)(v_1, \ldots, v_{q-1}, \cdot) \in \mathcal{L}(\mathbb{R}^n)$,
2. $F^{(q)}(u)(v_{\sigma(1)}, \ldots, v_{\sigma(q)}) = F^{(q)}(u)(v_1, \ldots, v_q)$, for all permutation σ of $\{1, 2, \ldots, q\}$.

From the above properties we can use the following notation:

(a) $F^{(q)}(u)(v_1, \ldots, v_q) = F^{(q)}(u)v_1 \ldots v_q$,
(b) $F^{(q)}(u)v^{q-1}F^{(p)}v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}$.

On the other hand, for $\bar{x} + h \in \mathbb{R}^n$ lying in a neighborhood of a solution $\bar{x}$ of $F(x) = 0$, we can apply Taylor's expansion and assuming that the Jacobian matrix $F'(\bar{x})$ is nonsingular, we have

$$F(\bar{x} + h) = F'(\bar{x}) \left[h + \sum_{q=2}^{p-1} C_q h^q \right] + O(h^p), \tag{4}$$

where $C_q = (1/q!)[F'(\bar{x})]^{-1} F^{(q)}(\bar{x})$, $q \geq 2$. We observe that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\bar{x}) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\bar{x})]^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

In addition, we can express F' as

$$F'(\bar{x} + h) = F'(\bar{x}) \left[I + \sum_{q=2}^{p-1} q C_q h^{q-1} \right] + O(h^p), \tag{5}$$

where I is the identity matrix. Therefore, $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. From (5), we obtain

$$[F'(\bar{x} + h)]^{-1} = \left[I + X_2 h + X_3 h^2 + X_4 h^3 + \cdots \right] [F'(\bar{x})]^{-1} + O(h^p), \tag{6}$$

where

$$\begin{aligned} X_2 &= -2C_2, \\ X_3 &= 4C_2^2 - 3C_3, \\ X_4 &= -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4, \\ &\vdots \end{aligned}$$

We denote $e^{(k)} = x^{(k)} - \bar{x}$ the error in the kth iteration. The equation

$$e^{(k+1)} = Le^{(k)^p} + O(e^{(k)^{p+1}}),$$

where L is a p-linear function $L \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$, is called the *error equation* and p is the *order of convergence*. Observe that $e^{(k)^p}$ is $(e^{(k)}, e^{(k)}, \cdots, e^{(k)})$.

The rest of this paper is organized as follows: Section 2 describes the new three-step iterative methods of order 4 obtained by composing the golden ratio methods of order 3, described by (1), with modified Newton's method. These new methods show to be very efficient, as they improve Newton's method even taking into account the number of operations included in each iteration. This is a key fact, as Newton-type methods usually need a great amount of operations in each iteration, which is important in systems of nonlinear equations.

Subsequently, we analyze the following question: how far can this composition be made in order to optimize the order of convergence and the computational effort? This is showed to be related directly with the size of the system, but the results are advantageous to high-order methods.

The last section is devoted to numerical results obtained by applying some of the obtained methods to several systems of nonlinear equations. From these results, we compare different methods, confirming the theoretical results.

2 Description of the methods and convergence analysis

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently differentiable function and let $\bar{x}$ be a zero of the nonlinear system $F(x) = 0$.

Let us consider a fixed point function $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that the iterative method $x^{(k+1)} = M(x^{(k)})$ converges to the solution $\bar{x}$ of $F(x) = 0$, with convergence order p. We define a new iteration function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by applying a modified Newton's iteration function to $M(x)$:

$$G(x) = M(x) - [F'(x)]^{-1}F(M(x)). \tag{7}$$

Then, the new iterative process is:

$$x^{(k+1)} = M(x^{(k)}) - [F'(x^{(k)})]^{-1} F(M(x^{(k)})), \tag{8}$$

which will be proved to have order of convergence $p+1$. This result, for the scalar case, is proved in [11]. So, an extension is made to the multidimensional case.

Theorem 2.1 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{x} \in \mathbb{R}^n$ that is a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular in $\bar{x}$. Let us also suppose that the method described by a fixed point function $M(x)$ converges to $\bar{x}$, with convergence order p. Then, the sequence obtained by the iterative process (8) converges to $\bar{x}$ with order $p+1$.*

Proof: From (4) and (5) we obtain

$$F(x^{(k)}) = F'(\bar{x}) \left[e^{(k)} + C_2 {e^{(k)}}^2 + C_3 {e^{(k)}}^3 + C_4 {e^{(k)}}^4 \right] + O({e^{(k)}}^5)$$

and

$$F'(x^{(k)}) = F'(\bar{x}) \left[I + 2C_2 e^{(k)} + 3C_3 {e^{(k)}}^2 + 4C_4 {e^{(k)}}^3 + 5C_5 {e^{(k)}}^4 \right] + O({e^{(k)}}^5),$$

where $C_k = (1/k!)[F'(\bar{x})]^{-1} F^{(k)}(\bar{x})$, $k = 2, 3, \ldots$, and $e^{(k)} = x^{(k)} - \bar{x}$.

From the above expression, we have

$$[F'(x^{(k)})]^{-1} = \left[I + X_2 e^{(k)} + X_3 {e^{(k)}}^2 + X_4 {e^{(k)}}^3 \right] [F'(\bar{x})]^{-1} + O({e^{(k)}}^4),$$

where $X_2 = -2C_2$, $X_3 = 4C_2^2 - 3C_3$ and $X_4 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4$.

On the other hand, by taking into account that $M(x)$ describes an iterative method of order p, we have

$$M(x^{(k)}) - \bar{x} = O({e^{(k)}}^p),$$

so

$$\begin{aligned} F(M(x^{(k)})) &= F(\bar{x}) + F'(\bar{x}) \left(M(x^{(k)}) - \bar{x} \right) + F''(\bar{x}) \left(M(x^{(k)}) - \bar{x} \right)^2 + \cdots \\ &= F'(\bar{x}) \left(M(x^{(k)}) - \bar{x} \right) + O\left({e^{(k)}}^{2p} \right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&[F'(x^{(k)})]^{-1}F(M(x^{(k)})) = \\
&= \left[I + X_2 e^{(k)} + X_3 {e^{(k)}}^2 + X_4 {e^{(k)}}^3\right][F'(\bar{x})]^{-1}\left[F'(\bar{x})\left(M(x^{(k)}) - \bar{x}\right) + O\left({e^{(k)}}^{2p}\right)\right] \\
&= M(x^{(k)}) - \bar{x} + X_2\left(M(x^{(k)}) - \bar{x}\right)e^{(k)} + O\left({e^{(k)}}^{p+2}\right).
\end{aligned}
$$

Then,

$$
x^{(k+1)} - \bar{x} = M(x^{(k)}) - \bar{x} - [F'(x^{(k)})]^{-1}F(M(x^{(k)})) = O\left({e^{(k)}}^{p+1}\right).
$$

□

As we have mentioned in the introduction, we consider two third-order methods based on golden ratio, whose associated fixed point functions are:

$$
M_i(x) = x - A_i[F'(x)]^{-1}F(\eta_i(x)), \tag{9}
$$

for $i = 1, 2$, where $A_1 = \frac{3+\sqrt{5}}{2}$, $A_2 = \frac{3-\sqrt{5}}{2}$ and

$$
\eta_i(x) = x - \tau_i[F'(x)]^{-1}F(x),
$$

being $\tau_1 = \frac{1}{\varphi}$, $\tau_2 = -\varphi$ and the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. These methods, that we will denote by G_1 and G_2, have convergence order three (see [16]) and their classical efficiency index is:

$$
I_G = 3^{\frac{1}{n^2+2n}},
$$

which is better than the one of Newton's method for all $n \geq 1$.

Moreover, the computational index of the golden ratio methods is

$$
(I_C)_G = 3^{\frac{1}{\frac{1}{3}n^3+3n^2+\frac{5}{3}n}}.
$$

When these methods are composed with the modified Newton's method of "frozen" Jacobian matrix, the resulting method (that we will denote by NG) have the following iterative expression:

$$
\begin{aligned}
z^{(k)} &= x^{(k)} - \frac{-1+\sqrt{5}}{2}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\
y(k) &= x^{(k)} - \frac{3+\sqrt{5}}{2}[F'(x^{(k)})]^{-1}F(z^{(k)}), \\
x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1}F(y^{(k)}),
\end{aligned} \tag{10}
$$

in the particular case of A_1 and τ_1. Let us note that, in this new method, only the Jacobian matrix at the iteration $x^{(k)}$ is evaluated. This is a key fact to the efficiency of the resulting method.

Certainly, the new linear system to be solved at the third step of each iteration holds the same matrix, the "frozen" Jacobian matrix $F'(x^{(k)})$. This improves the computational efficiency of the method, as only n^2 operations and n functional evaluations are added at each iteration. So, the corresponding index of NG is:

$$(I_C)_{NG} = 4^{\frac{1}{\frac{1}{3}n^3+4n^2+\frac{8}{3}n}}.$$

The relation between the efficiency indices of Newton's, Golden and NG methods is as follows: for $n > 1$,

$$I_{NG} > I_G > I_N$$

and respect to the computational efficiency index, for $5 > n \geq 2$,

$$(I_C)_{NG} > (I_C)_N$$

and

$$(I_C)_G > (I_C)_N.$$

Nevertheless, for $n \geq 5$,

$$(I_C)_{NG} > (I_C)_G > (I_C)_N.$$

It can be observed that the new method NG is better than the source method G only when the size of the system is greater than or equal to five.

Now, a question is stated: how many iterated compositions of the modified Newton's method have to be made in order to obtain an optimal relation between the order of convergence and the efficiency indices? Let us note that the computational and classical efficiency indices of the $(p-3)$-times iterated Newton-Golden method (NG_p), can be expressed only in terms of the order of convergence p and the size of the system:

$$I_{NG_p} = p^{\frac{1}{n^2+(p-1)n}}$$

and

$$(I_C)_{NG_p} = p^{\frac{1}{\frac{1}{3}n^3+pn^2+\left(p-\frac{4}{3}\right)n}}.$$

The optimal relation for each index is shown in Table 1, in terms of the size of the system. Note that the method with optimal order of convergence, in relation with the computational efficiency index, have needed less iterations than the respective one for the classical efficiency index.

Arrived at this point, it is useful to analyze this procedure from another point of view: does the total number of operations (or functional evaluations) actually decrease when the optimal order of convergence is close? Does it works for big-sized systems? The answer to these questions is found in Figure 1, when

n	optimal p for I	optimal p for I_C
3	4	3
4	5	3
5	6	4
10	8	5
20	12	7
30	16	8
40	20	10
50	23	11
100	37	18
500	129	55
1000	226	94

Table 1
Optimal iteration-method for different values of the size of the system n.

the size of the nonlinear system is $n = 99$. In order to generate this figure, we estimate the zeros of the nonlinear function (that will be studied in more detail in Section 3) $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$f_i(x) = x_i x_{i+1} - 1,\ i = 1, 2, \ldots, n-1$$
$$f_n(x) = x_n x_1 - 1.$$

We use the iterated methods NG_p: for each execution of NG_p, with increasing values of p, we calculate the total number of operations and functional evaluations (by the simple product of the number of iterations needed in each case and the number of operations and functional evaluations per iteration). It can be observed that when the order of the method is near of its optimal value, the total number of operations and functional evaluations is much lower than in previous executions, as the number of iterations needed to solve the nonlinear system has also decreased.

3 Numerical results

In this section we will check the effectiveness of some numerical methods in order to estimate the zeros of several nonlinear functions, some of them obtained by applying a finite differences scheme to boundary-value problems.

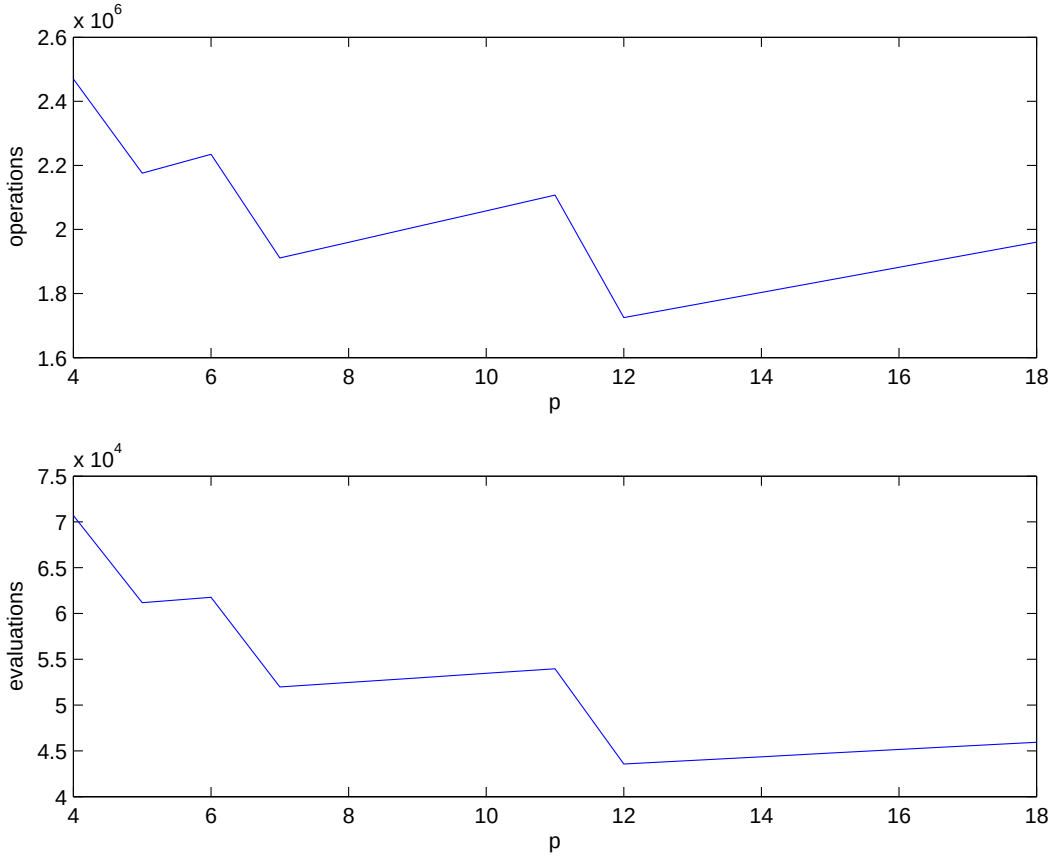

Fig. 1. Total number of operations and functional evaluations for NG_p-methods.

Numerical computations have been carried out in MATLAB, with variable precision arithmetic that uses floating point representation of 200 decimal digits of mantissa. Every iterate $x^{(k+1)}$ is obtained from the previous one, $x^{(k)}$, by adding one or more terms of the form $A^{-1}b$ where $x^{(k)} \in \mathbb{R}^n$, A is a real $n \times n$ matrix and $b \in \mathbb{R}^n$. The matrix A and the vector b are different according to the method used, but in any case the inverse calculation $-A^{-1}b$ is carried out solving the linear system $Ay = -b$, using Gaussian elimination with partial pivoting.

The stopping criterion used is $\left\|x^{(k+1)} - x^{(k)}\right\| + \left\|F\left(x^{(k)}\right)\right\| < 10^{-100}$. Therefore, we check that iterates converge to a limit and moreover that this limit is a solution of the system of nonlinear equations.

Tables 2, 3 and 4 show several results obtained by using the previously described methods and the Newton's and Jarratt's methods, in order to estimate the zeros of functions. Given an initial estimation $x^{(0)}$, we analyze, for every method, the number of iterations needed to converge to the solution, the approximated computational order of convergence, ρ, defined in (2), and the total number of operations (products/quotients) (TO) and functional evaluations (TFE).

Method	**Iterations**	ρ	**TO**	**TFE**
N	6	2.00	1999206	59400
J	4	3.83	2783616	78804
NG_4	6	3.75	2469621	70686
NG_8	4	7.65	1568028	41976
NG_{18}	3	17.40	1470051	34452

Table 2
Numerical results for the function (11).

The value of ρ that appears in Tables 2, 3 and 4 is the last coordinate of vector ρ when the variation between its coordinates is small. When this does not happen, the value of ρ is said to be not conclusive, and is denoted by "-" in the mentioned tables.

Firstly, let us consider the nonlinear function $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$\begin{aligned} f_i(x) &= x_i x_{i+1} - 1,\ i = 1, 2, \ldots, n-1 \\ f_n(x) &= x_n x_1 - 1. \end{aligned} \tag{11}$$

When n is odd, the exact zeros of $F(x)$ are $\bar{x}_1 = (1, 1, \ldots, 1)$ and $\bar{x}_2 = (-1, -1, \ldots, -1)$. Results appearing in Table 2 are obtained for $n = 99$ and all the methods converge to $\bar{x}_1$.

Let us observe that the optimal method, in terms of number of iterations and computational efficiency, is the iterated NG method of order 18, as was expected, being the total number of operations and functional evaluations much lower than the ones from classical Newton's method.

If we consider the nonlinear boundary value problem

$$\begin{gathered} y''(x) = y(x)^3 + sin(y'(x)^2), \quad x \in [0, 1] \\ y(0) = 0, \quad y(1) = 1 \end{gathered}$$

taken from [9], and use the finite differences method, we take the nodes $x_i = ih$, $i = 0, 1, \ldots, n$ where $h = \frac{1}{n}$, and use second order approximations for $y'(x_i)$ and $y''(x_i)$. By denoting the unknowns values $y(x_i)$ by y_i, $i = 0, 1, 2, \ldots, n$ the solution of the following nonlinear system provides us an estimation of the solution of the boundary value problem:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i^3 - \sin\left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 = 0, \qquad i = 1, 2, 3, \ldots, n-1. \tag{12}$$

Method	**Iterations**	ρ	**TO**	**TFE**
N	8	2.00	4488	1056
J	5	3.83	7480	1265
NG_4	5	3.45	4015	770
NG_5	4	4.83	3696	660
NG_{11}	3	10.92	4950	693

Table 3
Numerical results for the boundary problem (12) with $n = 9$.

Method	**Iterations**	ρ	**TO**	**TFE**
N	8	2.00	374408	21216
J	5	3.28	507280	26265
NG_4	5	4.08	260015	13770
NG_8	4	7.58	249628	11832
NG_{10}	3	9.42	202827	9180

Table 4
Numerical results for the boundary problem (12) with $n = 49$.

If we take $n = 9$ and the initial estimation $x^{(0)} = (0, 0.1, \ldots, 0.9, 1)$, the results obtained by applying different methods are showed in Table 3. We observe that, in this case, the optimal iterated method is NG_5, although the number of iterations made is not the least one. The fact is that, in terms of computational efficiency (see TO and TFE for NG_5), is better to get the solution in 4 iterations with NG_8 than in 3 with NG_{10}.

In order to know what happens when the size of the system increases, we observe the results showed in Table 4, obtained by applying the known and new methods on the system (12) with $n = 49$. The initial estimation is $x^{(0)} = (0, 0.02, \ldots, 0.98, 1)$ in this case. We observe that the optimal order is 10, as well in terms of number of iterations as in computational efficiency. This improves the theoretical results.

Finally, let us consider the boundary-value problem

$$y''(x) = -(1 + a^2(y')^2), \quad x \in [0, 1]$$

$$y(0) = 0, \quad y(1) = 0$$

taken from [19]. This problem arises in the study of finite deflections of an

Method	**Iterations**	ρ	**TO**	**TFE**
N	7	2.00	291207	17150
J	4	4.00	361816	19404
NG_4	4	4.00	185612	10192
NG_8	3	8.08	168021	8232
NG_{11}	3	6.25	189630	8673

Table 5
Numerical results for the boundary problem (13) with $n = 49$.

elastic string under a transverse load and its exact solution is

$$y(x) = ln\left(\frac{\cos\left(a(x-1/2)\right)}{\cos a/2}\right)\frac{1}{a^2}.$$

In order to get an estimation of this solution, we also use the second order finite differences method and, therefore, the following nonlinear system provides us an estimation of the solution of the original boundary value problem:

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+1+a^2\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)^2=0, \qquad i=1,2,3,\ldots,n-1. \quad (13)$$

The results obtained by applying different methods to system (13), with the initial estimation $x^{(0)} = (0.2, \ldots, 0.2)$, $n = 49$ and $a = 1/7$, are showed in Table 5. It is observed that, in practice, the optimal order of convergence is 8, which is a better result than the expected one. In fact, the exact error of this estimation is, in norm, $8.7994 \cdot 10^{-7}$, whereas if a smaller system is solved ($n = 9$, for example), the corresponding error is $9.8374 \cdot 10^{-6}$. In Figure 2 the exact error on each component of the estimation obtained for $n = 49$ is showed. Although the exact error seems to be high, it is necessary to take into account that the order of the finite differences method used to transform the boundary-value problem in a nonlinear system of equations is two; if a better estimation is needed, a higher order method should be used in this transformation.

In general, it can be concluded that the high-order methods generated are very efficient, specifically in the case of large systems. As we mentioned in the introduction, this can be the case when numerical applications are made (in particular, numerical solution of quadratic equations and nonlinear integral equations are needed in the study of dynamical models of chemical reactors [12], or in radioactive transfer [13]). The results of these numerical experiments show that the high order methods associated with a multiprecision arithmetic floating point are very useful.

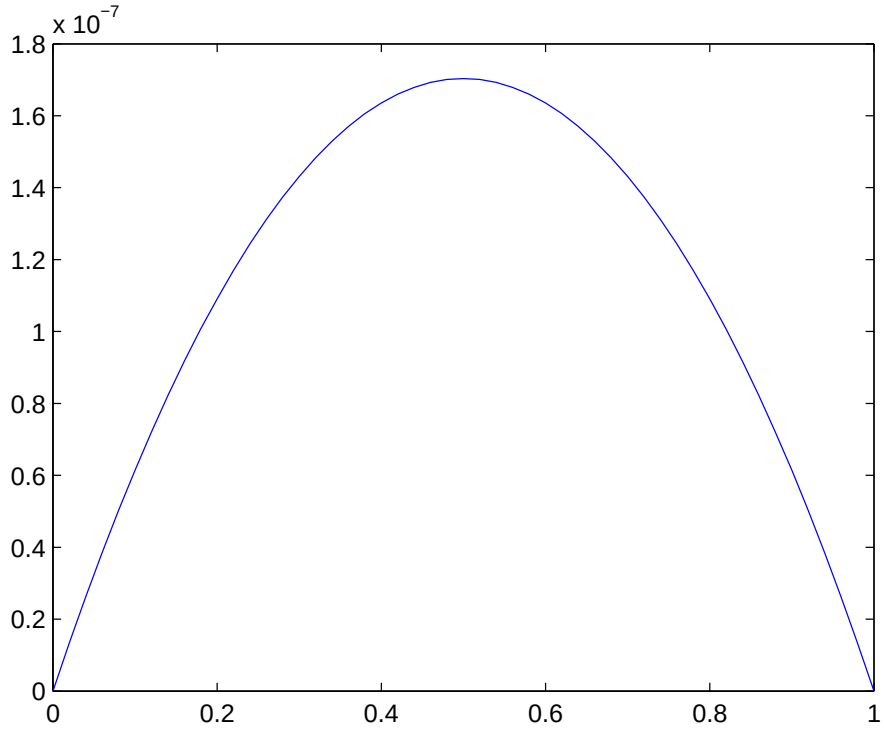

Fig. 2. Exact error made in the calculus of the deflections of an elastic string.

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