Factorization theorems for multiplication operators on Banach function spaces

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Abstract. Let X Y and Z be Banach function spaces over a measure space (Ω, Σ, μ). Consider the spaces of multiplication operators $X^{Y'}$ from X into the Köthe dual Y' of Y, and the spaces X^Z and $Z^{Y'}$ defined in the same way. In this paper we introduce the notion of factorization norm as a norm on the product space $X^Z \cdot Z^{Y'} \subseteq X^{Y'}$ that is defined from some particular factorization scheme related to Z. In this framework, a strong factorization theorem for multiplication operators is an equality between product spaces with different factorization norms. Lozanovskii, Reisner and Maurey-Rosenthal theorems are considered in our arguments to provide examples and tools for assuring some requirements. We analyze the class $d^*_{p,Z}$ of factorization norms, proving some factorization theorems for them when p-convexity/p-concavity type properties of the spaces involved are assumed. Some applications in the setting of the product spaces are given.

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1. Introduction

Factorization of operators through Banach function spaces is a common tool for solving problems in functional analysis. Often, these factorizations occur through a multiplication operator defined between Banach function spaces. Maurey-Rosenthal theorems through L^p spaces (see for instance [6, 8]), Pisier factorization theorems through Lorentz spaces (see [20, 9]), Nikishin theorem through weak L^p spaces ([26, Th. III.H.6]) are some relevant examples of these results, but this kind or arguments can be found in a lot of different settings, in which a multiplication operator plays a fundamental role (see for

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example [5]). Sometimes, the only information that gives the factorization theorem regarding the function appearing in it is continuity of the multiplication operator. However, a deeper knowledge about it improves the power of these results specially if it can be given as a new factorization of the multiplication operator, since it produces an enrichment of the factorization scheme for the original operator. This is the motivation of the study that we develop in this paper in which we introduce a technique for analyzing the factorization properties of the multiplication operators in a systematic way, following the research project that we started in [10].

From the technical point of view, as the reader will notice soon, we mimic in a sense the procedure for constructing reasonable topologies for tensor products in the representation theory of operator ideals by means of tensor norms (see [7]). The main differences are that we change tensor products of Banach spaces by products of function spaces, and the usual Banach space duality by the generalized duality for Banach function spaces (see [4, 17, 24]). Recently, a new effort has been made in order to develop the theory of products of Banach function spaces and general multiplication operators, that are a particular —but central— case of the general factorization norms and spaces that we develop here. The reader can find more information on that in the papers by Kolwicz, Leśnik and Maligranda (see [13] and the references therein), Schep ([24], see also [23]), Sukochev and Tomskova (see [22]) and Calabuig, Delgado and the author (see [4, 10]).

2. Factorization norms and strong factorization theorems

Let (Ω, Σ, μ) be a σ-finite measure space and let $X(\mu)$, $Y(\mu)$ and $Z(\mu)$ be Banach function spaces over μ (we shall write X, Y, Z for short if no explicit reference to the measure is needed). Let Y' be the Köthe dual of Y and X^Z, $Z^{Y'}$ and $X^{Y'}$ the corresponding spaces of multiplication operators from X to Z, Z to Y' and X to Y', respectively. In this paper we continue with the analysis of the topological products of function spaces that we started in [10] (see also [4]); with the aim of establishing factorization theorems for multiplication operators, we define and characterize the family $d^*_{p,Z}$ of what we call factorization norms for the product spaces $X^Z \cdot Z^{Y'}$. We show that the elements of the completion of the product space $X^Z \cdot Z^{Y'}$ with a factorization norm $d^*_{p,Z}$ always belong to $X^{Y'}$ and can be described in terms of a common factorization scheme. We also prove the main factorization theorems for this class of product spaces. We will use some classical factorization theorems (Lozanoskii, Reisner and Maurey-Rosenthal) in our arguments in order to assure some requirements that are needed. The class of norms $d^*_{p,Z}$ that we investigate here is modeled for satisfying suitable equalities between product spaces under p-convexity/p-concavity assumptions for the spaces involved. Such an equality is what we call a strong factorization theorem.

Let us fix some introductory ideas. Take three Banach function spaces X, Y and Z such that the spaces X^Z and $Z^{Y'}$ are saturated (see Section 2 for the definition). It is easy to check that $Z^{Y'} = Y^{Z'}$ (see Lemma 3.7 in [4], or [17]). The product of the space $X^Z \cdot Y^{Z'}$ is defined to be the subset of the space of (classes of μ-a.e. equal) measurable functions $L^0(\mu)$ given by the finite sums of functions $f \cdot g$, where $f \in X^Z$ and $g \in Y^{Z'}$. Note that such a function defines a multiplication operator from X to Y'. We analyze subspaces of $X^{Y'}$ instead of X^Y because for the definition of the norms that we define in the paper the generalized duality given by the bilinear form $Z \times Z' \to \mathbb{R}$ given by the integral is needed (see [4, 13, 17, 24]). However, the reader can notice that a great part of the results of the paper applies also to the case X^Y.

Two natural "extreme" norms can be defined in the space $X^Z \cdot Y^{Z'}$. The first one (that we will denote by ε_Z) is given by the restriction of the operator norm to the completion of such subspace of $X^{Y'}$. This is the weaker "reasonable" topology that we will consider; we use the symbol $X^Z \varepsilon_Z Y^{Z'}$ to denote the Banach function subspace of $X^{Y'}$ generated in this way. For the second one we define the function norm on $L^0(\mu)$ given by the formula

$$\pi(h) := \inf \sum_{i \geq 1} \|f_i\|_{X^Z} \cdot \|g_i\|_{Y^{Z'}}, \quad h \in L^0(\mu),$$

where the infimum is computed over all possible dominations of h as $|h| \leq \sum_{i \geq 1} |f_i g_i|$, $f_i \in X^Z$, $g_i \in Z^{Y'}$; if there is no such a domination, then $\pi(h) = \infty$ (see [25] for the general theory of function norms, and [10, 13, 24] for the properties of the π-norm, but note that the definitions are slightly different in these papers). This norm gives the strongest topology on the product $X^Z \cdot Y^{Z'}$ that we consider, and defines a function norm that generates the Banach function space $X^Z \pi Y^{Z'}$ in $L^0(\mu)$; under certain requirements an equivalent norm $\tilde{\pi}$ for this space can be computed as the infimum of the single product of norms of functions $f \in X^Z$ and $g \in Y^{Z'}$ that provide a decomposition of $|h|$ as $|h| = hg$, $h \in X^Z \pi Y^{Z'}$ (see [4, 10, 13, 24]). It is easy to check that for every $f \in X^Z \pi Y^{Z'}$, $\varepsilon_Z(f) \leq \pi(f)$. We will deal with norms α on $X^Z \cdot Y^{Z'}$ satisfying $\varepsilon_Z \leq \alpha \leq \pi$. We say that such a norm α is a *factorization norm* if $\alpha(h)$ can be computed as an infimum of products of the norms of suitable operators belonging to a fixed family that factorize the multiplication operator given by h; a *strong factorization theorem* is an equality $X^Z \alpha Z^{Y'} = X^Z \beta Z^{Y'}$ for a couple of factorization norms α and β.

Relevant known examples of strong factorization theorems for multiplication operators are the following. Some particular instances of Lozanovskii´s theorem ([16, 21]) can be written as the isometry $(L^\infty)^Y \pi Y^{L^1} = (L^\infty)^Y \tilde{\pi} Y' = L^1 = (L^\infty)^{L^1}$, whenever the adequate requirements are satisfied and $\tilde{\pi}$ is a norm. The same happens concerning Reisnerś theorem ([21]): if $1/p = 1/q + 1/r$, it gives the equality $(L^q)^Y \pi Y^{L^p} = (L^q)^Y \tilde{\pi} Y^{L^p} = L^r$ when the adequate requirements on convexity and concavity for Y are assumed and so $\tilde{\pi}$ is a norm. Also, the Maurey-Rosenthal theorem (see [6, Cor. 5] or [19, Ch.6]) for the case of multiplication operators gives the equality $X^{Y'} = X^{L^p} \pi (L^p)^{Y'}$ whenever X is order continuous and p-convex, and Y' is p-concave. Other example of the same methodological point of view for the case of sequence spaces can be found in the analysis of classical inequalities exposed in [2].

After this introductory section and the following one —where some notation and basic definitions are given—, we present our results in three parts. Section 4 is devoted to the analysis of the properties of the norm $d^*_{p,Z}$ as a factorization norm. The extreme cases (the norms ε_Z and π) are also considered as $d^*_{p,Z}$ norms. In Section 5 we give some applications and prove some factorization theorems related with the $d^*_{p,Z}$ norms. For instance, Theorem 5.1 establishes that, under the adequate convexity properties, the multiplication operators of the space defined by $d^*_{p,Z}$ satisfy a special factorization theorem that can be written as the equality $X^Z \pi Y^{Z'} = X^Z d^*_{p,Z} Y^{Z'}$ (Section 5.1). More applications involving the order continuity of the π norm are also given. Finally, we present in Section 5.2 the main results concerning the complementary equality $X^Z d^*_{p,Z} Y^{Z'} = X^{Y'}$.

3. Notation and basic concepts

Let (Ω, Σ, μ) be a measure space. Let $L^0(\mu)$ be the space of all (classes of μ-a.e. equal) real functions on Ω. A *Banach function space* over μ is a Banach space $X \subset L^0(\mu)$ with a norm $\|\cdot\|_X$ satisfying that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ μ–a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$. A Banach function space X is *order continuous* if increasing sequences that are bounded μ-a.e. are convergent in norm. We say that a Banach function space X has the *Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \leq f_n \uparrow f$ μ–a.e. and $\sup_n \|f_n\|_X < \infty$, $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$. A weaker property for X is given by what we call the *order semi-continuity*; a Banach function space X satisfies this property if for every $f, f_n \in X$ such that $0 \leq f_n \uparrow f$ μ–a.e., $\|f_n\|_X \uparrow \|f\|_X$.

Throughout the paper the Banach function spaces are considered to be over the same measure space. If X is a Banach function space, we denote by X' its Köthe dual or associate space, i.e. the Banach function space of functions that define multiplication operators from X to L^1 (see for example [15]). In general, we identify a function in X^Y with the multiplication operator that it defines. The generalized duality induced on X and X^Z by the bilinear map $X \times X^Z \to Z$ given by the product has been defined and studied by Maligranda and Persson in [17] and also by Calabuig and the authors in [4] (see also [10, 3, 13, 24]).

The space of multiplication operators X^Z is a Banach function space (with the natural operator norm) if X^Z is saturated, i.e. if there is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ–a.e. for all $f \in X^Z$. *Through the paper the saturation property is always required for all the spaces involved*; this fact will be explicitly mentioned only if we consider that it is specially relevant in the context. Our references for the definition and properties of Banach function spaces are [25, Ch. 15], considering the function norm ρ defined as $\rho(f) = \|f\|_X$ if $f \in X$ and $\rho(f) = \infty$ in other case, and [1, 14, 18].

The following notation for norms of sequences of functions involving the generalized duality will be used. Let (x_i) be a sequence in X. Let $1 \leq p \leq \infty$. Then we write

$$\pi_p((x_i)) := \big(\sum_{i\geq 1} \|x_i\|_X^p\big)^{1/p},$$

with the obvious modification if $p=\infty$. If $\pi_p((x_i)) < \infty$, we say that (x_i) is strongly p-summable. If (f_i) a sequence in X^Z, we define the function

$$w_{p,Z}^*((f_i)) := \sup_{x\in B_X} \big(\sum_{i\geq 1} \|f_i x\|_Z^p\big)^{1/p}.$$

In the case that $w_{p,Z}^*((f_i)) < \infty$, we say that (f_i) is weakly (p,Z)-summable; the "dual" definition and the corresponding analysis can be found in [10]. Also, if (f_i) is a sequence of functions in X^Y and $1 \leq p \leq q \leq \infty$, we will consider the operators defined in the natural way, that may change depending on for which spaces are defined, using the same symbol (f_i) to denote the corresponding operator. The reader will find the following three cases, that will be used without further explanations; of course, in each one of them the sequence (f_i) must satisfy an adequate boundedness condition in order the operator to be well defined.

(1) $(f_i): X \to \ell^p(Y)$, $\quad (f_i)(x) := (f_i x), \quad x \in X.$
(2) $(f_i): \ell^q(X) \to \ell^p(Y)$, $\quad (f_i)(x_i) := (f_i x_i), \quad (x_i) \in \ell^q(X).$
(3) $(f_i): \ell^p(X) \to Y$, $\quad (f_i)(x_i) := \sum_{i\geq 1} f_i x_i, \quad (x_i) \in \ell^p(X).$

Throughout the paper, if $1 \leq p \leq \infty$, p' denotes the (extended) real number satisfying $1/p + 1/p' = 1$. A Banach function space X is p-convex with constant $M^{(p)}(X)$ if for every sequence (x_i) in X,

$$\Big\|\big(\sum_{i\geq 1} |x_i|^p\big)^{1/p}\Big\|_X \leq M^{(p)}(X)\Big(\sum_{i\geq 1} \|x_i\|_X^p\Big)^{1/p}.$$

The space X is p-concave if there is a constant $M_{(p)}(X)$ such that for every sequence (x_i) in X,

$$\Big(\sum_{i\geq 1} \|x_i\|_X^p\Big)^{1/p} \leq M_{(p)}(X)\Big\|\big(\sum_{i\geq 1} |x_i|^p\big)^{1/p}\Big\|_X.$$

The constants $M^{(p)}(X)$ and $M_{(p)}(X)$ are the best ones in the above inequalities.

4. The $d_{p,Z}^*$-factorization norms

Consider a space Z such that X^Z and $Y^{Z'}$ are saturated. If $f \in X^Z$ and $g \in Y^{Z'} = Z^{Y'}$, the product fg defines an element of $X^{Y'}$. The function norm $\varepsilon_Z : L^0(\mu) \to \mathbb{R}^+ \cup \{\infty\}$ is defined by

$$\varepsilon_Z(h) := \inf\{\|\sum_{i\geq 1} |f_i g_i|\|_{X^{Y'}} : |h| \leq \sum_{i\geq 1} |f_i g_i|, \, f_i \in X^Z, \, g_i \in Y^{Z'}\},$$

where the infimum is computed over all possible $\|\cdot\|_{X^{Y'}}$-convergent series that dominates $|h|$ ($\varepsilon_Z(h)=\infty$ if no domination occurs). Now, define the space

$$X^Z\varepsilon_Z Y^{Z'} := \Big\{h\in L^0(\mu) : \textit{there exist}\,(f_i)\subset X^Z, (g_i)\subset Y^{Z'} \textit{ such that}$$
$$\textit{the series } \sum_{i\geq 1}|f_i g_i| \textit{ converges in}\, X^{Y'} \textit{ and}\, |h|\leq \sum_{i\geq 1}|f_i g_i|\Big\}.$$

Consequently, $X^Z\varepsilon_Z Z^{Y'}$ can be identified with the (normed) subspace of all the functions $h\in L^0(\mu)$ that satisfy that $\varepsilon_Z(h)$ is finite. Since the space $X^{Y'}$ is an ideal, we have in particular that $X^Z\varepsilon_Z Z^{Y'}\subseteq X^{Y'}$ isometrically. We also assume that $X^Z\varepsilon_Z Z^{Y'}$ is saturated; otherwise ε_Z maybe just a seminorm. In the case of the other product spaces defined in this section, saturation is also assumed.

Lemma 4.1. *Let $h\in X^Z\varepsilon_Z Y^{Z'}$. Then there is a μ-a.e. convergent series $\sum_{i\geq 1}|f_i^0 g_i^0|$ defined by functions $f_i^0\in X^Z$ and $g_i^0\in Y^{Z'}$ such that $|h| = \sum_{i\geq 1}|g_i^0 f_i^0|$ (convergence in $X^{Y'}$) and $\varepsilon_Z(h)=\|h\|_{X^{Y'}}$. Consequently, the space $(X^Z\varepsilon_Z Y^{Z'},\varepsilon_Z)$, where $X^Z\varepsilon_Z Y^{Z'} := \big\{h\in L^0(\mu) : \varepsilon_Z(h)<\infty\big\}$ is (isometrically) a Banach function subspace of $X^{Y'}$ that contains the closure in $X^{Y'}$ of the linear combinations of products of elements of X^Z and $Y^{Z'}$.*

Proof. Take $\delta>0$. Since $h\in X^Z\varepsilon_Z Y^{Z'}$, there is a series of functions $\sum_{i\geq 1}|f_i g_i|$ convergent in $X^{Y'}$ that dominates $|h|$, $f_i\in X^Z$, $g_i\in Y^{Z'}$, and $\|\sum_{i\geq 1}|f_i g_i|\|_{X^{Y'}}\leq \varepsilon_Z(h)+\delta$. Consider the function $h_0 = |h|/\sum_{i\geq 1}|f_i g_i|$. Clearly, $h_0\leq 1$ μ-a.e. and the functions $f_i^0 := f_i h_0$, $i=1,2,...$ belong to X^Z, since X^Z is in particular an ideal in $L^0(\mu)$. Define $g_i^0 := g_i$ for every i. Note that h and $\sum_{i\geq 1}|f_i^0 g_i^0|$ are elements of $X^{Y'}$, since it is an ideal, and

$$\|h\|_{X^{Y'}}\leq \|\sum_{i\geq 1}|f_i^0 g_i^0|\|_{X^{Y'}}\leq \|\sum_{i\geq 1}|f_i g_i|\|_{X^{Y'}}\leq \varepsilon_Z(h)+\delta.$$

The series $\sum_{i\geq 1}|f_i^0 g_i^0|$ converges in $X^{Y'}$ (and then also μ-a.e.) to $|h|$, since

$$\||h|-\sum_{i=1}^n|f_i^0 g_i^0|\|_{X^{Y'}}$$
$$=\Big\|h_0\big(\sum_{i\geq 1}|f_i g_i|-\sum_{i=1}^n|f_i g_i|\big)\Big\|_{X^{Y'}}\leq \|\sum_{i\geq 1}|f_i g_i|-\sum_{i=1}^n|f_i g_i|\|_{X^{Y'}}\|\to 0.$$

Consequently,

$$\|h\|_{X^{Y'}} = \|\sum_{i\geq 1}|f_i^0 g_i^0|\|_{X^{Y'}} = \varepsilon_Z(h).$$

This gives the result on the representation of $|h|$. Note also that the Riesz-Fischer property is satisfied. To see this, consider a sequence of functions $(h_k)\subset X^Z\varepsilon_Z Y^{Z'}$ such that $\sum_{k\geq 1}\varepsilon_Z(h_k)<\infty$. Since $X^{Y'}$ is complete, the function $h := \sum_{i\geq 1}h_i$ belongs to $X^{Y'}$ and $\sum_{k=1}^n h_i$ converges to $\sum_{i\geq 1}h_i$

in the norm of this space. Also, it can be found for each k a decomposition of $|h_k|$ as the $\|\cdot\|_{X^{Y'}}$-convergent sequence $\sum_{i\geq 1}|f_i^k g_i^k|$, where $f_i^k \in X^Z$ and $g_i^k \in Z^{Y'}$. Therefore, straightforward calculations show that the series $\sum_{i=1}^n \sum_{k=1}^n |f_i^k g_i^k|$ converges in $X^{Y'}$ to $\sum_{i\geq 1}\sum_{k\geq 1}|f_i^k g_i^k|$, and obviously $|h| \leq \sum_{i\geq 1}\sum_{k\geq 1}|f_i^k g_i^k|$ μ-a.e. This shows that $h \in X^Z \varepsilon_Z Y^{Z'}$, and so $X^Z \varepsilon_Z Y^{Z'}$ is a Banach space.

□

Definition 4.2. Let $1 \leq p \leq \infty$. Consider a couple of Banach function spaces (X, Y) and a Banach function space Z satisfying that X^Z and $Z^{Y'}$ are saturated. We define the function $d^*_{p,Z} : X^{Y'} \to \mathbb{R}^+ \cup \{\infty\}$ by the formula

$$d^*_{p,Z}(h) := \inf\{w^*_{p',Z}((f_i)) \cdot \pi_p((g_i)) : |h| \leq \sum_{i\geq 1}|f_i g_i|,\ f_i \in X^Z,\ g_i \in Z^{Y'}\}.$$

Lemma 4.3. *$d^*_{p,Z}$ defines a function norm on $X^{Y'}$ that is complete and such that $\varepsilon_Z \leq d^*_{p,Z} \leq \pi$. In particular, $X^Z d^*_{p,Z} Y^{Z'}$ is a Banach function space.*

Proof. Let us prove that $d^*_{p,Z}$ satisfies the triangular inequality. Let $\delta > 0$. Take a couple of functions f_1 and f_2 in $X^Z d^*_{p,Z} Y^{Z'}$ and note that we can always find two series of functions $\sum_{i\geq 1}|g_i^1 h_i^1|$ and $\sum_{i\geq 1}|g_i^2 h_i^2|$ of elements of X^Z and $Y^{Z'}$ dominating $|f_1|$ and $|f_2|$, respectively, and satisfying that $w^*_{p',Z}((g_i^j)) \leq (d^*_{p,Z}(f_j)+\delta)^{1/p'}$ and $\pi_p((h_i^j)) \leq (d^*_{p,Z}(f_j)+\delta)^{1/p}$, $j = 1, 2$. The domination of $|f_1 + f_2|$ given by $\sum_{i\geq 1}|g_i^1 h_i^1| + \sum_{i\geq 1}|g_i^2 h_i^2|$ satisfies that

$$\begin{aligned} &w^*_{p',Z}((g_i^1)\cup(g_i^2))\pi_p((h_i^1)\cup(h_i^2)) \\ &\leq (w^*_{p',Z}((g_i^1))^{p'} + w^*_{p',Z}((g_i^2))^{p'})(\pi_p((h_i^1))^p + \pi_p((h_i^2))^p) \\ &\leq (d^*_{p,Z}(f_1) + d^*_{p,Z}(f_2) + 2\delta)^{1/p}(d^*_{p,Z}(f_1) + d^*_{p,Z}(f_2) + 2\delta)^{1/p'} \\ &= d^*_{p,Z}(f_1) + d^*_{p,Z}(f_2) + 2\delta \end{aligned}$$

(here the notation $(g_i^1)\cup(g_i^2)$ means the sequence $(g_1^1, g_1^2, g_2^1, g_2^2, g_3^1, \ldots)$). Consequently, $d^*_{p,Z}$ is a (lattice) seminorm for the space of functions f satisfying that $d^*_{p,Z}(f) < \infty$. This, together with Lemma 4.1, implies that $d^*_{p,Z}$ is in fact a function norm. A straightforward argument using the inequalities above for infinite sequences of functions shows that $d^*_{p,Z}$ satisfies the Riesz-Fischer property, and so $X^Z d^*_{p,Z} Y^{Z'}$ is a Banach function space. Let us prove now the inequalities $\varepsilon_Z(f) \leq d^*_{p,Z}(f) \leq \pi(f)$ for every $f \in X^Z d^*_{p,Z} Z^{Y'}$. For such a function f and any series $\sum_{i\geq 1}|f_i g_i|$ dominating $|f|$

$$\varepsilon_Z(f) = \sup_{x\in B_X}\|fx\|_{Y'} \leq \sup_{x\in B_X, y\in B_Y}\sum_{i\geq 1}\int xy|f_i g_i|\,d\mu$$

$$\leq \sup_{x\in B_X}\Big(\sum_{i\geq 1}\|f_i x\|_Z^{p'}\Big)^{1/p'}\Big(\sum_{i\geq 1}\|g_i\|_{Z^{Y'}}^p\Big)^{1/p}.$$

Thus $\varepsilon_Z(f) \leq d^*_{p,Z}(f)$. Also, taking into account

$$\inf\{\pi_{p'}((f_i))\pi_p((g_i)) : |h| \leq \sum_{i\geq 1} |f_i g_i|,\ f_i \in X^Z,\ g_i \in Z^{Y'}\} = \pi(h),$$

(see the comments after Proposition 3.2 in [10]) we obtain $d^*_{p,Z} \leq \pi$. □

Theorem 4.4. *Let $1 < p < \infty$. The following assertions are equivalent for a multiplication operator $h \in X^{Y'}$.*

(1) $h \in X^Z\, d^*_{p,Z}\, Y^{Z'}$.

(2) *The function h defines a multiplication operator of $X^{Y'}$ that factorizes as*

$$\begin{tikzcd} X(\mu) \arrow[r, "h"] \arrow[d, "(f_i)"] & Y'(\mu) \\ \ell^{p'}(Z) \arrow[r, "(g_i)"] & \ell^1(Y') \arrow[u, "C"'] \end{tikzcd}$$

where $w^*_{p',Z}((f_i)) < \infty$, $\pi_p((g_i)) < \infty$ and C is defined by $C((y'_i)) := \sum_{i\geq 1} y'_i$, $(y'_i) \in \ell^1(Y')$.

(3) *h can be written as an almost everywhere sum of the product of a weakly (p', Z)-summable sequence (f_i) of elements of X^Z and other strongly p-summable sequence (g_i).*

Moreover, if (1), (2), (3) *hold, then $d^*_{p,Z}(h) = \inf \|(f_i)\|\|(g_i)\|$, where the infimum is computed over all suitable factorizations in* (2).

Proof. (1) $\Rightarrow$ (3). Since $d^*_{p,Z}(h)$ is finite, for any $1 < p < \infty$ there is a couple of sequences of functions (f_i^0) and (g_i^0) such that $|h| \leq \sum_{i\geq 1} |f_i^0 g_i^0|$, $w^*_{p',Z}((f_i^0)) < \infty$ and $\pi_p((g_i^0)) < \infty$. Let us show that we can obtain a couple of sequences (f_i) and (g_i) such that $h = \sum_{i\geq 1} f_i g_i$ μ-a.e., $w^*_{p',Z}((f_i)) < \infty$ and $\pi_p((g_i)) < \infty$. Write h as the sum of the positive and the negative parts $h = h^+ - h^-$. Let us define the functions of disjoint support $\varphi^+ = h^+/\sum_{i\geq 1} |f_i^0 g_i^0|$ and $\varphi^- = h^-/\sum_{i\geq 1} |f_i^0 g_i^0|$, that satisfy that $0 \leq \varphi^+ \leq 1$ and $0 \leq \varphi^- \leq 1$ μ-a.e. Consequently, $h = \sum_{i\geq 1} |f_i^0 g_i^0|\varphi^+ - \sum_{i\geq 1} |f_i^0 g_i^0|\varphi^-$. The μ-a.e. pointwise convergence of this sum is guaranteed by the convergence of $\sum_{i\geq 1} |f_i^0 g_i^0|$, since clearly

$$|\sum_{i\geq 1} |f_i^0 g_i^0|\varphi^+ - |f_i^0 g_i^0|\varphi^-| \leq \sum_{i\geq 1} |f_i^0 g_i^0|.$$

Now consider each of the elements $|f_i^0 g_i^0|\varphi^+$ and $|f_i^0 g_i^0|\varphi^-$ in this sum. There are functions $0 \leq f_i^1 \leq |f_i^0|$ and $0 \leq g_i^1 \leq g_i^0$ such that $f_i^1 g_i^1 = |f_i^0 g_i^0|\varphi^+$, and functions $0 \leq f_i^2 \leq |f_i^0|$ and $0 \leq g_i^2 \leq g_i^0$ such that $f_i^2 g_i^2 = |f_i^0 g_i^0|\varphi^-$.

Note that f_i^j and g_l^k can be chosen to be disjointly supported for every i and l whenever $j \neq k$. The same is true for f_i^j and f_l^k, and g_i^j and g_l^k, for $j \neq k$. Therefore, $h = \sum_{i \geq 1}(f_i^1 - f_i^2)(g_i^1 + g_i^2)$, since for every i,

$$(f_i^1 - f_i^2)(g_i^1 + g_i^2) = f_i^1 g_i^1 - f_i^2 g_i^1 + f_i^1 g_i^2 - f_i^2 g_i^2 = f_i^1 g_i^1 - f_i^2 g_i^2.$$

Moreover, $w^*_{p',Z}((f_i^1 - f_i^2)) \leq w^*_{p',Z}((f_i^0))$ and $\pi_p((g_i^1 + g_i^2)) \leq \pi_p((g_i^0))$, by the lattice properties of the Banach function spaces Z and X^Z.

Clearly, (3) implies (2) and (2) implies (1) as a consequence of the computation of the norms in the factorization, since

$$\|(f_i)\|_{L(X,\ell^{p'}(Z))} = \sup_{x \in B_X} \Big(\sum_{i \geq 1} \|f_i x\|_Z^{p'} \Big)^{1/p'} = w^*_{p',Z}((f_i))$$

and

$$\|(g_i)\|_{L(\ell^{p'}(Z),\ell^1(Y'))} = \Big(\sum_{i \geq 1} \|g_i\|^p_{Z^{Y'}} \Big)^{1/p} = \pi_p((g_i)).$$

Finally, note that the computation at the end of the proof of (1) $\Rightarrow$ (3) proves in particular that the infimum of the products $\|(f_i)\| \cdot \|(g_i)\|$ for all suitable factorizations gives $d^*_{p,Z}(h)$.

$\square$

Remark 4.5. Let us show how the factorization for the norms ε_Z and π can also be considered in a sense as the extreme cases of the factorization theorems for the $d^*_{p,Z}$ norms.

(a) *The factorization theorem for the ε_Z norm.* As a consequence of the isometry $X^{Y'}\pi Y'^{Y'} = X^{Y'}\pi L^\infty(\mu)$ (see [4, 17, 24]), we obtain the following result for the ε_Z norm for the particular case $Z = Y'$. For a function $h \in L^0(\mu)$, the following statements are equivalent.

(1) $h \in X^{Y'}$.

(2) There is a factorization for h as

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "(f_i)"'] & & Y' \\ & \ell^1(Y') \arrow[ur, "(g_i)"'] & \end{tikzcd},$$

where (f_i) satisfies $w^*_{1,Y'}((f_i)) < \infty$, and (g_i) is a sequence in $\ell^\infty(L^\infty)$.

Moreover, if (1), (2) hold, then $\|h\|_{X^{Y'}} = \inf \|(f_i)\| \|(g_i)\|$, where the infimum is computed over all suitable factorizations as the one given in (2). (Here the norms for the sequences are understood as their norms as operators)

This is a consequence of the following easy arguments. Take a function $h \in X^{Y'}$ and consider the following factorization through $\ell^1(Y')$ of the map $h : X \to Y'$ given by the sequence $(h, 0, 0, ...) \in \ell^1(X^{Y'})$ and the map $(g_i) : \ell^1(Y') \to Y'$ given by the sequence (g_i), where $g_1 = \chi_\Omega$, $g_i = 0$ for each $i = 2, ...$ Clearly, $\|h\|_{X^{Y'}} = \|(h, 0, 0, ...)\| \|(g_i)\|$.

On the other hand, take a couple of sequences (f_i) and (g_i) satisfying the requirements in (2). Since for every $x \in X$,

$$\|hx\| \leq \| \sum_{i\geq 1} |f_i g_i| x\|_{Y'} \leq \sup_i \|g_i\|_{L^\infty} \sum_{i\geq 1} \|f_i x\|,$$

the map is well defined and continuous. Moreover,

$$\|h\|_{X^{Y'}} \leq \| \sum_{i\geq 1} |f_i g_i| \|_{X^{Y'}} \leq \sum_{i\geq 1} \|f_i\|_{X^{Y'}} \|g_i\|_{L^\infty} \leq \|(f_i)\| \|(g_i)\|.$$

This, together with the particular factorization given at the beginning of the proof, gives the formula

$$\|h\|_{X^{Y'}} = \inf \|(f_i)\| \|(g_i)\|,$$

where the infimum is computed over all suitable factorizations, and finishes the proof.

(b) *The factorization theorem for the π norm.* Even in the case when we have no information about the coincidence of π and $\tilde{\pi}$ on $X^Z \cdot Y^{Z'}$, it is still possible to get a factorization theorem for the elements of the π product; in the case that the equivalence holds, then we have a strong factorization theorem in the sense that has been explained in the Introduction.

The following assertions are equivalent for a function $h \in X^{Y'}$.

(1) $h \in X^Z \, \pi \, Y^{Z'}$.

(2) There is a real number $1 < p < \infty$ such that there is a factorization of h as

$$\begin{tikzcd}
X(\mu) \arrow[rr, "h"] \arrow[d, "I"] & & Y'(\mu) \\
\ell^\infty(X) \arrow[r, "(f_i)"] & \ell^{p'}(Z) \arrow[r, "(g_i)"] & \ell^1(Y') \arrow[u, "C"']
\end{tikzcd}$$

where $I(x) := (x, x, x, ...)$, $x \in X$, and $C((y'_i)) = \sum_{i\geq 1} y'_i$, $(y'_i) \in \ell^1(Y')$.

(3) There is a real number $1 < p < \infty$ (and then for every such number p) such that h can be written as an almost everywhere sum of the product of a strongly p'-summable sequence (f_i) of elements of X^Z and other strongly p-summable sequence (g_i) of elements of $Z^{Y'}$.

Moreover, if $(1), (2), (3)$ hold, then $\pi(h) = \inf \|(f_i)\| \|(g_i)\|$, where the infimum is computed over all the factorizations as in (2).

The implication (1) $\Rightarrow$ (3) can be shown as (1) $\Rightarrow$ (3) in Theorem 4.4. For (2) $\Rightarrow$ (1), let us compute the norm of the factorization. By hypothesis, there are strongly p' and p summable sequences (f_i) and (g_i), respectively, such that $|h| \leq \sum_{i\geq 1} |f_i g_i|$, and define a factorization as the one in (2). But note that

$$\|(f_i)\|_{L(\ell^\infty(X),\ell^{p'}(Z))} = \sup_{\|x_i\|_X \leq 1} \Big(\sum_{i\geq 1} \|f_i x_i\|_Z^{p'} \Big)^{1/p'} = \Big(\sum_{i\geq 1} \|f_i\|_{X^Z}^{p'} \Big)^{1/p'} = \pi_{p'}((f_i)),$$

and, by duality of the spaces ℓ^p and $\ell^{p'}$,

$$\|(f_i)\|_{L(\ell^{p'}(Z),\ell^1(Y'))} = \sup_{(z_i)\in B_{\ell^{p'}(Z)}} \Big(\sum_{i\geq 1} \|g_i z_i\|_{Y'} \Big) = \Big(\sum_{i\geq 1} \|g_i\|_{Z^{Y'}}^{p} \Big)^{1/p} = \pi_p((g_i)).$$

The equivalent formula for the norm π given in the proof of Lemma 4.3 and these computations gives also $\pi(h) = \inf \|(f_i)\| \|(g_i)\|$. (3) $\Rightarrow$ (2) is direct, since (2) is just the factorization expression for (3).

5. Applications: p-convexity, p-concavity and strong factorization theorems for the $d^*_{p,Z}$ norms

5.1. Coincidence of the π norm and the $d^*_{p,Z}$ norm

In this section we use the results of the previous ones to obtain strong factorization theorems for multiplication operators in $X^Z d^*_{p,Z} Y^{Z'}$. The key for our results are the convexity properties of the spaces involved. As we shall show, under these requirements it is possible to improve the factorization for the elements of this product given by Theorem 4.4 in order to obtain a strong factorization theorem through Z.

Theorem 5.1. *Let $1 < p < \infty$. Let Z be a p'-convex space and let $Y^{Z'}$ be a p-convex space. Then $X^Z \tilde{\pi} Y^{Z'} = X^Z d^*_{p,Z} Y^{Z'}$. Consequently, if a function h allows a decomposition as the one given in* (3) *of Theorem* 4.4*, then it factorizes as*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y', \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

i.e. it can be written as a single product of a function $f \in X^Z$ and a function $g \in Z^{Y'}$. Moreover, in this case

$$\tilde{\pi}(h) \leq M^{(p')}(Z) M^{(p)}(Y^{Z'}) d^*_{p,z}(h) \leq M^{(p')}(Z) M^{(p)}(Y^{Z'}) \pi(h).$$

Proof. The inclusion $X^Z \pi Y^{Z'} \subseteq X^Z d^*_{p,Z} Y^{Z'}$ always holds, as a consequence of Lemma 4.3. On the other hand, if $h \in X^Z d^*_{p,Z} Y^{Z'}$, for every couple of sequences of functions (f_i) and (g_i) such that $|h| \le \sum_{i\ge 1} |f_i g_i|$, where $w^*_{p',Z}((f_i)) < \infty$ and $\pi_p((g_i)) < \infty$, we have

$$\begin{aligned}
&\|(\sum_{i\ge 1} |f_i|^{p'})^{1/p'}\|_{X^Z} \cdot \|(\sum_{i\ge 1} |g_i|^p)^{1/p}\|_{Y^{Z'}} \\
&= \sup_{x\in B_X} \|(\sum_{i\ge 1} |f_i|^{p'})^{1/p'} x\|_Z \cdot \|(\sum_{i\ge 1} |g_i|^p)^{1/p}\|_{Y^{Z'}} \\
&\le M^{(p')}(Z) M^{(p)}(Y^{Z'}) \sup_{x\in B_X} \big(\sum_{i\ge 1} \|f_i x\|_Z^{p'}\big)^{1/p'} \big(\sum_{i\ge 1} \|g_i\|^p_{Y^{Z'}}\big)^{1/p}.
\end{aligned}$$

Now, taking into account that

$$|h| \le \sum_{i\ge 1} |f_i g_i| \le \big(\sum_{i\ge 1} |f_i|^{p'}\big)^{1/p'} \big(\sum_{i\ge 1} |g_i|^p\big)^{1/p}$$

μ-a.e., we obtain

$$\pi(h) \le \tilde{\pi}(h) \le M^{(p')}(Z) M^{(p)}(Y^{Z'}) \sup_{x\in B_X} \big(\sum_{i\ge 1} \|f_i x\|_Z^{p'}\big)^{1/p'} \big(\sum_{i\ge 1} \|g_i\|^p_{Y^{Z'}}\big)^{1/p}.$$

This gives the result. $\square$

The canonical example of the situation described in Theorem 5.1 is given when $Z = L^{p'}(\mu)$; this is a consequence of the fact that $(L^{p'}(\mu))' = L^p(\mu)$ and $Y^{Z'}$ is p-convex whenever Z' is p-convex. However, there are more cases for other spaces for which this also hold, as we show in what follows. Recall that the saturation requirement is always assumed for X^Z, $Z^{Y'}$ and $X^{Y'}$; it holds in all the examples that we explain.

Example 5.2. Clearly, the most interesting examples are found when the convexity properties of the space $Z^{Y'}$ improve the ones of Y'. Let us show some of them.

(1) *Spaces of integrable functions with respect to a vector measure.* Let $1 \le p < \infty$ and let m be a (countably additive) Banach space valued vector measure. Consider the class of spaces $L^p(m)$ and $L^p_w(m)$ of p-integrable and weakly p-integrable functions, respectively. Take any Banach function space X and consider the case $Y' = L^1(m)$; remark that this example is rather general, since every order continuous Banach function space with a weak order unit can be written (order isometrically) as an $L^1(m)$ of a vector measure m (see for instance [19, Prop. 3.9]). The Banach function spaces $L^p(m)$ and $L^p_w(m)$ are always p-convex; the reader can find more information on these spaces in [19, Ch.3]. Take p such that $1 < p \le p' < \infty$ and $Z = L^{p'}_w(m)$. Then $Z^{Y'} = (L^{p'}_w(m))^{L^1(m)} = L^p(m)$, that is p-convex (see for instance [4, Ex. 4.3], or [19, Prop. 3.43]). Therefore, Theorem 5.1 gives that every function h of

$$X^{L^{p'}_w(m)} d^*_{p, L^{p'}_w(m)} {L^{p'}_w(m)}^{L^1(m)} = X^{L^{p'}_w(m)} d^*_{p, L^{p'}_w(m)} L^p(m)$$

factorizes through $L^{p'}_w(m)$ as $h = fg$, where $f \in X^{L^{p'}_w(m)}$ and $g \in {L^{p'}_w(m)}^{L^1(m)} = L^p(m)$. Similar examples can be found for p-th powers of Banach function spaces, see for instance [4, Prop. 4.2] or [19, 17].

(2) *Orlicz spaces.* The space of multiplication operators between Orlicz spaces can be sometimes represented as other Orlicz space. For the following particular choice of Young functions, this can be found in [4, Th.6.7]. Consider three Young functions Φ, Φ_0, Φ_1 satisfying:

(i) $\Phi(st) \leq \frac{1}{2}\big(\Phi_0(s) + \Phi_1(t)\big)$, for all $s, t \geq 0$, and

(ii) $\Phi^{-1}(t) \leq \Phi_0^{-1}(t)\Phi_1^{-1}(t)$, for all $t \geq 0$.

Then, $(L^{\Phi_0})^{L^{\Phi}} = L^{\Phi_1}$. On the other hand, the following characterization holds for p-convex Orlicz spaces defined on non-atomic measure spaces: An Orlicz space L_φ is p-convex (with p-convexity constant one) if and only if $\varphi(s^{1/p})$ is convex ([11, Th. 5.1.]). Using both results we can construct the following example, that illustrates also Theorem 5.1.

Consider the Young functions Φ, Φ_0, Φ_1 satisfying the requirements given above. Take a Banach function space X such that $X^{L^{\Phi_0}}$ is saturated, $Z = L^{\Phi_0}$ and Y such that $Y' = L^{\Phi}$ (for instance, if $(L^{\Phi})'$ has the Fatou property, take $Y = (L^{\Phi})'$). Then $Z^{Y'} = L^{\Phi_1}$. Suppose also that $\Phi_1(s^{1/p})$ and $\Phi_0(s^{1/p'})$ are convex functions. Then an application of Theorem 5.1 gives that every function h belonging to

$$X^{L^{\Phi_0}} d^*_{p,L^{\Phi_0}} ({L^{\Phi_0}}^{L^{\Phi}}) = X^{L^{\Phi_0}} d^*_{p,L^{\Phi_0}} L^{\Phi_1}$$

factorizes through an scheme

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & L^{\Phi}, \\ & L^{\Phi_0} \arrow[ur, "g"'] & \end{tikzcd}$$

where $f \in X^{L^{\Phi_0}}$ and $h \in L^{\Phi_1}$. Note that this example generalizes the case $\Phi_1(s) = s^p$ and $\Phi_0(s) = s^{p'}$, that gives the spaces $Z = L^{\Phi_0} = L^{p'}$, $Y = L^\infty$ and $Z^{Y'} = L^{\Phi_1} = L^p$.

(3) *Lorentz function spaces.* Recall that given $1 \leq p < \infty$, the Lorentz space $\Lambda_{p,w}$ is the subspace of the classes of functions f of L^0 that satisfy that the norm

$$\|f\|_{p,w} := \Big(\int_I f^{*p}(s)w(s)ds\Big)^{1/p}$$

is finite, where $I = (0,1]$ or $I = (0,\infty)$, and f^* is the decreasing rearrangement of f. It is known that these spaces are always p-convex with p-convexity constant 1 (see for instance [12, Th. 3] and the references therein). Consider the following case. Fix $1 < p < \infty$. Let $Z = \Lambda_{p',w}$ for some function w. Take numbers $1 \leq r, t < \infty$ satisfying that $r't = p$ and $t \leq p'$, and take $Y' = L^t$. Then it is known (see [4, Prop. 5.3] and the comments that follow its proof), that since $\Lambda_{p',w}$ is t-convex (recall $t \le p'$), the equalities

$$(\Lambda_{p',w})^{L^t} = \Big(\Big((\Lambda_{p',w})^{1/t}\Big)'\Big)^t = \Big((\Lambda_{p'/t,w})'\Big)^t$$

hold, where the symbol X^q represents the q-th power of the space X, see [4, 17, 19] for more information. Assume also that $\Lambda_{p'/t,w}$ is r-concave (conditions for this to hold are given in [12, Th. 7]). Then $(\Lambda_{p'/t,w})'$ is r'-convex and so $((\Lambda_{p'/t,w})')^t$ is $r't$-convex, i.e. p-convex. Consequently, if X is a Banach function space such that $X^{\Lambda_{p',w}}$ is saturated, under all the requirements exposed we obtain by Theorem 5.1 that for every function h belonging to

$$\Big(X^{\Lambda_{p',w}}\Big) d^*_{p,\Lambda_{p',w}} \Big((\Lambda_{p'/t,w})'\Big)^t$$

there are functions $f \in X^{\Lambda_{p',w}}$ and $g \in \Big((\Lambda_{p'/t,w})'\Big)^t$ such that $h = fg$. A particular choice of parameters for which the example is meaningful are given by $p = p' = 2$, $t = \sqrt{2}$ and $r' = \sqrt{2}$. For w being the constant function 1, this example gives also the case of L^p spaces, as in (2).

Let us finish this section by giving some applications related with the order continuity of the product spaces. Some interesting results follow when π is equivalent to $\tilde{\pi}$ (i.e. there exists a constant $C > 0$ such that $\tilde{\pi}(z) \le C \cdot \pi(z)$ for all $z \in X\pi Y$). For instance, this happens when there is a factorization for every function $f \in X^{Y'}$ as the one given by a strong factorization theorem (see Introduction and [24]). The following result can be applied in this setting. A general factorization theorem also holds for the π norm, even if it is not equivalent to $\tilde{\pi}$, as we have explained in Remark 4.5(b).

Proposition 5.3. *Let X and Y be saturated Banach function spaces such that X is order continuous, $X^{Y'}$ is saturated and the norm π on $X\pi Y$ is equivalent to $\tilde{\pi}$. Then, $X\pi Y$ is order continuous. Moreover, in this case, $X\pi Y$ has the Fatou property if and only if $(X^{Y'})'$ is order continuous.*

Proof. First note that the hypothesis guarantees that $X\pi Y$ is a saturated Banach function space. . Given $z \in X\pi Y$ such that $0 \le z$ μ–a.e. and $E_i \downarrow \emptyset$ in Ω, we have to prove that $\pi(z\chi_{E_i}) \to 0$ as $i \to \infty$. Denote by A the support of z. Since π and $\tilde{\pi}$ are equivalent, $\tilde{\pi}(z) < \infty$. So, there exist $0 \le x \in X$ and $0 \le y \in Y$ such that $z = |xy|$. Assume without loss of generality that $\pi = \tilde{\pi}$. Then

$$\pi(z\chi_{E_i}) \le \|x\chi_{E_i}\|_X \|y\|_Y \to 0.$$

This implies that $X\pi Y$ is order continuous.

Concerning the Fatou property, the result is consequence of the following fact: if L is an order continuous Banach function space, then it has the Fatou property if and only is L'' is order continuous. Its proof is a simple exercise taking into acount that simple functions are dense in order continuous Banach function spaces and that a Banach function space has the Fatou property if and only if $L = L''$.

$\square$

The order continuity of the spaces X and Y is not a necessary condition for $X\pi Y$ to be order continuous: it is enough to consider the case $X\pi L^\infty(\mu) = X$.

Corollary 5.4. *Let Z be a p'-convex space and let $Y^{Z'}$ be an order continuous p-convex space such that all the elements of $X^{Y'}$ factorize as in Theorem 4.4. Then $X^Z\tilde{\pi}Y^{Z'} = X^{Y'}$, and if X^Z is order continuous, then $X^{Y'}$ is order continuous.*

The proof is a direct consequence of the equality $X^Z d^*_{p,Z} Y^{Z'} = X^{Y'}$ given by Theorem 4.4 that is given by hypothesis, Theorem 5.1 and Proposition 5.3.

Remark 5.5. Clearly, by Lemma 4.3, under the hypothesis of each of the strong factorization theorems given in the introduction (Lozanovskii, Reisner and Maurey-Rosenthal), all the product spaces $X^Z\pi Y^{Z'}$, $X^Z d^*_{p,Z} Y^{Z'}$ and $X^Z \varepsilon_Z Y^{Z'}$ coincide with $X^{Y'}$.

5.2. Coincidence of the ε norm and the $d^*_{p,Z}$ norm

In this section we analyze the complementary case of strong factorization theorems for the $d^*_{p,Z}$ norm. We provide conditions under which the spaces $X^Z d^*_{p,Z} Y^{Z'}$ and $X^{Y'}$ coincide. This can happen even if there is no coincidence with the space $X^Z\pi Z^{Y'}$. Let us show some examples of this situation.

Example 5.6. *Banach function spaces satisfying $X^Z d^*_{p,Z} Y^{Z'} = X^{Y'}$ but*

$$X^Z d^*_{p,Z} Y^{Z'} \neq X^Z \pi Y^{Z'}.$$

(1) Let us consider first a simple case: take $X = \ell^2$, $Z = \ell^3$ and $Y' = \ell^1$. We have that $X^Z = (\ell^2)^{\ell^3} = \ell^\infty$, $Z^{Y'} = (\ell^3)^{\ell^1} = \ell^{3/2}$ and $X^{Y'} = (\ell^2)^{\ell^1} = \ell^2$, and then

$$X^Z \pi Z^{Y'} = (\ell^2)^{\ell^3} \pi (\ell^3)^{\ell^1} = \ell^\infty \pi \ell^{3/2} = \ell^{3/2},$$

which do not coincide with $X^{Y'} = \ell^2$. However, let us show that $X^Z d^*_{2,Z} Z^{Y'} = X^{Y'}$. Take an element $(\lambda_i) \in \ell^2 = (\ell^2)^{\ell^1}$. Then it can be written as the product $(\delta_i) \cdot (\gamma_i) = \sum_{i=1}^\infty \delta_i \lambda_i e_i$, where $\delta_i = 1$ for every i and $(\gamma_i) = (\lambda_i)$ ((e_i) is the canonical basis). But notice that

$$w^*_{2,Z}((\delta_i e_i)) = \sup_{(\tau_i)\in B_{\ell^2}} (\sum_{i=1}^\infty \|\tau_i e_i\|^2_{\ell^3})^{1/2} = \|(\tau_i)\|_{\ell^2} = 1, \quad \textit{and}$$

$$\pi_2((\gamma_i e_i)) = (\sum_{i=1}^\infty \|\gamma_i e_i\|^2)^{1/2} = \|(\lambda_i)\|_{\ell^2}.$$

Therefore $X^Z d^*_{2,Z} Z^{Y'} = (\ell^2)^{\ell^3} d^*_{2,\ell^3} (\ell^3)^{\ell^1} = \ell^2 = X^{Y'}$.

(2) Consider a measure space (Ω, Σ, μ) and a measurable partition $\{A_i\}_{i=1}^{\infty}$, $0 < \mu(A_i)$ for each $i \in \mathbb{N}$ and call μ_i to the restriction of μ to the set A_i. Take three sequences of Banach function spaces $\{X_i(\mu_i)\}_{i=1}^{\infty}$, $\{Z_i(\mu_i)\}_{i=1}^{\infty}$ and $\{Y_i'(\mu_i)\}_{i=1}^{\infty}$, $i \in \mathbb{N}$. Assume also that $X_i^{Z_i} \tilde{\pi} Y_i^{Z_i'} = X_i^{Y_i'}$ isometrically for every i.

Consider the Banach function spaces $X(\mu) = \oplus_{p'} X_i(\mu_i)$, $Z(\mu) = \oplus_q Z_i(\mu_i)$ and $Y(\mu) = \oplus_1 Y_i'(\mu_i)$, where $1 < p' < q < \infty$. Note that for the case $p = 2$, $q = 3$ and $X_i(\mu_i) = Z_i(\mu_i) = Y'(\mu_i) = \mathbb{R}$, that satisfy the requirements above, we obtain the example given in (1).

By the definition of the spaces, $X^{Y'} = (\oplus_{p'} X_i(\mu_i))^{(\oplus_1 Z_i(\mu_i))} = \oplus_p X_i^{Z_i}$. Fix $h \in X^{Y'}$; for every $x \in X$ we can consider the decompositions of h and x as the pointwise sums $\sum_{i=1}^{\infty} h_i$ and $\sum_{i=1}^{\infty} x_i$, where $h_i = h\chi_{A_i}$ and $x_i = x\chi_{A_i}$. By hypothesis for each $0 < \varepsilon < 1$ there is a couple of sequences of elements $(f_i)_{i=1}^{\infty}$ and $(g_i)_{i=1}^{\infty}$ with $f_i \in X_i^{Z_i}$ and $g_i \in Z_i^{Y_i'}$ such that $h_i = f_i g_i$, $\|f_i\| = 1$ and $(1-\varepsilon)\|g_i\| \leq \|h_i\|$, $i \in \mathbb{N}$.

We have that

$$w_{p',Z}^{*}((f_i)) = \sup_{x \in B_X} \Big(\sum_{i=1}^{\infty} \|x_i f_i\|_{Z_i}^{p'} \Big)^{1/p'} \leq \sup_{x \in B_X} \Big(\sum_{i=1}^{\infty} \|x_i\|_{X_i}^{p'} \Big)^{1/p'} \leq 1,$$

and

$$(1-\varepsilon)\pi_p((g_i)) = \Big(\sum_{i=1}^{\infty} \|g_i\|^p (1-\varepsilon) \Big)^{1/p} \leq \Big(\sum_{i=1}^{\infty} \|h_i\|^p \Big)^{1/p} = \|h\|_{X^{Y'}}.$$

This proves the equality

$$\begin{aligned} X^Z d_{p,Z}^{*} Y^{Z'} &= \Big((\oplus_{p'} X_i(\mu_i))^{\oplus_q Z_i(\mu_i)} \Big) d_{p,Z}^{*} \Big((\oplus_q Z_i(\mu_i))^{\oplus_1 Y_i'(\mu_i)} \Big) \\ &= \oplus_p X_i(\mu_i)^{Y_i'(\mu_i)} = (\oplus_{p'} X_i(\mu_i))^{\oplus_1 Y_i'(\mu_i)} = X^{Y'}, \end{aligned}$$

and, as we have shown in (1), this space do not coincide in general with the π-product.

In a sense, this example gives the rule for giving a criterion under which the inequality $d_{p,Z}^{*} \leq \varepsilon_Z$ on $X^Z d_{p,Z}^{*} Y^{Z'}$ holds. In order to do that, we introduce the following concavity type notion, that is weaker than the inequality $\pi \leq \varepsilon_Z$. We say that $X^{Y'}$ satisfies a *lower* (p, Z)*-estimate* for $1 \leq p \leq \infty$ if there is a constant $K > 0$ such that for every function h in the space we have

$$\inf \Big(\sum_{i=1}^{\infty} (\|f_i\| \|g_i\|)^p \Big)^{1/p} \leq K \|h\|$$

(changing the p-sum by the supremum in the case $p = \infty$ as usual) where the infimum is computed over all decomposition of h as a sum of disjoint products $f_i g_i$ of functions $f_i \in X^Z$ and $g_i \in Z^{Y'}$. Note that if $X^Z \tilde{\pi} Y^{Z'} = X^{Y'}$, this can be obtained as a consequence of the fact that $X^{Y'}$ has a lower p-estimate in the usual sense. Recall that a Banach lattice X is said to have a lower

p-estimate if and only if there is a constant $k > 0$ such that for every element $x \in X$,

$$\Big(\sum_{i \geq 1} \|x_i\|_X^p \Big)^{1/p} \leq k \|x\|_X$$

for all the decompositions of x as a pointwise sum of disjoint elements $x_i \in X$ (see for instance Definition 1.f.4. [15]). Notice that p-concavity implies lower p-estimate.

We say that $d^*_{p,Z}(h)$ *can be computed over disjoint sums* if it coincides with the infimum of $w^*_{p',Z}((f_i)) \cdot \pi((g_i))$ when only disjoint sums of products $f_i g_i$ satisfying $|h| \leq \sum_{i \geq 1} |f_i g_i|$ are considered.

Theorem 5.7. *Let $1 < p < \infty$ and let X be a Banach function space that satisfies a lower p'-estimate. If $X^{Y'}$ satisfies a lower (p, Z)-estimate, then $X^Z d^*_{p,Z} Y^{Z'} = X^{Y'}$. Moreover, under the assumption that $d^*_{p,Z}$ can be computed over disjoint sums the converse is also true.*

Proof. Take $\varepsilon > 0$. Consider a disjoint representation of h as disjoint products of elements of $X^Z \cdot Y^{Z'}$, $h = \sum_{i=1}^\infty f_i g_i$ such that

$$\Big(\sum_{i=1}^\infty (\|f_i\| \|g_i\|)^p \Big)^{1/p} \leq K \|f\| (1 + \varepsilon)$$

and write it as $h = \sum_{i=1}^\infty \frac{f_i}{\|f_i\|} g_i \|f_i\|$. Then, if for every $x \in X$ we write x_i for $x \chi_{A_i}$, where A_i is the support of each f_i, by the lower p'-estimate of X we obtain

$$\sup_{x \in B_X} \Big(\sum_{i=1}^\infty \|x \frac{f_i}{\|f_i\|}\|^{p'} \Big)^{1/p'} \leq \sup_{x \in B_X} \Big(\sum_{i=1}^\infty \|x_i\|^{p'} \Big)^{1/p'} \leq k,$$

and

$$\Big(\sum_{i=1}^\infty (\| \|f_i\| g_i \|)^p \Big)^{1/p} = \Big(\sum_{i=1}^\infty (\|f_i\| \|g_i\|)^p \Big)^{1/p} \leq \|h\| K (1 + \varepsilon).$$

Since this can be done for every ε, we obtain that $d^*_{p,Z}(h) \leq k K \varepsilon_Z(h)$, that implies the result. A direct computation using the formula of the $d^*_{p,Z}$ norm gives the converse result. $\square$

Corollary 5.8. *Let $1 < p < \infty$, let X satisfy a lower p'-estimate and let $X^{Y'}$ satisfy a lower p-estimate. Assume also that there is a measurable partition $\{A_i\}_{i=1}^\infty$ of Ω such that the restriction spaces X_i, Z_i and Y_i' satisfy $X_i^{Z_i} \pi Z_i^{Y_i'} = X_i^{Y_i'}$ isometrically for every $i \in \mathbb{N}$. Then $X^Z d^*_{p,Z} Y^{Z'} = X^{Y'}$.*

Remark 5.9. In this paper only the $d^*_{p,Z}$ factorization norm has been defined and studied. In the same way, it is possible to define the norms $g^*_{p,Z}$ by computing the infimum over the products $\pi_{p'}((f_i)) w^*_{p,Z}((g_i))$ and $m^*_{p,Z}$, computing in this case the infimum over all the products $w^*_{p',Z}((f_i)) w^*_{p,Z}((g_i))$. Similar results to the ones that have been shown for the norm $d^*_{p,Z}$ regarding equality between product spaces associated to the convexity properties of the spaces (Theorem 5.1) can be expected also in these cases.

Remark 5.10. For general operators on Banach function spaces, the Maurey-Rosenthal factorization theory provides factorization of operators through multiplication operators. Consequently, the results that have been obtained here gives additional information for the development of this theory and for finding more applications in the theory of operators between Banach spaces. More information about this research program can be found for instance in [5, 6, 8, 9, 19].

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