An Application of Bilinear Integration to the Quantum Scattering

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Abstract

Scattering theory has its origin in Quantum Mechanics. From the mathematical point of view it can be considered as a part of perturbation theory of self-adjoint operators on the absolutely continuous spectrum. In this work we deal with the passage from the time-dependent formalism to the stationary state scattering theory. This problem involves applying Fubini's Theorem to a spectral measure integral and a Lebesgue integral of functions that take values in spaces of operators. In our approach, we use bilinear integration in tensor product of spaces of operators with suitable topologies and generalize the results previously stated in the literature.

Key words: Scattering Theory, Time-dependent, Stationary theory, bilinear integration, tensor products.

2000 Mathematics Subject Classification: Primary: 28A25 46A32 Secondary:46N50.(Review)

1 Introduction

Scattering theory can be considered as a part of the more general perturbation theory in physics [17]. The main idea is that detailed information about an unperturbed self-adjoint operator H_0 (the free hamiltonian) enables us to draw

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conclusions about another self-adjoint operator H (the total hamiltonian $H = H_0 + V$ where V is the potential) provided that H_0 and H differ little from one to another in an appropriate sense.

Scattering theory requires classification of the spectrum based on measure theory. Given a self-adjoint operator H with the spectral family E_H in a Hilbert space $\mathcal{H}$, this can be spanned as a direct sum as folllows:

$$\mathcal{H} = \mathcal{H}^{(p)} \oplus \mathcal{H}^{(sc)} \oplus \mathcal{H}^{(ac)},$$

where the subspace $\mathcal{H}^{(p)}$ is spanned by eigenvectors of H (the typical example is given by the Hamiltonian for the harmonic oscillator that has a quadratic potential; this Hamiltonian has pure point spectrum that corresponds with the energy of bound states in Quantum Mechanics), and the subspaces $\mathcal{H}^{(sc)}$ and $\mathcal{H}^{(ac)}$ are distinguished by the conditions that the measures $X \longmapsto (E_H(X)f, f)$, where X is a Borel subset of $\mathbb{R}$, are absolutely or singularly continuous with respect to the Lebesgue measure, for every $f \in \mathcal{H}^{(ac)}$ or $f \in \mathcal{H}^{(sc)}$. Typically the singularly continuous part is absent, that is, in physical applications hamiltonians do not have the singular continuous spectrum so that $\mathcal{H}^{(sc)} = \{0\}$. Scattering theory is concerned with the structure of the absolutely continuous part $\mathcal{H}^{(ac)}$.

There are two main approaches to the mathematical formulation of quantum mechanical scattering theory, the time dependent and the time-independent or stationary scattering theory.

In the time-dependent scattering theory, we consider the time evolution of an incident particle (wave packet) under the influence of the interaction with a scattering center or with another particle. This interaction (force) is represented by the existence of a potential. The asymptotic behavior of such a wave packet in the remote past and the distant future is considered that of the free particles. The assumption is that far away from the influence area of the potential, particles must behave as a free particles with no change in the values of the observables that represent the particle. The operator which connects the two asymptotic states is the scattering operator or S-operator that is in deep relation with the physical observables of the scattering process.

In the time-independent or stationary scattering theory, one studies solutions of the time-independent Schrödinger equation with a parameter that belongs to the continuous part of the spectrum of the total hamiltonian operator. These solutions lie outside the Hilbert space and are characterized by certain asymptotic properties partly motivated by physical considerations.The observables, in particular the S-operator, are obtained from the asymptotic properties of such solutions [4].

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It has been well known for a long time that these two methods are mathematically very different. The connections between them has been a problem studied since the seventies. Of fundamental importance is the task to establish conditions for which the final objects of the calculations (the S operator) are identical in both cases. This question is not easy to answer because of the nature of the calculations in the stationary scattering formalism. This theory uses mathematical manipulations that must first be interpreted in some sense before they can be made rigorous so that it is possible to compare with the time-dependent method, which is a very well developed mathematical theory.

The recent developments in scattering theory can be found, for example, in [22] and references therein. Specially important is the work developed by M. Sh. Birman and D. R. Yafaev in stationary scattering theory and for the time-dependent theory, the work developed by Werner O. Amrein, Vladimir Georgescu, J.M. Jauch and K. B. Shina (see for example [2,3,5]). We can cite also the book from Berthier [7] and the work of J. Dereziński and C. Gérard (see for example [10] and references therein).

In this paper we focus our attention in the passage from time-dependent to the stationary formalism. Our starting point is the paper from W.O. Amrein, V. Georgescu, J.M. Jauch [4]. The principal problem to solve in this passage is the following: the basic quantities in the time-dependent theory (e.g. the wave operator) will be expressed in terms of a Bochner integral of certain operators over the time available. These formulas have been known for a long time. Operators in the Bochner integral can be expressed as a spectral integrals via the Spectral Theorem. Then the passage is achieved if we are able to interchange the two integrals and evaluate the time integral. The main problem of mathematical nature is under which conditions we can interchange the two integrals and verify that the conditions are in fact satisfied for the integrals that we encounter in scattering theory.

In order to develop this alternative approach, we have to change the definition of wave operators replacing the unitary groups by the corresponding resolvents $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ ([22]). In this stationary approach, in place of the limits in time, one has to study the boundary values in a suitable topology of the resolvents as the spectral parameter z tends to the real axis. An important advantage of the stationary approach is that it gives convenient formulas for the wave operators and the scattering matrix. The temporal asymptotics of the time-dependent Schrödinger equation is closely related to the asymptotics at large distances of solutions of the stationary Schrödinger equation:

$$-\Delta\Psi + V(x)\Psi = \lambda\Psi. \tag{1}$$

In other words, from the physical point of view, in the time-dependent for-

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malism we consider that the particle being scattered have to behave as a free particle in the far past and in the far future ($t \to \pm\infty$). In the time-independent formalism we consider that the particle being scattered have to behave as a free particle far away from the scattering center, where the influence of the potential is negligible and the total hamiltonian is practically the free hamiltonian. In terms of boundary values of the resolvent, the scattering solution, or eigenfunction of the continuous spectrum, can be constructed using the Lippmann-Schwinger equation (see for example [22]).

The paper is divided into four sections and two appendices. After the introduction, we show in Section 2 the main results on bilinear integration in tensor products of locally convex spaces in order to develop our theory. Section 3 will be devoted to the introduction of the formalism necessary for the passage from the time-dependent scattering to the stationary theory and Section 4 is devoted to the main theorem of the paper. In Appendix A, we establish some technical facts about measurability in spaces of operators and in Appendix B, we state the necessary elements of the theory of tensor products of Banach spaces in order to the correct understanding of the ideas and proofs developed in the paper.

2 Bilinear integration

Following [4], the passage from the time-dependent formalism to the time-independent one depends on the correct formulation of a Fubini's Theorem. In this section we will introduce several definitions in a very general framework that will be useful for this purpose. All the definitions and results respect to vector measures can be found in [11] and definitions and results related to bilinear integration in tensor products can be found in [14,15].

If X and Y are Banach spaces, then $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from X into Y. We write $\mathcal{L}(X,X)$ as $\mathcal{L}(X)$. The space $\mathcal{L}(X,Y)$ (or $\mathcal{L}(X)$) equipped with the strong operator topology is written as $\mathcal{L}_s(X,Y)$ (or $\mathcal{L}_s(X)$), otherwise, $\mathcal{L}(X,Y)$ and $\mathcal{L}(X)$ are assumed to have the uniform operator topology.

From here to the end of this section, let X_j and Y_j, $j = 1,2$, be Banach spaces and $(\Omega, \mathcal{S})$ a measurable space. Consider the tensor product $X_2\widehat{\otimes}_\tau Y_2$ completed for one fixed topology τ and suppose the product $X_2' \otimes Y_2'$ separates points in $X_2\widehat{\otimes}_\tau Y_2$. If $\tau = \tau_\varepsilon$, the topology defined by the injective tensor norm (for the definition of the injective tensor norms, and in general reasonable tensor norms, see for example [9,20]), this is true if either X_2 or Y_2 has the approximation property.

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Definition 2.1 A set function $m : \mathcal{S} \to \mathcal{L}(Y_1, Y_2)$ is called an *operator valued measure* if $m(\cdot)y_1 : \mathcal{S} \to Y_2$ is a σ-additive vector measure for each $y_1 \in Y_1$, in other words if $m(\cdot)$ is σ-additive in $\mathcal{L}(Y_1, Y_2)$ with the strong operator topology.

Remark 2.2 A σ-additive scalar measure μ on $\mathcal{S}$ can be considered as an operator valued measure $\mu : \mathcal{S} \to \mathcal{L}(Y_1, Y_2)$ with $Y_1 = Y_2 = \mathbb{K}$, if we define $\mu(S)y_1 = \mu(S) \cdot y_1$ with $S \in \mathcal{S}$ and $y_1 \in Y_1$. A σ-additive vector valued measure $m : \mathcal{S} \to Y_2$ can be considered as an operator valued measure $\mu : \mathcal{S} \to \mathcal{L}(Y_1, Y_2)$ with $Y_1 = \mathbb{K}$, the scalar field of Y_2, if we define $\mu(S)y_1 = y_1 \cdot \mu(S)$ with $S \in \mathcal{S}$ and $y_1 \in Y_1$, that is, the notion of operator valued measure subsumes those of σ-additive scalar and vector measures.

Definition 2.3 A function $f : \Omega \to X_2$ is said to be *integrable* with respect to a measure $m : \mathcal{S} \to Y_2$ in $X_2 \widehat{\otimes}_\tau Y_2$ if for every $x_2' \in X_2'$ and $y_2' \in Y_2'$, the scalar valued function

$$\langle f, x_2' \rangle : \omega \longmapsto \langle f(\omega), x_2' \rangle, \quad \omega \in \Omega,$$

is integrable with respect to the scalar valued measure

$$\langle m, y_2' \rangle : S \longmapsto \langle m(S), y_2' \rangle, \quad S \in \mathcal{S},$$

and for every $S \in \mathcal{S}$ there exists $(f \otimes m)(S) \in X_2 \widehat{\otimes}_\tau Y_2$ such that

$$\langle (f \otimes m)(S), x_2' \otimes y_2' \rangle = \int_S \langle f, x_2' \rangle d\langle m, y_2' \rangle, \quad \forall x_2' \in X_2', \ y_2' \in Y_2' \qquad (2)$$

Because $X_2' \otimes Y_2'$ separates points in $X_2 \widehat{\otimes}_\tau Y_2$, the operator $(f \otimes m)(S)$ is uniquely determined by (2). In the case that $Y_2 = \mathbb{K}$, the X_2-valued function f is sometimes said to be *Pettis m-integrable* in X_2 [1].

As we shall see in next sections, we have to deal with functions that have range in a space of parametrized operators between fixed Hilbert spaces and integrals of these with respect to spectral measures. This is the reason that motivates the next definition:

Definition 2.4 Let κ be a tensor product topology on $X_1 \otimes Y_1$. A function

$$f : \Omega \to \mathcal{L}(X_1, X_2)$$

is said to be *integrable* with respect to a measure

$$m : \mathcal{S} \to \mathcal{L}_s(Y_1, Y_2)$$

in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$, if for every $x_1 \in X_1$ and $y_1 \in Y_1$ the X_2-valued function

$$f x_1 : \omega \longmapsto f(\omega)x_1, \quad \omega \in \Omega,$$

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is integrable with respect to the Y_2-valued measure

$$my_1 : S \longmapsto m(S)y_1, \quad S \in \mathcal{S},$$

in the sense of the previous definition and for every $S \in \mathcal{S}$, there exists an operator $(f \otimes m)(S) \in \mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$ such that

$$[(f \otimes m)(S)](x_1 \otimes y_1) = \int_S (fx_1) \otimes d(my_1), \; x_1 \in X_1 \;\; y_1 \in Y_1, \tag{3}$$

for a suitable tensor product topology κ.

A bounded linear operator on $X_1 \widehat{\otimes}_\kappa Y_1$ in uniquely determined by its values on the dense subspace $X_1 \otimes Y_1$, so the definition makes sense. We use the same definition if $Y_1 = Y_2 = \mathbb{C}$.

If $\{(f \otimes m)(S) : S \in \mathcal{S}\}$ is an equicontinuous family of operators, then

$$S \longmapsto (f \otimes m)(S), \quad S \in \mathcal{S}$$

is σ-additive for the strong operator topology in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$.

In Section 4, we have Hilbert spaces $X_1 = Y_2 = \mathcal{D}(A)$, $X_2 = \mathcal{H}$, $Y_1 = \mathbb{C}$ with respect to a selfadjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$ defined in $\mathcal{H}$. Then we must deal with the problem of determining whether or not the tensor product

$$\mathcal{D}(A) \otimes \mathcal{H} \otimes \mathcal{D}(A)$$

separates a subspace of operators belonging to $\mathcal{L}(\mathcal{D}(A), \mathcal{H} \widehat{\otimes}_\tau \mathcal{D}(A))$ for an appropriate locally convex tensor product topology τ with respect to the duality

$$\langle T, \psi \otimes \phi \otimes \xi \rangle = \langle T\psi, \tilde{\phi} \otimes \tilde{\xi} \rangle, \quad \psi, \xi \in \mathcal{D}(A), \; \phi \in \mathcal{H}.$$

The tilde denotes the continuous linear functional corresponding to the given element of Hilbert space via the inner product. We deal with this question in detail in Appendix B.

Let Σ, Ω be two nonempty sets. If E is a subset of the cartesian product $\Sigma \times \Omega$, then

$$E(\sigma) = \{\omega : (\sigma, \omega) \in E\}, \qquad E(\omega) = \{\sigma : (\sigma, \omega) \in E\}$$

are the sections of E corresponding to fixed $\sigma \in \Sigma$ and $\omega \in \Omega$.

The proof of the following observation is a straightforward consequence of the definitions.

Proposition 2.5 *Let* $(\Sigma, \mathcal{E}, \mu)$ *be a finite measure space,* $m : \mathcal{S} \to \mathcal{L}_s(Y_1, Y_2)$ *an operator valued measure, and* $f : \Sigma \times \Omega \to \mathcal{L}_s(X_1, X_2)$ *be an operator*

valued $\mu \otimes m$-integrable function in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$, for suitables tensor topologies κ and τ.

(1) *If there exist a set $\Sigma_0 \subset \Sigma$ of full μ-measure such that for every $\sigma \in \Sigma_0$ the function $\omega \to f(\sigma, \omega)$, $\omega \in \Omega$ is m-integrable in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$, then the function*

$$\sigma \to \int_{E(\sigma)} f(\sigma, \omega) \otimes dm(\omega), \quad \sigma \in \Sigma_0$$

is μ-integrable in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$ with respect to the separating family $X_1 \otimes Y_1 \otimes X_2' \otimes Y_2'$ of continuous linear functionals on $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$ and

$$(f \otimes (\mu \otimes m))(E) = \int_\Sigma \left(\int_{E(\sigma)} f(\sigma, \omega) \otimes dm(\omega) \right) d\mu(\sigma), \ E \in \mathcal{E} \otimes \mathcal{S} \quad (4)$$

(2) *If there exists a set $\Omega_0 \subset \Omega$ of full m-measure such that for every $\omega \in \Omega_0$ the function $\sigma \to f(\sigma, \omega)$, $\sigma \in \Sigma$ is Pettis μ-integrable in $\mathcal{L}_s(X_1, X_2)$, then the function*

$$\omega \to \int_{E(\omega)} f(\sigma, \omega) d\mu(\sigma), \quad \omega \in \Omega_0$$

is m-integrable in $\mathcal{L}(X_1 \widehat{\otimes}_\kappa Y_1, X_2 \widehat{\otimes}_\tau Y_2)$ and

$$(f \otimes (\mu \otimes m))(E) = \int_\Omega \left(\int_{E(\omega)} f(\sigma, \omega) d\mu(\sigma) \right) dm(\omega), \ E \in \mathcal{E} \otimes \mathcal{S} \quad (5)$$

(3) *The existence of either*

$$(f \otimes (\mu \otimes m))(E) = \int_\Sigma \left(\int_{E(\sigma)} f(\sigma, \omega) \otimes dm(\omega) \right) d\mu(\sigma), \ E \in \mathcal{E} \otimes \mathcal{S} \quad (6)$$

or

$$(f \otimes (\mu \otimes m))(E) = \int_\Omega \left(\int_{E(\omega)} f(\sigma, \omega) d\mu(\sigma) \right) \otimes dm(\omega), \ E \in \mathcal{E} \otimes \mathcal{S} \quad (7)$$

suffices to ensure the $\mu \otimes m$-integrability of f and the equality of the integrals.

Remark 2.6 The notion of Pettis integrability can be found in several books, for example in [1].

Now let $X_2 = Y$, $Z = X_1 = Y_2$ and $Y_1 = X$ and let π be the projective tensor product topology defined in Appendix B. Suppose that $\mathcal{L}(Z, Y) \widehat{\otimes}_\pi \mathcal{L}(X, Z)$ is embedded in $\mathcal{L}(Z \widehat{\otimes}_\kappa X, Y \widehat{\otimes}_\tau Z)$ by the continuous extension map $A \otimes B \to (A \otimes B)\widetilde{\ }$ defined for $A \in \mathcal{L}(Z, Y)$ and $B \in \mathcal{L}(X, Z)$ by

$$(A \otimes B)\widetilde{\ }(z \otimes x) = (Az) \otimes (Bx), \ z \in Z, \ x \in X$$

This is always true if X, Y, Z are Hilbert spaces and κ, τ are Hilbert space tensor product topologies. Then the continuous linear map given by $J : A \otimes B \to AB$, $A \in \mathcal{L}_s(Z, Y)$ and $B \in \mathcal{L}_s(X, Z)$, is defined on the dense linear subspace $\mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z)$ of $\mathcal{L}(Z, Y) \widehat{\otimes}_\pi \mathcal{L}(X, Z)$. There may be other subspaces of $\mathcal{L}(Z \widehat{\otimes}_\kappa X, Y \widehat{\otimes}_\tau Z)$ where J is defined. Such a kind of map J will be fundamental in our discussion in Section 4 about the interchange of spectral integrals and time integrals as a previous step for our Fubini's Theorem.

3 Mathematical framework for the scattering problem

In this section we are going to focus on the problem of the passage from time-dependent to the stationary formalism. Following the reference [4], we would like to be able to integrate the operator valued function f_ϵ defined by

$$f_\epsilon(t, \lambda) = e^{-i(\lambda - i\epsilon)t} V_t^* V \in \mathcal{L}(\mathcal{D}(A), \mathcal{H})$$

for $t \geq 0$ and $\lambda \in \mathbb{R}$, where $V_t = e^{-it(A+V)}$ for $t \in \mathbb{R}$. Here V_t is a parametrized family of unitary operators, A is a self-adjoint operator (the free Hamiltonian) and V represents the potential, a real symmetric operator. The domain $\mathcal{D}(A)$ of A is equipped with the graph norm

$$\phi \longmapsto (\|A\phi\|^2 + \|\phi\|^2)^{\frac{1}{2}}, \quad \phi \in \mathcal{D}(A), \tag{8}$$

under which it becomes a Hilbert space. We shall require that $\mathcal{D}(A)$ is included in the domain $\mathcal{D}(V)$ of the operator V and that $A + V$ is selfadjoint as an operator with domain $\mathcal{D}(A)$.

The treatment of the function f_ϵ above is representative of the many identities established in [4] concerning the connection between stationary state and time-dependent scattering theory by applying a version of Fubini's Theorem and taking the limit as $\epsilon \to 0+$.

If F is the spectral measure associated with A, then informally, we have

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$$\int_{\mathbb{R}} f_\epsilon(t,\lambda)\, d(F\psi)(\lambda) = \int_{\mathbb{R}} e^{-i(\lambda-i\epsilon)t} V_t^* V\, d(F\psi)(\lambda)$$

$$= V_t^* V \int_{\mathbb{R}} e^{-i(\lambda-i\epsilon)t}\, d(F\psi)(\lambda) \tag{9}$$

$$= e^{-\epsilon t} V_t^* V e^{-itA}\psi$$

and

$$\int_0^\infty \int_{\mathbb{R}} f_\epsilon(t,\lambda)\, d(F\psi)(\lambda)\, dt = \int_0^\infty e^{-\epsilon t} V_t^* V e^{-itA}\psi\, dt$$

for each $\psi \in \mathcal{D}(A)$. On reversing the order of integration, we need to integrate the $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$-valued function $\lambda \longmapsto \int_0^\infty V_t^* V e^{-i(\lambda-i\epsilon)t}\, dt$, $\lambda \in \mathbb{R}$ with respect to the $\mathcal{D}(A)$-valued measure $F\psi$.

Because F is a spectral measure, the classical approaches to bilinear integration [6,12], are not well adapted to integrating an operator valued function with respect to the vector measure $F\psi$. Typically, the $\mathcal{H}$-valued measure $B \longmapsto (F\psi)(B)$, $B \in \mathcal{B}(\mathbb{R})$, will have infinite variation on every set of positive measure, so the theory of [6,12] is inapplicable.

The approach of the present work is suggested by equation (9) where for each $t \in \mathbb{R}_+ = [0,\infty)$, the operator $V_t^* V$ is taken outside the integral, that is, the integrals with respect to t and λ are "decoupled" when the iterated integral is written in this order. "Decoupling" is a feature of many recent applications of bilinear integration to diverse areas of analysis, see [16]. If we use tensor product notation [14], then we have

$$\int_0^\infty \int_{\mathbb{R}} f_\epsilon(t,\lambda) \otimes d(F\psi)(\lambda)\, dt = \int_0^\infty e^{-\epsilon t} V_t^* V \otimes e^{-itA}\psi\, dt.$$

Applying the *evaluation map* $J : T \otimes x \longmapsto Tx$, informally we have

$$\int_0^\infty \int_{\mathbb{R}} f_\epsilon(t,\lambda)\, d(F\psi)(\lambda)\, dt := J \int_0^\infty \int_{\mathbb{R}} f_\epsilon(t,\lambda) \otimes d(F\psi)(\lambda)\, dt$$

$$= J \int_0^\infty e^{-\epsilon t} V_t^* V \otimes (e^{-itA}\psi)\, dt$$

$$= \int_0^\infty e^{-\epsilon t} J(V_t^* V \otimes (e^{-itA}\psi))\, dt$$

$$= \int_0^\infty e^{-\epsilon t} V_t^* V e^{-itA}\psi\, dt$$

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Writing the iterated integrals in the opposite order:

$$\int\limits_{\mathbb{R}} \int\limits_0^\infty f_\epsilon(t,\lambda)\, dt\, d(F\psi)(\lambda) := J \int\limits_{\mathbb{R}} \left(\int\limits_0^\infty f_\epsilon(t,\lambda) dt \right) \otimes d(F\psi)(\lambda)$$

$$= J \int\limits_{\mathbb{R}} \left(\int\limits_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda}\, dt \right) \otimes d(F\psi)(\lambda) \qquad (10)$$

$$= \int\limits_{\mathbb{R}} J \left(\left(\int\limits_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda}\, dt \right) \otimes d(F\psi)(\lambda) \right)$$

$$= \int\limits_{\mathbb{R}} \left(\int\limits_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda}\, dt \right) d(F\psi)(\lambda). \qquad (11)$$

Making sense of the last equations poses a problem. Our solution is to *define* the integral (11) of the operator valued function $\lambda \longmapsto \int_0^\infty V_t^* V e^{-i(\lambda - i\epsilon)t}\, dt$, $\lambda \in \mathbb{R}$ with respect to the vector measure $F\psi$ to be equal to the element of $\mathcal{H}$ on the right hand side of equation (10) and then appeal to a vector version of Fubini's theorem for tensor product valued integrals.

The evaluation map $J : T \otimes x \longmapsto Tx$, $x \in \mathcal{D}(A)$, $T \in \mathcal{L}(\mathcal{D}(A), \mathcal{H})$ extends linearly to the vector space $\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$ of all finite linear combinations $\sum_j c_j(T_j \otimes x_j)$ of tensor products $T_j \otimes x_j$. Integrating a function involves taking the limit of integrals of a sequence of elementary functions, so the integral $\int_0^\infty e^{-\epsilon t} V_t^* V \otimes (e^{-itA}\psi)\, dt$ will belong to some suitable *completion* $\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes} \mathcal{D}(A)$ of the linear space $\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$. The choice of a suitable complete linear tensor product space $\mathcal{L}(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes} \mathcal{D}(A)$ is fundamental to our approach.

To motivate our solution, we look at a simple (non-physical) example. Suppose that A and V are *bounded* selfadjoint operators acting on the Hilbert space $\mathcal{H}$. Let $V_t = e^{-it(A+V)}$ be the unitary group generated by the bounded linear operator $-i(A+V)$. We are now seeking a linear space $\mathcal{L}(\mathcal{H}) \widetilde{\otimes} \mathcal{H}$ in which the tensor product integral

$$\int\limits_0^\infty e^{-\epsilon t} V_t^* V \otimes (e^{-itA}\psi)\, dt \qquad (12)$$

belongs, and for which the evaluation map $J : \mathcal{L}(\mathcal{H}) \otimes \mathcal{H} \to \mathcal{H}$ has a continuous linear extension from $\mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$ to $\mathcal{L}(\mathcal{H}) \widetilde{\otimes} \mathcal{H}$. An obvious candidate is obtained by noting that for $T_j \in \mathcal{L}(\mathcal{H})$, $x_j \in \mathcal{H}$, $j = 1, \ldots, n$, we have

$$\| J(\sum_{j=1}^n T_j \otimes x_j) \| \leq \sum_{j=1}^n \| T_j x_j \| \leq \sum_{j=1}^n \| T_j \| . \| x_j \|,$$

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for all $n = 1, 2, \ldots$. The *projective tensor product* $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$ of the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ with the Hilbert space $\mathcal{H}$ is obtained by taking the completion of the linear space $\mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$ with respect to the norm

$$\|u\|_\pi = \inf \left\{ \sum_j \|T_j\|.\|x_j\| : u = \sum_j T_j \otimes x_j \right\}, \quad u \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{H}.$$

The infimum is over all possible representations of u in the linear space $\mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$. It is well known that any element u of $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$ can be represented as $u = \sum_{j=1}^\infty T_j \otimes x_j$ with $\sum_{j=1}^\infty \|T_j\|.\|x_j\| < \infty$ [9,20]. Then the evaluation map $J : T \otimes x \longmapsto Tx$ has a unique continuous linear extension $\widetilde{J} : \mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H} \to \mathcal{H}$ given by

$$Ju = \sum_{j=1}^\infty T_j x_j$$

for any representation $u = \sum_{j=1}^\infty T_j \otimes x_j$ with $\sum_{j=1}^\infty \|T_j\|.\|x_j\| < \infty$.

We now see that under the assumptions that A and V are bounded selfadjoint operators, the integral (12) is actually the Bochner integral of the $\mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$-valued function

$$t \longmapsto e^{-\epsilon t} V_t^* V \otimes (e^{-itA} \psi), \ t > 0, \tag{13}$$

in the *projective* tensor product $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$. The projective tensor product norm is the *strongest* reasonable cross norm [11, Chap. 8].

Because A and V are bounded linear operators, the $\mathcal{L}(\mathcal{H})$-valued functions $t \longmapsto V_t^* V$, $t > 0$, and $t \longmapsto e^{-itA}$, $t > 0$ are continuous for the *uniform operator topology*. It follows that the $\mathcal{L}(\mathcal{H}) \otimes \mathcal{H}$-valued function

$$t \longmapsto V_t^* V \otimes (e^{-itA} \psi), \quad t > 0,$$

is continuous for the projective tensor product norm $\| \cdot \|_\pi$ and

$$\int_0^\infty e^{-\epsilon t} \|V_t^* V\|.\|(e^{-itA} \psi)\| \, dt \leq \|V\| \|\psi\| / \epsilon.$$

Consequently, the function (13) is Bochner integrable in the projective tensor product $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$. Moreover, the continuous $\mathcal{H}$-valued function

$$t \longmapsto e^{-\epsilon t} V_t^* V (e^{-itA} \psi), \ t > 0,$$

is Bochner integrable in $\mathcal{H}$ and the equalities

$$J \int_0^\infty e^{-\epsilon t} V_t^* V \otimes e^{-itA} \psi \, dt = \int_0^\infty e^{-\epsilon t} J \left(V_t^* V \otimes e^{-itA} \psi \right) dt$$

$$= \int_0^\infty e^{-\epsilon t} V_t^* V \left(e^{-itA} \psi \right) dt$$

hold [11, Theorem II.2.6]. Next, we see that the integral

$$\int_{\mathbb{R}} \left(\int_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda} \, dt \right) \otimes d(F\psi)(\lambda) \tag{14}$$

exists as an element of the projective tensor product $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$ and that the equality

$$\int_{\mathbb{R}} \left(\int_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda} \, dt \right) \otimes d(F\psi)(\lambda) = \int_0^\infty e^{-\epsilon t} V_t^* V \otimes e^{-itA} \psi \, dt$$

is valid, so that the equality

$$J \int_{\mathbb{R}} \left(\int_0^\infty e^{-\epsilon t} V_t^* V e^{-it\lambda} \, dt \right) \otimes d(F\psi)(\lambda) = \int_0^\infty e^{-\epsilon t} V_t^* V (e^{-itA} \psi) \, dt \tag{15}$$

also holds.

In fact, for $x, h, \xi \in \mathcal{H}$, the scalar version of Fubini's Theorem implies that

$$\int_S \left(\int_0^\infty e^{-\epsilon t} \langle V_t^* V x, h \rangle e^{-it\lambda} \, dt \right) d\langle F\psi, \xi \rangle(\lambda)$$

$$= \int_0^\infty e^{-\epsilon t} \langle V_t^* V x, h \rangle \left(\langle e^{-itA} F(S)\psi, \xi \rangle \right) dt$$

$$= \left\langle \int_0^\infty e^{-\epsilon t} V_t^* V \otimes (e^{-itA} F(S)\psi) \, dt, x \otimes h \otimes \xi \right\rangle,$$

so the integral (14) must belong to $\mathcal{L}(\mathcal{H}) \widehat{\otimes}_\pi \mathcal{H}$ too and equation (15) is valid.

The above argument breaks down for unbounded A and V because the $t \longmapsto V_t^* V$, $t > 0$, and $t \longmapsto e^{-itA}$, $t > 0$, are no longer continuous for the *uniform operator topology*. We can only expect them to be continuous in the *strong operator topology* on their respective domains. Moreover, strong measurability in the *uniform operator norm* no longer holds, so applying the Bochner integral in the Banach space $\mathcal{L}(\mathcal{H})$ is no longer an option.

For unbounded densely defined selfadjoint operators A and V, we modify the preceding argument, essentially by replacing the uniform operator topology of $\mathcal{L}(\mathcal{H})$ in the projective tensor product $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\pi \mathcal{H}$ by the strong operator topology on the appropriate space of operators. The projective tensor product $\mathcal{L}(\mathcal{H})\widehat{\otimes}_\pi \mathcal{H}$ is auxiliary to the final definition of the integral (11) but it is a feature of the "decoupling" approach to bilinear integration that allows the integration of operator valued functions with respect to spectral measures. An argument analogous to the one above holds, but first, certain technical difficulties which we now describe must be overcome. This is essentially the purpose of the next section.

4 Fubini's Theorem for the passage from time-dependent to the stationary formalism

This last section contains the main result of this paper: adopting a new mathematical perspective (the bilinear integration in tensor products) we want to extend and generalize the results obtained in [4] and open a new path for revisiting these problems. We start with the Theorem 3 of this reference:

Theorem 4.1 (W.O. Amrein, V. Georgescu & J.M. Jauch) *Let:*

I. *F_λ be a spectral family defining a self-adjoint operator $A = \int \lambda dF_\lambda$ in a separable Hilbert space $\mathcal{H}$.*
II. *(a,b) and (c,d) two (finite or infinite) intervals on the real line and*

$$u : (a,b) \times (c,d) \longrightarrow \mathbb{C}$$

a complex valued function denoted by $u(\lambda,t)$, $\lambda \in (a,b)$ and $t \in (c,d)$.
III. *B_t ($t \in \mathbb{R}$) a family of (not necessarily bounded) linear operators in $\mathcal{H}$.*
IV. *$\psi \in \mathcal{D}_A$ a fixed vector in the domain of A.*

Assume:

1. *The integrals $\int_a^b u(\lambda,t)dF_\lambda\psi$ and $\int_a^b u(\lambda,t)dF_\lambda A\psi$ exist for all $t \in (c,d)$.*
2. *For all $t \in (c,d)$ one has $\mathcal{D}_A \subset \mathcal{D}_{B_t}$, and there exist positive constants $\alpha(t)$, $\beta(t)$, such that for every $\varphi \in \mathcal{D}_A$*

$$\|B_t\varphi\| \le \alpha(t)\|A\varphi\| + \beta(t)\|\varphi\|$$

3. *For all $\varphi \in \mathcal{D}_A$ and for all $\lambda \in (a,b)$ the function $t \longmapsto u(\lambda,t)B_t\varphi$ is Bochner integrable on (c,d).*
4. *There exists a function $v : (c,d) \longrightarrow \mathbb{R}$ such that*
(a) *$|u(\lambda,t)| \le v(t)$ for all $\lambda \in (a,b)$ and $t \in (c,d)$.*
(b) *$t \longmapsto v(t)\left(\alpha(t)\|A\psi\| + \beta(t)\|\psi\|\right)$ is Lebesgue integrable in (c,d).*

Then the existence of one of the integrals below entails the existence of the other one and $W = W'$

$$W = \int\limits_c^d B_t \left(\int\limits_a^b u(\lambda, t) dF_\lambda \psi \right) dt, \qquad W' = \int\limits_a^b \left(\int\limits_c^d u(\lambda, t) B_t dt \right) dF_\lambda \psi.$$

The integrals W and W' above are defined as a type of limit of Riemann sums.

In Theorem 4.4 below, our assumptions imply the *existence* of the integrals W and W' in the sense described below and it follows that $W = W'$. First we need some results that we state in the form of Lemma 4.2 and Proposition 4.3 to prove Theorem 4.4.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $(\cdot|\cdot)_\mathcal{H}$, linear in the first variable and antilinear in the second. Suppose that $A : \mathcal{D}(A) \to \mathcal{H}$ a selfadjoint operator with domain $\mathcal{D}(A)$, equipped with the graph norm (8), under which it becomes a separable Hilbert space with inner product $(\cdot|\cdot)_{\mathcal{D}(A)}$. Then the corresponding Hilbert space norms are given by

$$\|\phi\|_\mathcal{H} = (\phi|\phi)_\mathcal{H}^{\frac{1}{2}}, \ \phi \in \mathcal{H} \text{ and } \|\psi\|_{\mathcal{D}(A)} = (\psi|\psi)_{\mathcal{D}(A)}^{\frac{1}{2}}, \ \psi \in \mathcal{D}(A).$$

If $T : \mathcal{D}(A) \to \mathcal{H}$ is a continuous linear map, then its Hilbert space adjoint $T^* : \mathcal{H} \to \mathcal{D}(A)$ is defined by the formula

$$(T\psi|\phi)_\mathcal{H} = (\psi|T^*\phi)_{\mathcal{D}(A)}, \quad \phi \in \mathcal{H}, \ \psi \in \mathcal{D}(A).$$

Let $\mu : \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}_+$ be a σ-finite measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of $\mathbb{R}_+$. A function $f : \mathbb{R}_+ \to X$ with values in a Banach space X is said to be *strongly μ-measurable* if it is the limit μ-almost everywhere of X-valued Borel simple functions.

A strongly μ-measurable function $f : \mathbb{R}_+ \to X$ is *Bochner μ-integrable* in X if and only if

$$\int\limits_{\mathbb{R}_+} \|f(t)\| \, d\mu(t) < \infty.$$

Equivalently, there exist X-valued Borel simple functions s_j, $j = 1, 2, \ldots$, converging to f μ-a.e. such that

$$\int\limits_{\mathbb{R}_+} \|s_j - s_k\|_X \, d\mu \longrightarrow 0 \text{ as } j, k \to \infty.$$

Then $\int_B f \, d\mu = \lim_{n \to \infty} \int_B s_n \, d\mu$ for each $B \in \mathcal{B}(\mathbb{R}_+)$. In particular, f is Pettis μ-integrable and the integrals agree.

Let X, Y be Banach spaces. A function $f : \mathbb{R}_+ \to \mathcal{L}(X, Y)$ is said to be *strongly μ-measurable* in $\mathcal{L}_s(X, Y)$ if and only if there exist $\mathcal{L}(X, Y)$-valued Borel simple functions s_j, $j = 1, 2, \ldots$, converging to f μ-a.e.

A function $f : \mathbb{R}_+ \to \mathcal{L}(X, Y)$ is said to be *strongly Bochner μ-integrable* in $\mathcal{L}(X, Y)$ if and only if there exist $\mathcal{L}(X, Y)$-valued Borel simple functions s_j, $j = 1, 2, \ldots$, converging to f μ-a.e. such that

$$\int_{\mathbb{R}_+} \|s_j x - s_k x\|_Y \, d\mu \longrightarrow 0 \text{ as } j, k \to \infty,$$

for every $x \in X$. The function f is necessarily strongly μ-measurable in $\mathcal{L}_s(X, Y)$. Then

$$\int_S f \, d\mu = \lim_{j \to \infty} \int_S s_j \, d\mu, \quad S \in \mathcal{B}(\mathbb{R}_+),$$

in the strong operator topology and

$$\int_{\mathbb{R}_+} \|f(t)x\|_Y \, d\mu(t) = \lim_{j \to \infty} \int_{\mathbb{R}_+} \|s_j(t)x\|_Y \, d\mu(t)$$

is finite for each $x \in X$, see Appendix A.

Lemma 4.2 *Let $\mu : \mathcal{B}(\mathbb{R}_+) \to \mathbb{R}_+$ be a σ-finite measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of $\mathbb{R}_+$. Suppose that $B : \mathbb{R}_+ \to \mathcal{L}(\mathcal{D}(A), \mathcal{H})$ with the following properties:*

(i) *the $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$-valued function $t \longmapsto B(t)$, $t \in \mathbb{R}_+$, is strongly μ-measurable in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H})$ and*

(ii) *there exist μ-integrable functions $t \mapsto \alpha(t)$, $t \in \mathbb{R}_+$, and $t \mapsto \beta(t)$, $t \in \mathbb{R}_+$ such that*

$$\|B(t)\psi\|_{\mathcal{H}} \leq \alpha(t)\|A\psi\|_{\mathcal{H}} + \beta(t)\|\psi\|_{\mathcal{H}}$$

for all $\psi \in \mathcal{D}(A)$ and $t \in \mathbb{R}_+$.

Then the following statements hold.

(a) $\sup_{\|\Psi\|_\infty \leq 1} \int_0^\infty \|B(t)\Psi(t)\|_{\mathcal{H}} \, d\mu(t) < \infty$. *The supremum is taken over all simple functions $\Psi : \mathbb{R}_+ \to \mathcal{D}(A)$ uniformly norm bounded in $\mathcal{D}(A)$ by one.*

(b) *The $\mathcal{H}$-valued function $t \longmapsto B(t)\Psi(t)$, $t > 0$, is Bochner μ-integrable in $\mathcal{H}$ for every uniformly bounded, strongly μ-measurable function $\Psi : \mathbb{R}_+ \to \mathcal{D}(A)$.*

(c) $\int_0^\infty \|B(t)^*\phi\|_{\mathcal{D}(A)} \, d\mu(t) < \infty$ *for every $\phi \in \mathcal{H}$.*

Proof. For any Borel simple function $\Psi : \mathbb{R}_+ \to \mathcal{D}(A)$, by (ii) we have

$$\|B(t)\Psi(t)\|_{\mathcal{H}} \leq \alpha(t)\|A\Psi(t)\|_{\mathcal{H}} + \beta(t)\|\Psi(t)\|_{\mathcal{H}}$$

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$$\leq (\alpha(t) + \beta(t)) \max(\|A\Psi(t)\|_{\mathcal{H}}, \|\Psi(t)\|_{\mathcal{H}})$$
$$\leq (\alpha(t) + \beta(t))\|\Psi(t)\|_{\mathcal{D}(A)},$$

so that (a) follows from condition (ii). Part (b) follows by approximating Ψ pointwise in $\mathcal{D}(A)$ by a uniformly bounded sequence of $\mathcal{D}(A)$-valued Borel simple functions and appealing to Part (a). Part (c) follows from the observation that

$$\sup_{\|\Psi\|_\infty \leq 1} \int_0^\infty |(\Psi(t)|B(t)^*\phi)_{\mathcal{D}(A)}| \, d\mu(t)$$

$$\leq \|\phi\|_{\mathcal{H}}. \sup_{\|\Psi\|_\infty \leq 1} \int_0^\infty \|B(t)\Psi(t)\|_{\mathcal{H}} \, d\mu(t) < \infty, \quad \phi \in \mathcal{H}.$$

$\square$

Denote by $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$ the locally convex space tensor product induced from the linear map defined by $S \otimes x \longmapsto x \otimes S^*$ into the projective tensor product $\mathcal{D}(A)\widetilde{\otimes}_\pi \mathcal{L}_s(\mathcal{H}, \mathcal{D}(A))$. Then the linear map induced by the evaluation map $J : S \otimes x \mapsto Sx$ is the restriction to $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H}) \otimes \mathcal{D}(A)$ of a unique continuous linear map from $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$ into the weak topology of $\mathcal{H}$. We denote the continuous extension by J as well. The locally convex space $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$ is sequentially complete and

$$J : \mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A) \longrightarrow \mathcal{H}$$

is a continuous linear map for the weak topology of $\mathcal{H}$. The details of this construction and the following basic result are given in Appendix B.

Proposition 4.3 $\mathcal{D}(A) \otimes \mathcal{H} \otimes \mathcal{D}(A)$ *separates the lcs* $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$.

As in Definition 2.3, we say that an $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})$-valued function

$$f : \mathbb{R}_+ \longrightarrow \mathcal{L}_s(\mathcal{D}(A),\mathcal{H})$$

is $F\psi$-integrable in $\mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$ for $\psi \in \mathcal{D}(A)$, if $\langle f\phi, h\rangle$ is $\langle F\psi, \eta\rangle$-integrable for each $\phi \in \mathcal{D}(A)$, $h \in \mathcal{H}$ and $\eta \in \mathcal{D}(A)$ and for each $S \in \mathcal{B}(\mathbb{R})$, there exists a vector

$$(f \otimes (F\psi))(S) \in \mathcal{L}_s(\mathcal{D}(A),\mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$$

such that

$$\langle (f \otimes (F\psi))(S), \phi \otimes h \otimes \eta\rangle = \int_S \langle f(\sigma)\phi, h\rangle \, d\langle F\psi, \eta\rangle(\sigma)$$

for every $\phi \in \mathcal{D}(A)$, $h \in \mathcal{H}$ and $\eta \in \mathcal{D}(A)$. By virtue of Proposition 4.3, the integral is well defined, because if $T_1, T_2 \in \mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes}_\tau \mathcal{D}(A)$ satisfy

$$\langle (T_1, \phi \otimes h \otimes \eta \rangle = \langle (T_2, \phi \otimes h \otimes \eta \rangle$$

for every $\phi \in \mathcal{D}(A)$, $h \in \mathcal{H}$ and $\eta \in \mathcal{D}(A)$, then necessarily $T_1 = T_2$.

Here we are considering the tensor product space

$$\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes}_\tau \mathcal{D}(A)$$

as a dense subspace of the linear space $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H} \widetilde{\otimes}_\kappa \mathcal{D}(A))$ of operators for a suitable locally convex tensor product topology κ on $\mathcal{H} \otimes \mathcal{D}(A)$, see Remark B.10.

If $f : \mathbb{R}_+ \to \mathcal{L}_s(\mathcal{D}(A), \mathcal{H})$ is $F\psi$-integrable in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes}_\tau \mathcal{D}(A)$ for $\psi \in \mathcal{D}(A)$, then we define

$$\int_S f(\sigma) \, d(F\psi)(\sigma) := J[f \otimes (F\psi))(S)].$$

We can also write this definition more suggestively as

$$\int_S f(\sigma) \, d(F\psi)(\sigma) := J \int_S f(\sigma) \otimes d(F\psi)(\sigma),$$

where $J(T \otimes x) = Tx$ for every $T \in \mathcal{L}_s(\mathcal{D}(A), \mathcal{H})$ and $x \in \mathcal{D}(A)$. Because J maps the space $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes}_\tau \mathcal{D}(A)$ to $\mathcal{H}$, the integral $\int_S f(\sigma) \, d(F\psi)(\sigma)$ is an element of $\mathcal{H}$ for each $S \in \mathcal{B}(\mathbb{R})$.

Theorem 4.4 *Let $\psi \in \mathcal{D}(A)$. Suppose $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$ is a measurable function for which $u(\,\cdot\,, t)$ is $(F\psi)$-integrable and $F(A\psi)$-integrable in $\mathcal{H}$ for each $t > 0$.*

Suppose that the $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$-valued function $t \longmapsto B(t)$, $t \in \mathbb{R}_+$, is strongly μ-measurable in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H})$ and there exists a measurable function $v : \mathbb{R}_+ \to [0, \infty)$ with the following properties:

(i) *$|u(\lambda, t)| \le v(t)$ for all $\lambda \in \mathbb{R}$, $t \in \mathbb{R}_+$,*
(ii) *The function $t \longmapsto v(t) \|B(t)^* \phi\|_{\mathcal{D}(A)}$, $t > 0$ is integrable for each $\phi \in \mathcal{H}$.*

Then,

(1) *for each $S \in \mathcal{B}(\mathbb{R})$, the function*

$$t \longmapsto B(t) \int_S u(\lambda, t) \, d(F\psi)(\lambda), \quad t > 0,$$

is integrable in $\mathcal{H}$,

(2) *the function* $t \longmapsto u(\lambda, t) B(t) \phi$, $t > 0$, *is integrable in* $\mathcal{H}$ *for each* $\phi \in \mathcal{D}(A)$
     and $\lambda \in \mathbb{R}$,

(3) *for each* $T \in \mathcal{B}(\mathbb{R}_+)$, *the* $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$-*valued function* $\lambda \longmapsto \int_T u(\lambda, t) B(t) \, dt$,
     $\lambda \in \mathbb{R}$ *is* $(F\psi)$-*integrable in* $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \widetilde{\otimes}_\tau \mathcal{D}(A)$.

Moreover, the equality

$$\int_T B(t) \left(\int_S u(\lambda, t) \, d(F\psi)(\lambda) \right) dt = \int_S \left(\int_T u(\lambda, t) B(t) \, dt \right) d(F\psi)(\lambda)$$

holds for every $S \in \mathcal{B}(\mathbb{R})$, $T \in \mathcal{B}(\mathbb{R}_+)$.

If, furthermore, there exist measurable functions $t \mapsto \alpha(t)$, $t > 0$, *and* $t \mapsto \beta(t)$, $t > 0$ *such that*

$$\|B(t)\phi\| \leq \alpha(t)\|A\phi\| + \beta(t)\|\phi\|, \ \text{ for all } \phi \in \mathcal{D}(A)$$

and $\int_{\mathbb{R}_+} v(t)(\alpha(t) + \beta(t)) \, dt < \infty$, *then the integrals in* (1) *and* (2) *are Bochner integrals.*

Proof. The spectral measure F is the resolution of the identity of the self-adjoint operator A, so for every Borel subset B of $\mathbb{R}$, we have $F(B)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $F(B)A\psi = AF(B)\psi$ for every $\psi \in \mathcal{D}(A)$.

We first observe that $\int_S u(\lambda, t) \, d(F\psi)(\lambda) \in \mathcal{D}(A)$ for each $t > 0$ and $S \in \mathcal{B}(\mathbb{R})$ because $F\psi$ is countably additive with respect to the topology defined by the graph norm of A and $u(\cdot, t)$ is $(F\psi)$-integrable in $\mathcal{D}(A)$. Moreover, by condition (i),

$$\left\| \int_S u(\lambda, t) \, d(F\psi)(\lambda) \right\|_{\mathcal{D}(A)} \leq v(t)\|F\psi\|_{sv(\mathcal{D}(A))}$$

with respect to the $\mathcal{D}(A)$-semivariation $\|F\psi\|_{sv(\mathcal{D}(A))}$ [11, Chapter 1] of the vector measure $F\psi$. Consequently,

$$\left| \left\langle B(t) \int_S u(\lambda, t) \, d(F\psi)(\lambda), \phi \right\rangle \right| = \left| \left\langle \int_S u(\lambda, t) \, d(F\psi)(\lambda), B(t)^*\phi \right\rangle \right|$$
$$\leq v(t)\|F\psi\|_{sv(\mathcal{D}(A))}\|\|B(t)^*\phi\|_{\mathcal{D}(A)}.$$

By condition (ii), the function $t \longmapsto B(t) \int_S u(\lambda, t) \, d(F\psi)(\lambda)$ is scalarly integrable in $\mathcal{H}$ and so integrable in $\mathcal{H}$, because $\mathcal{H}$ is reflexive [11]. This establishes (1). A similar estimate proves (2).

Because

$$\int_{\mathbb{R}} \left| \left\langle \int_T u(\lambda, t) B(t) \phi \, dt, h \right\rangle \right| d|\langle F\psi, \eta \rangle|$$

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$$\leq \|\eta\|_{\mathcal{D}(A)}\|\phi\|_{\mathcal{D}(A)}\|F\psi\|_{sv(\mathcal{D}(A))} \int\limits_0^\infty v(t)\|B(t)^*h\|_{\mathcal{D}(A)}\, dt,$$

the scalar-valued function $\lambda \longmapsto \langle \int_T u(\lambda,t)B(t)\phi\, dt, h\rangle$, $\lambda \in \mathbb{R}$ is $\langle F\psi, \eta\rangle$-integrable. Moreover, $(\lambda, t) \longmapsto \langle u(\lambda,t)B(t)\phi, h\rangle$ is $\langle F\psi, \eta\rangle \otimes dt$-integrable on $\mathbb{R} \times \mathbb{R}_+$ by the Fubini-Tonelli theorem and we have

$$\int\limits_S \langle \int\limits_T u(\lambda,t)B(t)\phi\, dt, h\rangle\, d\langle F\psi, \eta\rangle(\lambda) = \int\limits_T \langle B(t)\phi, h\rangle \left(\int\limits_S u(\lambda,t)\, d\langle F\psi, \eta\rangle(\lambda)\right) dt.$$

For each $S \in \mathcal{B}(\mathbb{R})$, the function

$$t \longmapsto B(t) \otimes \left(\int\limits_S u(\lambda,t)\, d(F\psi)(\lambda)\right), \quad t > 0, \tag{16}$$

is Bochner integrable in the quasicomplete space $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$. To see this, observe that the function given by (16) certainly has values in the tensor product $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$. For the continuous seminorm p_h, $h \in \mathcal{H}$, on $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \otimes_\tau \mathcal{D}(A)$ given by

$$p_h(\alpha) = \inf\{\sum_j \|\psi_j\|.\|T_j^*h\|_{\mathcal{D}(A)} : \alpha = \sum_j T_j \otimes \psi_j\}, \quad \alpha \in \mathcal{L}_s(\mathcal{D}(A), H) \otimes \mathcal{D}(A),$$

we have

$$p_h\left(B(t) \otimes \left(\int\limits_S u(\lambda,t)\, d(F\psi)(\lambda)\right)\right) \leq v(t)\|B(t)^*h\|_{\mathcal{D}(A)}\|F\psi\|_{sv},$$

so by condition ii) above, it follows that

$$\int\limits_0^\infty p_h\left(B(t) \otimes \left(\int\limits_S u(\lambda,t)\, d(F\psi)(\lambda)\right)\right) dt \leq \|F\psi\|_{sv} \int\limits_0^\infty v(t)\|B(t)^*h\|_{\mathcal{D}(A)}\, dt < \infty.$$

It is not hard to see that the function (16) is the pointwise limit in $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H}) \otimes \mathcal{D}(A)$ of simple functions, from which Bochner integrability follows.

As in Proposition 2.5, it follows that the $\mathcal{L}(\mathcal{D}(A), \mathcal{H})$-valued function $\lambda \longmapsto \int_T u(\lambda,t)B(t)\, dt$, $\lambda \in \mathbb{R}$ is $F\psi$-integrable in the lcs $\mathcal{L}_s(\mathcal{D}(A), \mathcal{H})\widetilde{\otimes}_\tau \mathcal{D}(A)$ and we have

$$\int\limits_T B(t)\left(\int\limits_S u(\lambda,t)\, d(F\psi)(\lambda)\right) dt = \int\limits_T J\left[B(t) \otimes \left(\int\limits_S u(\lambda,t)\, d(F\psi)(\lambda)\right)\right] dt$$

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$$= J \int_T B(t) \otimes \left(\int_S u(\lambda, t) \, d(F\psi)(\lambda) \right) dt$$

$$= J \int_S \left(\int_T u(\lambda, t) B(t) \, dt \right) \otimes d(F\psi)(\lambda)$$

$$= \int_S \left(\int_T u(\lambda, t) B(t) \, dt \right) d(F\psi)(\lambda).$$

The final assertion is proved in Lemma 4.2. □

Appendix

A   Measurability in spaces of operators on a separable Banach space

Let $(\Omega, \mathcal{S}, \mu)$ be any σ-finite measure space. A function $f : \Omega \to X$ with values in a Banach space X is said to be *strongly μ-measurable* if it is the limit μ-almost everywhere of X-valued $\mathcal{S}$-simple functions. In the case that X is a locally convex space which is not metrizable, we run into difficulties with strong measurability, because all fundamental families of seminorms are uncountable. We see how to deal with the situation in the special case of the strong operator topology on the space of continuous linear operators between separable Banach spaces. The idea goes back to the work of G.Y.H. Chi [8] in 1975.

Suppose that X and Y are separable Banach spaces. If $f : \Omega \to \mathcal{L}(X, Y)$ is a function for which $\langle fx, y' \rangle : \omega \longmapsto \langle f(\omega)x, y' \rangle$, $t \geq 0$, is μ-measurable for each $x \in X$ and $y' \in Y^*$. Then the sets

$$\{\omega \in \Omega : \Re \langle f(\omega)x, y' \rangle < a\}, \quad \{\omega \in \Omega : \Im \langle f(\omega)x, y' \rangle < b\}$$

are μ-measurable for each $a, b \in \mathbb{R}$, so that $f^{-1}(B)$ is μ-measurable for every Borel set B in the weak operator toplogy of $\mathcal{L}_s(X, Y)$.

If we set $\Omega_n = f^{-1}(nB_1)$ for $n = 1, 2, \ldots$, with

$$B_1 = \{T \in \mathcal{L}(X, Y) : \|T\|_{\mathcal{L}(X,Y)} \leq 1\},$$

then each set Ω_n, $n = 1, 2, \ldots$, is μ-measurable with $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and each operator belonging to the range $f(\Omega_n)$ of f on Ω_n is bounded by n in the uniform operator norm.

Because X is assumed to be separable, we can find a dense subset

$$\{x_k : k = 1, 2, \ldots\}$$

of the closed unit ball of X. For each $T \in \mathcal{L}(X, Y)$, set $p_k(T) = \|Tx_k\|_Y$, $k = 1, 2, \ldots$. By an $\epsilon/2$ argument, the strong operator topology on each ball nB_1, $n = 1, 2, \ldots$, is given by the family $\{p_k : k = 1, 2, \ldots\}$ of seminorms. Let $p = \sum_{k=1}^{\infty} 2^{-k} p_k$. Then p is a norm on $\mathcal{L}(X, Y)$ that gives the strong operator topology to each ball nB_1, $n = 1, 2, \ldots$.

Because Y is assumed to be separable, the Pettis Measurability theorem [11] ensures that the Y-valued function $f_n x : \omega \longmapsto f(\omega)x$, $\omega \in \Omega_n$, is strongly μ-measurable for each $x \in X$ and $n = 1, 2, \ldots$. Hence, for each $n = 1, 2, \ldots$, we can choose a sequence of $\mathcal{L}(X, Y)$-valued μ-measurable simple functions $s_{j,n}$, $j = 1, 2, \ldots$, such that $p(f(\omega) - s_{j,n}(\omega)) \to 0$ as $j \to \infty$ for μ-almost all $\omega \in \Omega_n \setminus \Omega_{n-1}$, where $\Omega_0 = \emptyset$. Let

$$s_j = \sum_{n=1}^{\infty} s_{j,n} \chi_{\Omega_n \setminus \Omega_{n-1}} \quad j = 1, 2, \ldots .$$

Then s_j, $j = 1, 2, \ldots$ is a sequence of $\mathcal{L}(X, Y)$-valued μ-measurable simple functions such that $s_j \to f$ μ-a.e. in the strong operator topology as $j \to \infty$.

If, in addition, $\int_{\Omega} \|f(\omega)x\|_Y \, d\mu(\omega) < \infty$ for each $x \in X$, we can choose s_j, $j = 1, 2, \ldots$ such that we also have

$$\int_{\Omega} \|f(\omega)x - s_j(\omega)x\|_Y \, d\mu \longrightarrow 0 \text{ as } j \to \infty.$$

Remark 1 Another proof is obtained from the theory of Lusin spaces [21]. Under the given assumptions, $\mu \circ f^{-1}$ is a Borel measure for the weak operator topology on $\mathcal{L}_s(X, Y)$. Because $\mathcal{L}_s(X, Y) = \bigcup_{n=1}^{\infty} nB_1$ is the countable union of complete metrizable sets (Polish spaces), $\mathcal{L}_s(X, Y)$ is itself a Lusin space [21]. It follows that $\mu \circ f^{-1}$ is a Radon measure for the strong operator topology [21]. There exists a pairwise disjoint collection $\{K_n : n = 1, 2, \ldots\}$ of compact metrizable subsets in the strong operator topology of $\mathcal{L}_s(X, Y)$ such that

$$\mu \circ f^{-1} \left(\mathcal{L}_s(X, Y) \setminus \bigcup_{n=1}^{\infty} K_n \right) = 0.$$

On each set $\Omega_n = \mu \circ f^{-1}(K_n)$, using a metric on the compact set K_n, we find $\mathcal{S}$-measurable K_n-valued simple functions $s_{j,n}$, $j = 1, 2, \ldots$, converging

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uniformly to f on Ω_n. Piecing these together gives $\mathcal{L}_s(X, Y)$-valued simple functions s_j, $j = 1, 2, \ldots$, converging μ-a.e. to f on Ω.

B   Tensor products of spaces of operators

In this appendix we present the elements of the theory of tensor products of Banach spaces necessary for the correct understanding of the ideas presented in this paper. A basic reference in this subject can be found in [19]. The conditions we require are satisfied by Hilbert spaces, but stating the results for Banach spaces isolates the essential elements of the proof. As indicated in Section 3, consideration of the strong operator topology in place of the uniform operator topology introduces some technical complications. In this section we will deal with these questions and we clarify what we want to say when, in the above section, we have talked about suitable topologies.

Let X_j, Y_j, $j = 1, 2$, be Banach spaces. The *projective tensor product* $\mathcal{L}(X_1, Y_1) \widehat{\otimes}_\pi \mathcal{L}(X_2, Y_2)$ has the norm

$$\|T\|_\pi = \inf \left\{ \sum_{j=1}^\infty \|S_j\|.\|T_j\| : T = \sum_j S_j \otimes T_j \right\},$$

where the infimum is over all norm absolutely convergent representations of T with $S_j \in \mathcal{L}(X_1, Y_1)$ and $T_j \in \mathcal{L}(X_2, Y_2)$, $j = 1, 2, \ldots$.

For the projective tensor product $\mathcal{L}_s(X_1, Y_1) \widehat{\otimes}_\pi \mathcal{L}_s(X_2, Y_2)$ with the strong operator topology on $\mathcal{L}(X_j, Y_j)$, $j = 1, 2$, we take the completion of $\mathcal{L}(X_1, Y_1) \otimes \mathcal{L}(X_2, Y_2)$ with respect to the fundamental family of seminorms r_{x_1, x_2}, $x_1 \in X_1$, $x_2 \in X_2$, defined by

$$r_{x_1, x_2}(T) = \inf \left\{ \sum_{j=1}^n \|S_j x_1\|.\|T_j x_2\| : T = \sum_{j=1}^n S_j \otimes T_j \right\},$$

where the infimum is over all representations of T with $S_j \in \mathcal{L}(X_1, Y_1)$ and $T_j \in \mathcal{L}(X_2, Y_2)$, $j = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ [19, §41.2 (4)].

Let τ be the topology of $\mathcal{L}(X_1, Y_1) \otimes \mathcal{L}(X_2, Y_2)$ defined from $\mathcal{L}_s(Y_1', X_1') \otimes_\pi \mathcal{L}_s(X_2, Y_2)$, that is, let $j : \mathcal{L}(X_1, Y_1) \otimes \mathcal{L}(X_2, Y_2) \to \mathcal{L}(Y_1', X_1') \otimes \mathcal{L}(X_2, Y_2)$ be the linear map defined by

$$j(A \otimes B) = A' \otimes B, \quad A \in \mathcal{L}(X_1, Y_1), \ B \in \mathcal{L}(X_2, Y_2).$$

Then we define τ to be the coarsest topology on $\mathcal{L}(X_1, Y_1) \otimes \mathcal{L}(X_2, Y_2)$ for which j is continuous into $\mathcal{L}_s(Y_1', X_1') \otimes_\pi \mathcal{L}_s(X_2, Y_2)$. Thus, τ has a fundamental

family of seminorms r_{y_1', x_2}, $y_1' \in Y_1'$, $x_2 \in X_2$, defined by

$$r_{y_1', x_2}(T) = \inf \left\{ \sum_{j=1}^{n} \|S_j' y_1'\|_{X_1'} . \|T_j x_2\|_{Y_2} : T = \sum_{j=1}^{n} S_j \otimes T_j \right\},$$

where the infimum is over all representations of T with $S_j \in \mathcal{L}(X_1, Y_1)$ and $T_j \in \mathcal{L}(X_2, Y_2)$, $j = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$.

Let X, Y, Z be Banach spaces. The composition map $J : \mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z) \to \mathcal{L}(X, Y)$ is the linear map defined by

$$J(A \otimes B) = AB, \quad A \in \mathcal{L}(Z, Y), \ B \in \mathcal{L}(X, Z).$$

Lemma B.1 *The linear map $J : \mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z) \to \mathcal{L}(X, Y)$ is continuous for the topology τ on $\mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z)$ and the weak operator topology of $\mathcal{L}(X, Y)$.*

Proof. For any representation $T = \sum_{j=1}^{n} A_j \otimes B_j \in \mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z)$ and $x \in X$, $y' \in Y'$, we have

$$\begin{aligned}
|\langle J(T)x, y' \rangle| &= \left| \sum_{j=1}^{n} \langle A_j B_j x, y' \rangle \right| \\
&\leq \sum_{j=1}^{n} |\langle B_j x, A_j' y' \rangle| \\
&\leq \sum_{j=1}^{n} \|B_j x\|_Z \|A_j' y'\|_{Z'},
\end{aligned}$$

so that

$$|\langle J(T)x, y' \rangle| \leq r_{y', x}(T). \tag{17}$$

Hence, J is τ-$\sigma(\mathcal{L}(X, Y), X \otimes Y')$–continuous. $\qquad\qquad \square$

Denote $\mathcal{L}(Z, Y) \widetilde{\otimes}_\tau \mathcal{L}(X, Z)$ the quasicompletion of the lcs $\mathcal{L}(Z, Y) \otimes_\tau \mathcal{L}(X, Z)$ and $\mathcal{L}_w(X, Y)$ denotes $\mathcal{L}(X, Y)$ equipped with the weak operator topology.

Lemma B.2 *If Y is a reflexive Banach space, then J is the restriction to $\mathcal{L}(Z, Y) \otimes \mathcal{L}(X, Z)$ of a continuous linear map*

$$\tilde{J} : \mathcal{L}(Z, Y) \widetilde{\otimes}_\tau \mathcal{L}(X, Z) \to \mathcal{L}_w(X, Y),$$

Proof. The first part is a consequence of the proof of Lemma B.1. If Y is reflexive, then $\mathcal{L}_w(X, Y)$ is quasicomplete because Y is weakly quasicomplete

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[18, §23.3(2)] and X is barrelled, so we may appeal to [19, §39.6(5)]. A continuous linear map has a unique extension to the quasicompletion of its domain [18, §23.2(4)] $\qquad\Box$

Lemma B.3 *The linear map $j : \mathcal{L}(Z,Y) \otimes \mathcal{L}(X,Z) \to \mathcal{L}(X,Z) \otimes \mathcal{L}(Y',Z')$ defined by $j(S \otimes T) = T \otimes S'$ extends to an embedding $\widetilde{j} : \mathcal{L}(Z,Y) \widetilde{\otimes}_\tau \mathcal{L}(X,Z) \to \mathcal{L}_s(X,Z) \widetilde{\otimes}_\pi \mathcal{L}_s(Y',Z')$ of locally convex spaces. If Y and Z are reflexive, then $\widetilde{j}$ is an isomorphism.*

Proof. $\{T_\beta\}_\beta$ is τ-Cauchy iff $\{jT_\beta\}_\beta$ is π-Cauchy. If Y and Z are reflexive, then every linear operator $T \in \mathcal{L}(Y',Z')$ is the dual of an element of $\mathcal{L}(Z,Y)$. The map $\widetilde{j}$ is onto if Y and Z are reflexive because the inverse map $k : \mathcal{L}(X,Z) \otimes \mathcal{L}(Y',Z') \to \mathcal{L}(Z,Y) \otimes \mathcal{L}(X,Z)$ also extends to a one-to-one map $\widetilde{k} : \mathcal{L}_s(X,Z) \widetilde{\otimes}_\pi \mathcal{L}_s(Y',Z') \to \mathcal{L}(Z,Y) \widetilde{\otimes}_\tau \mathcal{L}(X,Z)$. $\qquad\Box$

Let $\mathfrak{B}(U \times V)$ denote the linear space of all separately continuous bilinear forms on the cartesian product $U \times V$ of the locally convex spaces U and V. If E and F are locally convex spaces, then $\mathfrak{B}_e(E'_s \times F'_s)$ denotes the space $\mathfrak{B}(E'_s \times F'_s)$ of bilinear forms equipped with the topology of bi-equicontinuous convergence, see [19, p. 167].

Lemma B.4 *Let E and F be quasicomplete locally convex spaces. Then the space $\mathfrak{B}_e(E'_s \times F'_s)$ of bilinear forms is quasicomplete.*

Proof. By [19, §40.4(5)], $\mathfrak{B}_e(E'_s \times F'_s)$ is topologically isomorphic to $\mathcal{L}_e(E'_k, F)$ where E'_k has the Mackey topology of uniform convergence on weakly compact absolutely convex subsets of E. We show that $\mathcal{L}_e(E'_k, F)$ is quasicomplete.

Let $\mathcal{M}$ denote the family of equicontinuous subsets of E'. Then $(E'_k)' = E$ and $E = E(\mathfrak{T}_\mathcal{M})$ is quasicomplete, so by [19, §39.6(3) p. 297ff], the space $\mathcal{L}_e(E'_k, F) = \mathcal{L}_\mathcal{M}(E'_k, F)$ is quasicomplete. $\qquad\Box$

Proposition B.5 [19, §43.2(12)] *Let E be a quasicomplete locally convex space with a fundamental system of absolutely convex neighbourhoods U of zero such that every $\widetilde{E}_U$ has the approximation property. Then for every quasicomplete lcs F, the canonical map*

$$\widetilde{\psi} : E \widetilde{\otimes}_\pi F \to \mathfrak{B}_e(E'_s \times F'_s)$$

from the quasicompletion $E \widetilde{\otimes}_\pi F$ of $E \otimes_\pi F$ into the linear space $\mathfrak{B}_e(E'_s \times F'_s)$ of bilinear maps is one-one.

Proof. Because $\mathfrak{B}_e(E'_s \times F'_s)$ is quasicomplete, the canonical map $\widetilde{\psi}$ is well-defined. The proof in [19, §43.2(12)] works in this case too. $\qquad\Box$

Lemma B.6 *Let X and Z be Banach spaces and $E = \mathcal{L}_s(X, Z)$. If $x_1, \ldots, x_n$, $n = 1, 2, \ldots$, are elements of X, $\epsilon_1, \ldots, \epsilon_n > 0$ and*

$$U_{x_1,\ldots,x_n}^{\epsilon_1,\ldots,\epsilon_n} = \{T \in E : \|Tx_j\|_Z \leq \epsilon_j, \ j = 1, \ldots, n \ \},$$

then $E_{U_{x_1,\ldots,x_n}^{\epsilon_1,\ldots,\epsilon_n}}$ is a Banach space norm equivalent to $\mathcal{L}(\mathrm{span}\{x_1, \ldots, x_n\}, Z)$ for the uniform norm, with the norm of X on $\mathrm{span}\{x_1, \ldots, x_n\}$.

Lemma B.7 *Let X and Z be Banach spaces. If Z has the approximation property then the space $E = \mathcal{L}_s(X, Z)$ of linear operators with the strong operator topology has a fundamental system of absolutely convex neighbourhoods U of zero such that every normed space E_U is complete and has the approximation property.*

Proof. If $x_1, \ldots, x_n, n = 1, 2, \ldots$, are elements of X, then $\mathcal{L}(\mathrm{span}\{x_1, \ldots, x_n\}, Z)$ is topologically isomorphic to Z^F where F is a basis of the finite dimensional space $\mathrm{span}\{x_1, \ldots, x_n\}$. By [19, 43.4(3)], Z^F has the approximation property. Clearly, Z^F is complete. By Lemma B.6, $U_{x_1,\ldots,x_n}^{\epsilon_1,\ldots,\epsilon_n}$ are the required neighbourhoods of zero. $\square$

Lemma B.8 *Let X, Y, Z be Banach spaces. If $Z \otimes X \otimes Y' \otimes Z'$ acts on $\mathcal{L}_s(X, Z) \otimes \mathcal{L}_s(Y', Z')$ via the unique linear extension of the map*

$$T \otimes U \longmapsto \langle Tx, z' \rangle \langle Uy', z \rangle, \quad T \in \mathcal{L}(X, Z), \ U \in \mathcal{L}_s(Y', Z'), \tag{18}$$

for all $z \in Z$, $x \in X$, $y' \in Y'$ and $z' \in Z'$, then

$$Z \otimes X \otimes Y' \otimes Z' \subset (\mathcal{L}_s(X, Z) \otimes_\pi \mathcal{L}_s(Y', Z'))'$$

Proof. For every $z \in Z$, $x \in X$, $y' \in Y'$ and $z' \in Z'$, the bound

$$|\langle T, z \otimes x \otimes y' \otimes z' \rangle| = |\sum_j \langle T_j x, z' \rangle \langle U_j y', z \rangle|$$
$$\leq \|z'\| . \|z\| \sum_j \|T_j x\|_Z \|U_j y'\|_{Z'}$$

holds for $T = \sum_j T_j \otimes U_j \in \mathcal{L}_s(X, Z) \otimes \mathcal{L}_s(Y', Z')$, so

$$|\langle T, z \otimes x \otimes y' \otimes z' \rangle| \leq r_{y',x}(T) \|z'\| . \|z\|.$$

Hence, $z \otimes x \otimes y' \otimes z'$ is a continuous linear functional on $\mathcal{L}_s(X, Z) \otimes_\pi \mathcal{L}_s(Y', Z')$ $\square$

Proposition B.9 *Let X, Y, Z be Banach spaces. If Z is a reflexive Banach space with the approximation property, then the linear space $Z \otimes X \otimes Y' \otimes$*

Z' separates the quasicompletion $\mathcal{L}(Z,Y)\widetilde{\otimes}_\tau\mathcal{L}(X,Z)$ via the continuous linear extension of the map

$$S \otimes T \longmapsto \langle Sz, y' \rangle \langle Tx, z' \rangle, \quad S \in \mathcal{L}(Z,Y),\ T \in \mathcal{L}(X,Z) \tag{19}$$

for every $z \in Z$, $x \in X$, $y' \in Y'$ and $z' \in Z'$.

Proof. By Lemma B.3, the lcs $\mathcal{L}(Z,Y)\widetilde{\otimes}_\tau\mathcal{L}(X,Z)$ may be identified via the embedding $\widetilde{j}$ with a linear subspace of the quasicompletion $\mathcal{L}_s(X,Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y',Z')$ of the projective tensor product of $\mathcal{L}_s(X,Z)$ with $\mathcal{L}_s(Y',Z')$. By Lemma B.8,

$$Z \otimes X \otimes Y' \otimes Z' \subset (\mathcal{L}_s(X,Z) \otimes_\pi \mathcal{L}_s(Y',Z'))',$$

with the identification defined by formula (18). Moreover, the equality

$$\langle \alpha, z \otimes x \otimes y' \otimes z' \rangle = \langle \widetilde{j}\alpha, z \otimes x \otimes y' \otimes z' \rangle, \quad \alpha \in \mathcal{L}(Z,Y)\widetilde{\otimes}_\tau\mathcal{L}(X,Z),$$

holds for all $z \in Z$, $x \in X$, $y' \in Y'$ and $z' \in Z'$. The left hand side of the equation above is defined by formula (19). It suffices to show that $Z \otimes X \otimes Y' \otimes Z'$ separates points of the lcs $\mathcal{L}_s(X,Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y',Z')$.

According to Lemma B.7, the space $E = \mathcal{L}_s(X,Z)$ has a fundamental system of absolutely convex neighbourhoods U of zero such that every E_U is a Banach space with the approximation property. Moreover, $E = \mathcal{L}_s(X,Z)$ and $F = \mathcal{L}_s(Y'_b, Z'_b)$ are quasicomplete by [19, 39.6(5)], so appealing to Proposition B.5, the canonical map $\widetilde{\psi} : E\widetilde{\otimes}_\pi F \to \mathfrak{B}_e(E'_s \times F'_s)$ is one-one. The dual E' of $E = \mathcal{L}_s(X,Z)$ can be identified with $X \otimes Z'$ and the dual F' of $F = \mathcal{L}_s(Y'_b, Z'_b)$ can be identified with $Y' \otimes Z''$. [19, 39.7(2)]. The assumption that the Banach space Z is reflexive means $Z'' = Z$.

For $S \in \mathcal{L}_s(Y', Z')$ and $T \in \mathcal{L}(X,Z)$, we have

$$\psi(T \otimes S)(x \otimes z', y' \otimes z) = \langle T \otimes S, z \otimes x \otimes y' \otimes z' \rangle$$

and by linearity and continuity we obtain

$$\widetilde{\psi}(\alpha)(x \otimes z', y' \otimes z) = \langle \alpha, z \otimes x \otimes y' \otimes z' \rangle$$

for every element α of the quasicompletion $\mathcal{L}_s(X,Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y',Z')$ of $\mathcal{L}_s(X,Z)\otimes_\pi\mathcal{L}_s(Y',Z')$.

Consequently, if $\alpha \in \mathcal{L}_s(X,Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y',Z')$ and $\langle \alpha, z \otimes x \otimes y \otimes z' \rangle = 0$ for every tensor $z \otimes x \otimes y' \otimes z' \in Z \otimes X \otimes Y' \otimes Z'$, then $\widetilde{\psi}(\alpha) = 0$. Because $\widetilde{\psi}$ is one-one, $\alpha = 0$ and so $Z \otimes X \otimes Y' \otimes Z'$ separates points of the lcs $\mathcal{L}_s(X,Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y',Z')$ and of the embedded space $\mathcal{L}(Z,Y)\widetilde{\otimes}_\tau\mathcal{L}(X,Z)$. $\qquad\square$

Remark B.10 The linear space $\mathcal{L}_s(X, Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y', Z')$ may be identified with a subspace of the space $L(X \otimes Y', Z_\sigma\widetilde{\otimes}_\pi Z'_\sigma)$ of *all* linear maps from the tensor product $X \otimes Y'$ into the quasicompletion $Z_\sigma\widetilde{\otimes}_\pi Z'_\sigma$. Because $Z' \otimes Z$ separates $Z_\sigma\widetilde{\otimes}_\pi Z'_\sigma$ [19], the tensor product $X \otimes Y' \otimes Z' \otimes Z$ separates $L(X \otimes Y', Z_\sigma\widetilde{\otimes}_\pi Z'_\sigma)$ and hence, $\mathcal{L}_s(X, Z)\widetilde{\otimes}_\pi\mathcal{L}_s(Y', Z')$. Our Hilbert space integrals are therefore consistent with Definition 2.4.

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