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## RESEARCH ARTICLE

### *Study of the dynamics of third-order iterative methods on quadratic polynomials*

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In this paper we analyze the dynamical behavior of the operators associated to multi-point interpolation iterative methods and frozen derivative methods, for solving nonlinear equations, applied on second degree complex polynomials. We obtain that, in both cases, the Julia set is connected and separates the basins of attraction of the roots of the polynomial. Moreover, the Julia set of the operator associated to multi-point interpolation methods is the same as the Newton operator, although it is more complicated for the frozen derivative operator. We explain these differences by obtaining the conjugacy function of each method and by showing that the operators associated to Newton's method and multi-point interpolation methods are both conjugate to powers of  $z$ .

**Keywords:** Nonlinear equations, iterative methods, complex dynamics, conjugacy map, basin of attraction

**AMS Subject Classification:** 65H10

## 1. Introduction

Many engineering applications involve nonlinear equations  $f(x) = 0$  whose solution can not be found by means of analytical methods. To approximate the solution of these equations we use iterative methods. This means that the output of the method is a sequence of images  $\{x_0, R(x_0), R^2(x_0), \dots, R^n(x_0), \dots\}$  for the initial condition  $x_0$ , where  $R$  is a rational function that represents the fixed point operator of the iterative scheme. Therefore, it can be seen as a discrete dynamical system and we can study it from this point of view.

There is an extensive literature on the study of iteration of rational mappings  $R$  of a complex variable (see [8], [9], for example) and, as it is well known, the Newton's method (see [4], [10]) applied on polynomials is a rational function. In this case, the Riemann sphere  $\hat{\mathbb{C}}$  is also considered as the domain of the rational mapping  $R$  associated to the iterative method.

To our knowledge, the study on the dynamics of Newton's method has been extended to other point-to-point iterative schemes used for solving nonlinear equa-

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tions, with convergence order up to three (see, for example [1], [2] and, more recently, [12] and [15]).

S. Amat et al. in [3] make a brief raid into the study of the dynamics of the Potra-Ptak method (see [16]) defined on the real numbers and applied on polynomials of second and third degrees. Nevertheless, this study, interesting in itself, does not allow to see all the richness of the dynamics of the method when it is defined on the complex numbers.

Now, let us recall some basic concepts on complex dynamics. Given a rational function  $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ , the *orbit of a point*  $z_0 \in \hat{\mathbb{C}}$  is defined as:

$$z_0, R(z_0), R^2(z_0), \dots, R^n(z_0), \dots$$

and we are interested in the study of the asymptotic behavior of the orbits depending on the initial condition  $z_0$ , that is, we are interested in the study of the phase plane of the map defined by the iterative method.

To obtain this phase space, we need to classify the initial conditions from the asymptotic behavior of their orbits.

Let us consider  $\alpha \in \mathbb{C}$  a root of  $f$ ,  $f(\alpha) = 0$ . The *basin of attraction* of  $\alpha$  is defined as the set of pre-images of any order:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \rightarrow \alpha, n \rightarrow \infty\}.$$

A point  $z_0 \in \hat{\mathbb{C}}$  is called a *fixed point* of  $R$  if  $R(z_0) = z_0$ .  $z_0$  is a *periodic point* of period  $p > 1$  if  $R^p(z_0) = z_0$  and  $R^k(z_0) \neq z_0$ ,  $k < p$ . A *pre-periodic point* is a point  $z_0$  that is not periodic but there exists a  $k > 0$  such that  $R^k(z_0)$  is periodic. A point  $z_0$  such that  $R'(z_0) = 0$  is called *critical point*.

On the other hand, a fixed point  $z_0$  is called *attractor* if  $|R'(z_0)| < 1$ , *superattractor* if  $|R'(z_0)| = 0$ , and *repulsor* if  $|R'(z_0)| > 1$ .

The set of points  $z \in \hat{\mathbb{C}}$  such that their families  $\{R^n(z)\}_{n \in \mathbb{N}}$  are normal in some neighborhood  $U(z)$ , is the *Fatou set*,  $\mathcal{F}(R)$ , that is, the Fatou set is composed by the points whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in  $\hat{\mathbb{C}}$  is the *Julia set*,  $\mathcal{J}(R)$ ; therefore, the Julia set includes all repelling fixed points, periodic orbits and their pre-images. That means that the basin of attraction of any fixed point belongs to the Fatou set. On the contrary, the boundaries of the basins of attraction belong to the Julia set.

### 1.1 The Newton's Method

The Newton's method is the best known algorithm to find the roots of a nonlinear function  $f(z) = 0$ , where  $f \in C^1(\hat{\mathbb{C}})$  is defined on the Riemann sphere  $\hat{\mathbb{C}}$ . The Newton's operator is

$$N_f(z) = z - \frac{f(z)}{f'(z)}, \quad (1)$$

which satisfies that  $f(z) = 0$  if and only if  $N_f(z) = z$ . So, to find the roots of  $f(z)$  is equivalent to find the fixed points of the operator  $N_f(z)$ . Actually, the global analysis of convergence of Newton's method on  $f(z)$  is equivalent to compute individual orbits of the dynamical systems generated by the Newton's map  $N_f(z)$ .

The equation (1) on a polynomial  $p(z)$ ,  $N_p(z)$ , verifies the following properties:

1. The roots of  $p(z)$  correspond to the finite fixed points of  $N_p$ .

2. The point at the infinity is a repelling fixed point.
3. As the derivative of the iteration function is

$$N'_p(z) = \frac{p(z)p''(z)}{p'(z)^2}, \quad (2)$$

the simple roots of  $p(z)$  are superattracting fixed points. Multiple roots are attracting fixed points, but not superattracting.

For simplicity, we begin studying the Newton's method on quadratic polynomials. It is known that any quadratic polynomial can be transformed into  $p(z) = z^2 + c$  by an affine map without qualitatively changing the dynamics of the Newton's iteration function.

**THEOREM 1.1** [14] *Let  $f$  be an analytic function on the Riemann sphere, and  $A(z) = \alpha z + \beta$ , with  $\alpha \neq 0$ , an affine map. If  $g(z) = \lambda(f \circ A)(z)$ , where  $\lambda \in \mathbb{C} - \{0\}$ , then the Newton's function  $N_f$  is analytically conjugated to  $N_g$  by  $A$ , that is,  $A \circ N_g \circ A^{-1}(z) = N_f(z)$ .*

Then, if the two roots are simple, we obtain that the two basin of attraction of the roots are separated by the perpendicular bisector of the line segment from one root to the other. This bisector is the Julia set for this polynomial, that it is connected. On the other hand, when the polynomial has a double root, the dynamical plane consists in one basin of attraction.

**THEOREM 1.2** [4] *Let  $p(z)$  be a quadratic polynomial with simple roots. The Newton's operator  $N_p(z)$  is globally, analytically conjugate to the quadratic polynomial  $z^2$ .*

P. Blanchard, in [4], proves this result by considering the conjugacy map

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}, \quad (3)$$

with the following properties:

$$(i)h(\infty) = 1, \quad (ii)h(i\sqrt{c}) = 0, \quad (iii)h(-i\sqrt{c}) = \infty.$$

Then,

$$(h \circ N_p \circ h^{-1})(z) = z^2. \quad (4)$$

So, for quadratic polynomials with simple roots, the Newton's operator is always conjugate to the rational map  $z^2$ , satisfying the following properties:

1. The dynamics of this operator gives the unit circle  $S^1(z) = \{z \in \hat{\mathbb{C}} : |z| = 1\}$  as the invariant Julia set.
2. The Fatou set is defined by the two basins of attraction of the superattracting fixed points: 0 and  $\infty$ .

In this paper, we study the dynamics of two families of iterative methods of order three: the multi-point interpolation methods (Section 2) and the frozen derivative methods (Section 3) on quadratic polynomials defined on the complex plane.

## 2. The Multi-Point Interpolation Methods

Now, we are going to introduce a known family of iterative methods and we analyze its dynamical behavior for simple and multiple roots.

From quadrature formulas applied to the integral

$$f(x) = f(x_k) + \int_{x_k}^x f'(t) dt,$$

some authors (see [7] and [11]) have derived a family of multi-point interpolation methods, whose iterative scheme is

$$x_{k+1} = x_k - \frac{f(x_k)}{\sum_{j=1}^m A_j f'(\eta_j(x_k))}, \quad (5)$$

where  $\eta_j(x_k) = x_k - \tau_j \frac{f(x_k)}{f'(x_k)}$ ,  $j = 1, 2, \dots, m$ , with  $\tau_j$  the knots in  $[0, 1]$  and  $A_j$  the weight of the interpolatory quadrature formula used. So, they satisfy the relationships:

$$\sum_{j=1}^m A_j = 1 \quad \text{and} \quad \sum_{j=1}^m A_j (1 - \tau_j) = \frac{1}{2}. \quad (6)$$

These identities allow the authors assure that the resulting methods have, at least, order of convergence three for simple roots.

Now, we state the Scaling Theorem for this iterative scheme, whose fixed point operator is denoted by  $M_f$ .

**THEOREM 2.1** *Let  $f$  be an analytic function on the Riemann sphere, and  $A(z) = \alpha z + \beta$ , with  $\alpha \neq 0$ , an affine map. If  $g(z) = \lambda(f \circ A)(z)$ , where  $\lambda \in \mathbb{C} - \{0\}$ , then the fixed point operator  $M_f$  is analytically conjugated to  $M_g$  by  $A$ , that is,  $A \circ M_f \circ A^{-1}(z) = M_g(z)$ .*

Lets begin studying the dynamics of the operator associated to the family (5) on quadratic polynomials with simple roots  $p(z) = z^2 + c$ ,  $c \in \mathbb{C}$  and  $c \neq 0$ .

$$M_p(z) = z - \frac{f(z)}{\sum_{j=1}^m A_j f'(\eta_j(z))} = \frac{-z^3 + cz + (cz + z^3) \sum_{j=1}^m A_j \tau_j}{-2z^2 + (z^2 + c) \sum_{j=1}^m A_j \tau_j}$$

and, by using (6), we obtain:

$$M_p(z) = \frac{3cz - z^3}{c - 3z^2}. \quad (7)$$

The fixed points of  $M_p(z)$  are the roots of the polynomial:

$$M_p(z) = z \Rightarrow z = \pm i\sqrt{c}$$

and these are also the critical points, so that  $M'_p(z) = \frac{3(z^2 + c)^2}{(c - 3z^2)^2} = 0$ . As in Newton's method, these roots are the only critical points, and they are superattractor fixed points.

Therefore, similarly to the Newton's method, the Fatou set consists of the basins of attraction of the two roots of the polynomial. That means that these methods never fail on quadratic polynomials when they are applied on an open set of the complex plane. The dynamical plane of the operator (7) is the same as the one of Newton's method. This result is deduced from the following statement.

**THEOREM 2.2** *Let  $p(z)$  be a quadratic polynomial with distinct roots. The fixed point operator  $M_p(z)$  associated to the family (5) verifies:*

- i)  $M_p(z)$  is globally, analytically conjugated to the cubic polynomial  $z^3$ .
- ii) The Julia set of this operator is the connected unit circle  $S^1(z) = \{z \in \hat{\mathbb{C}} : |z| = 1\}$ .
- iii) The Fatou set is defined by the two basins of attraction of the superattracting fixed points: 0 and  $\infty$ .

**Proof:** We consider the conjugacy map (3), with properties  $h(\infty) = 1$ ,  $h(i\sqrt{c}) = 0$  and  $h(-i\sqrt{c}) = \infty$ . So,

$$h^{-1}(z) = i\sqrt{c} \frac{z + 1}{1 - z}$$

and, therefore

$$(h \circ M_p \circ h^{-1})(z) = z^3 \quad (8)$$

is a cubic polynomial of degree three that has superattracting fixed points at 0 and  $\infty$  separated by the unit circle. As this map does not depend on the parameter  $c$ , the Julia set is the unit circle and, therefore, it is connected for every  $c$ . Moreover, (8) implies that the origin is a zero of order three.  $\square$

There are some well known iterative methods belonging to family (5).

- The Midpoint Method was developed by Özban in [13] and can be written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'\left(\frac{x_k + y_k}{2}\right)},$$

where  $y_k$  is the Newton's iteration. The corresponding operator is:

$$M_f(z) = z - \frac{f(z)}{f'\left(z - \frac{f(z)}{2f'(z)}\right)}. \quad (9)$$

- Similarly, Frontini and Sormani derived in [11] the Trapezoidal Method

$$x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k) + f'(y_k)},$$

yields to the operator

$$Tr_f(z) = z - \frac{2f(z)}{f'(z) + f'\left(z - \frac{f(z)}{f'(z)}\right)}. \quad (10)$$

- Finally, Frontini and Sormani also presented in [11] the Simpson Method:

$$x_{k+1} = x_k - \frac{6f(x_k)}{f'(x_k) + 4f'\left(\frac{x_k+y_k}{2}\right) + f'(y_k)}$$

provides the operator

$$S_f(z) = z - \frac{6f(z)}{f'(z) + 4f'\left(z - \frac{f(z)}{2f'(z)}\right) + f'\left(z - \frac{f(z)}{f'(z)}\right)}. \quad (11)$$

As we have seen these three operators (9), (10) and (11) acting on the family of quadratic polynomial  $p(z) = z^2 + c$ ,  $c \in \mathbb{C}$  give the operator  $M_p(z) = \frac{3cz - z^3}{c - 3z^3} = S_p(z) = Tr_p(z)$ . Consequently, their dynamics have been studied in the previous section.

**Remark** The dynamics of the Midpoint, Trapezoidal, Simpson and any other iterative method of family (5) on quadratic polynomials, defined on the complex plane, is the same as the one of Newton's method.

## 2.1 Family of interpolation methods for multiple roots

It is known that methods of family (5) converge linearly in case of multiple roots. Nevertheless, in this case it is possible to modify this family in order to attain quadratic convergence.

To get this aim, in a similar way as in the modified Newton's method, we replace  $\eta_j(x_k)$  in (5) by

$$\eta_j^*(x) = x - q\tau_j \frac{f(x)}{f'(x)}, \quad (12)$$

being  $q$  the multiplicity of the root  $\alpha$ , that satisfies  $\lim_{x \rightarrow \alpha} \eta_j^*(x) = \alpha$  and  $\lim_{x \rightarrow \alpha} \eta_j^*(x) = 1 - \tau_j$  (see [6]).

The following result establishes that the modified family

$$x_{k+1} = x_k - \frac{f(x_k)}{\sum_{j=1}^m A_j f'(\eta_j^*(x_k))}, \quad (13)$$

has order of convergence two.

**THEOREM 2.3** Let  $\alpha$  be a root of  $f(x)$  with multiplicity  $q > 1$  and  $M^*(x)$  the fixed point operator of the family (13). Then  $(M^*)'(\alpha) = 0$ .

**Proof:** The derivative of  $M^*(x)$  is

$$(M^*)'(x) = 1 - \frac{\sum_{j=1}^m A_j f'(\eta_j^*(x)) f'(x)}{\left(\sum_{j=1}^m A_j f'(\eta_j^*(x))\right)^2} + \frac{f(x) \sum_{j=1}^m A_j f''(\eta_j^*(x)) \eta_j^*(x)}{\left(\sum_{j=1}^m A_j f'(\eta_j^*(x))\right)^2}.$$

Then, by using  $\lim_{x \rightarrow \alpha} \eta_j^*(x) = \alpha$ , we obtain

$$\lim_{x \rightarrow \alpha} (M^*)'(x) = 0.$$

□

In the particular case of  $f(z) = p(z) = z^2$ , we obtain  $M_p^*(z) = 0$ , for all  $z \in \mathbb{C}$ . Therefore, the dynamical plane has only one basin of attraction corresponding to the double root  $\alpha = 0$ .

### 3. The Frozen Derivative Methods

In this section we study the dynamics of a family of multi-point iterative methods obtained from Newton's method by replacing  $f(z)$  by a linear combination of values of  $f(z)$  in different points. Specifically, the general scheme is

$$x_{k+1} = x_k - \frac{\sum_{j=1}^m B_j f(\eta_j(x_k))}{f'(x_k)}, \quad (14)$$

where  $\eta_j(x_k) = x_k - \tau_j \frac{f(x_k)}{f'(x_k)}$ , for  $j = 1, 2, \dots, m$ ,  $\tau_j$  and  $B_j$  are parameters to be chosen in  $[0, 1]$  and  $\mathbb{R}$  respectively, and  $m$  is a positive integer (see [5]). These parameters satisfy the following relationships:

$$\sum_{j=1}^m B_j (1 - \tau_j) = 1, \quad \sum_{j=1}^m B_j \tau_j^2 = 1. \quad (15)$$

Distinct values of these parameters yields to different methods, as we will see in the following; moreover, they play an important role in the order of convergence of the method, which is at least three for simple roots and  $m \geq 2$ .

The fixed point operator of these methods is:

$$O_p(z) = z - \frac{\sum_{j=1}^m B_j f(\eta_j(z))}{f'(z)}. \quad (16)$$

The Scaling Theorem for this operator can be established in the following way.

**THEOREM 3.1** *Let  $f$  be an analytic function on the Riemann sphere, and  $A(z) = \alpha z + \beta$ , with  $\alpha \neq 0$ , an affine map. If  $g(z) = \lambda(f \circ A)(z)$ , where  $\lambda \in \mathbb{C} - \{0\}$ , then the fixed point operator  $O_f$  is analytically conjugated to  $O_g$  by  $A$ , that is,  $A \circ O_f \circ A^{-1}(z) = O_g(z)$ .*



Now, we are going to study its dynamics for quadratic polynomials defined on the complex plane,  $p(z) = z^2 + c$ ,  $c \in \mathbb{C}$ ,  $c \neq 0$ .

$$\begin{aligned} O_p(z) &= z - \frac{\sum_{j=1}^m B_j f(\eta_j(z))}{f'(z)} = \\ &= \frac{8z^4 - (c^2 + 2cz^2 + z^4) \sum_{j=1}^m B_j \tau_j^2 - (4cz^2 + 4z^4) \sum_{j=1}^m B_j (1 - \tau_j)}{8z^3}. \end{aligned}$$

By using the relationships (15) between the parameters, we obtain a common expression of the operator for any member of the family (14):

$$O_p(z) = \frac{3z^4 - 6cz^2 - c^2}{8z^3}. \quad (17)$$

This operator has four fixed points: two of them are the roots of the polynomial. The other two are called *strange fixed points*.

$$O_p(z) = z \Rightarrow z = \pm i\sqrt{c}, \pm i\sqrt{\frac{c}{5}}.$$

The dynamical properties of a complex analytical function are often determined for the dynamics of its critical points. In this case,

$$O'_p(z) = \frac{3(z^2 + c)^2}{8z^4},$$

allows us to deduce that the only critical points are the roots of the polynomial. Moreover,  $O'_p(\pm i\sqrt{c}) = 0$  implies that these roots are superattractor critical points. The other roots of  $O_p(z)$  are repulsive fixed points ( $O'_p(\pm i\sqrt{\frac{c}{5}}) = 6$ ); so, they are in the Julia set.

As in Newton's method, the Fatou set consists of the basins of attraction of the two roots of the polynomial. That means that these methods never fail for quadratic polynomials when they are applied on an open set of the complex plane. The dynamical plane of the operator (17) is shown in the Figure 1.

From Theorem 1.2, we know that Newton's iteration function on any quadratic polynomial is conjugated to  $z^2$ . In this case, we prove that  $O_p$  on quadratic polynomials has a more complicated expression:

**THEOREM 3.2** *Let  $p(z)$  be a quadratic polynomial with simple roots. The operator  $O_p(z)$  has the following properties:*

- i)  $O_p(z)$  is globally, analytically conjugated to the rational map  $z^3 \frac{z+2}{2z+1}$ .
- ii) The unit circle  $S^1(z)$  is included in the invariant Julia set. Moreover, this set is connected.
- iii) The Fatou set is defined by the two basins of attraction of the superattracting fixed points: 0 and  $\infty$ .

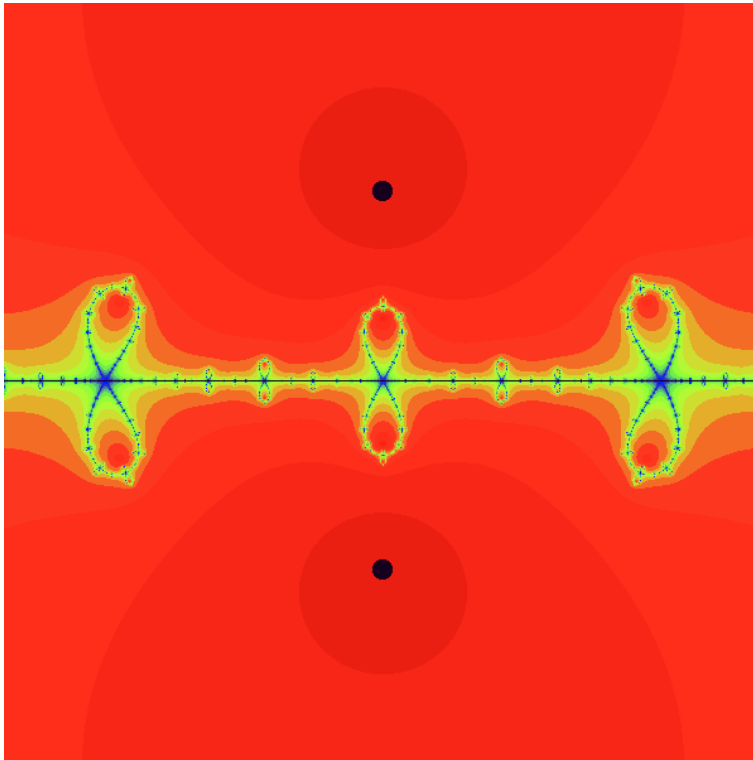


Figure 1. Dynamical plane for the frozen derivative methods on quadratic polynomials

**Proof:** As before, we consider the conjugacy map (3) and its inverse

$$h^{-1}(z) = i\sqrt{c}\frac{z+1}{1-z}.$$

Therefore,

$$B(z) = (h \circ O_p \circ h^{-1})(z) = z^3 \frac{z+2}{2z+1} \quad (18)$$

is a rational map of degree three that has superattracting fixed points at 0 and  $\infty$ . As in the previous case, this map does not depend on the parameter  $c$ . Moreover, for every  $z \in S^1$ ,  $B(z) \in S^1$  since

$$\forall z \in S^1, |z| = 1 \Rightarrow \left| z^3 \frac{z+2}{2z+1} \right| = \left| \frac{z+2}{2z+1} \right|$$

and

$$\left| \frac{z+2}{2z+1} \right| = \left| \frac{(z+2)(2\bar{z}+1)}{(2z+1)(2\bar{z}+1)} \right| = \left| \frac{z+4+4\bar{z}}{5+4Re(z)} \right| = 1,$$

since  $|z+4+4\bar{z}| = 5+4Re(z)$ .

Therefore, the unit circle is invariant under this function  $B : S^1 \rightarrow S^1$ , and it separates the two basins of attraction of the two superattractor fixed points: 0 and  $\infty$ .  $\square$

It is known that the Julia set separates the two basins of attraction of the two superattractor fixed points: 0 and  $\infty$ . Because of the rational factor in (18), this set

is more complicated than the one of Newton's method; nevertheless, all the points in the unit circle belongs to the Julia set (see Figure 2).



Figure 2. Julia set for the frozen derivative methods on quadratic polynomials

There are some well known iterative methods belonging to family (14). For example, the Potra-Pták's method (see [16]) is a particular case for  $m = 2$ ,  $B_j = 1$ ,  $j = 1, 2$   $\tau_1 = 0$ ,  $\tau_2 = 1$ .

There are other two methods, related with the golden ratio number, coming from the general formula (14), that are also of order three (see [5])

$$y_k = x_k - \frac{2}{1 + \sqrt{5}} \frac{f(x_k)}{f'(x_k)}, \quad x_{k+1} = x_k - \frac{3 + \sqrt{5}}{2} \frac{f(y_k)}{f'(x_k)},$$

and

$$y_k = x_k + \frac{1 + \sqrt{5}}{2} \frac{f(x_k)}{f'(x_k)}, \quad x_{k+1} = x_k - \frac{3 - \sqrt{5}}{2} \frac{f(y_k)}{f'(x_k)}.$$

As we have proved before, when we apply these three different methods on quadratic polynomial defined on the complex plane,  $p(z) = z^2 + c$ , we obtain the same operator  $O_p(z) = \frac{3z^4 - 6cz^2 - c^2}{8z^3}$ .

**Remark** The dynamics of methods of family (14) on quadratic polynomials defined on the complex plane are the same. The dynamical plane is shown in Figure 2.

We are going to complete the study of the frozen derivative family (14) in case of multiple roots (analyzed in [6]). This family converges linearly for multiple roots.

To reach the quadratic convergence the fixed point operator  $O(z)$  must be

$$O_p(z) = z - D \frac{\sum_{j=1}^m B_j f(\eta_j^*(z))}{f'(z)},$$

$$\text{where } D = \frac{q}{\sum_{j=1}^m B_j (1 - \tau_j)^q}.$$

It is easy to see that this operator applied on the polynomial  $p(z) = z^2$  is  $O_p(z) = 0$ . So, as in the previous section, the dynamical plane has only one basin of attraction.

#### 4. Conclusions

The dynamical behavior of iterative methods for solving nonlinear equations is an important tool to measure their stability and reliability. Unfortunately, only Newton's method have been studied in detail until now. In this paper, we analyze the dynamical properties of two rich families of iterative schemes on quadratic polynomials.

We have obtained the conjugacy function of the operators associated to multi-point interpolation methods and frozen derivative methods applied on quadratic polynomials.

We have proved that the dynamical plane for the operator of the multi-point interpolation methods on quadratic polynomials for different roots is the same than in the Newton case. This is explained from the fact that the two operators are conjugated to powers of  $z$  (Theorems 1.2 and 2.2). For multiple roots we obtain a variant of the interpolation family which has order of convergence two (Theorem 2.3). In this case the dynamical plane has only one basin of attraction.

Finally, the Julia set in the dynamical plane for the operator of the frozen derivative methods is more complicated. In Theorem 3.2 we show that this operator is conjugated to the product of a monomial and a rational function. For multiple roots the family obtained in [6], applied to the polynomial  $z^2$ , has the same dynamical plane as in the previous case.

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