DEPARTAMENT DE MATEMÀTICA APLICADA

Dynamics of strongly continuous semigroups associated to certain differential equations

TESI DOCTORAL REALITZADA PER:

Javier Aroza Benlloch

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Als meus pares
Ja saps que a tu

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Podría empezar y acabar como decía Charles Bukowski en *Post office* (Cartero, 1971): “Esto se presenta como un libro de ficción y no está dedicado a nadie”. En cambio, no podía dejar pasar la oportunidad de agradecer su tiempo y paciencia a toda esa gente que ha compartido conmigo este periodo de mi vida.

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Nules, a 30 de julio de 2015
Javier Aroza Benlloch

Aquesta memòria ha estat elaborada en l'Institut Universitari de Matemàtica Pura i Aplicada (IUMPA) de la Universitat Politècnica de València, durant el període de gaudi d'un contracte com a personal investigador en el projecte "Análisis funcional, teoría de operadores y aplicaciones", Programa Prometeu per a grups d'investigació d'excel·lència (PROMETEO/2008/101, Generalitat Valenciana) durant el període 2009-2012, i una beca de movilitat per estudiants en Programes de Doctorat amb l'Esment cap a l'Excel· lència, referència MHE2011-00201, per tal d'optar al grau de Doctor amb menció "Doctor Internacional" fent una estada en el estranger durant l'any 2012.

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Resumen

La presente memoria "Dinámica de semigrupos fuertemente continuos asociadas a ciertas ecuaciones diferenciales" es analizar, desde el punto de vista del análisis funcional, la dinámica de las soluciones de ecuaciones de evolución lineales. Estas soluciones pueden ser representadas por semigrupos fuertemente continuos en espacios de Banach de dimensión infinita. El objetivo de nuestra investigación es proporcionar condiciones globales para obtener caos, en el sentido de Devaney, y propiedades de estabilidad de semigrupos fuertemente continuos, los cuales son soluciones de ecuaciones de evolución lineales.

Este trabajo está compuesto de tres capítulos principales. El Capítulo 0 es introductorio y define la terminología básica y notación usada, además de presentar los resultados básicos que usaremos a lo largo de esta tesis. Los Capítulos 1 y 2 describen, de forma general, un semigrupo fuertemente continuo inducido por un semiflujo en espacios de Lebesgue y en espacios de Sobolev, los cuales son solución de una ecuación diferencial lineal en derivadas parciales de primer orden. Además, algunas caracterizaciones de las principales propiedades dinámicas, incluyendo hiperciclicidad, mezclante, débil mezclante, caos y estabilidad, se obtienen a lo largo de estos capítulos. El Capítulo 3 describe las propiedades dinámicas de una ecuación en diferencias basada en el llamado modelo de nacimiento-muerte y analiza las condiciones previamente probadas para este modelo, mejorándolas empleando una estrategia diferente.

La finalidad de esta tesis es caracterizar propiedades dinámicas para este tipo de semigrupos fuertemente continuos de forma general, cuando sea posible, y extender estos resultados a otros espacios. A lo largo de esta memoria, estos resultados son comparados con los resultados previos dados por varios autores en años recientes.

Resum

La present memòria “Dinàmica de semigrups fortament continus associats a certes equacions diferencials” és analitzar, des del punt de vista de l’anàlisi funcional, la dinàmica de les solucions d’equacions d’evolució lineals. Aquestes solucions poden ser representades per semigrups fortament continus en espais de Banach de dimensió infinita. L’objectiu de la nostra investigació es proporcionar condicions globals per obtenir caos, en el sentit de Devaney, i propietats d’estabilitat de semigrups fortament continus, els quals són solucions d’equacions d’evolució lineals.

Aquest treball està compost de tres capítols principals. El Capítol 0 és introductori i defineix la terminologia bàsica i notació utilitzada, a més de presentar els resultats bàsics que utilitzarem al llarg d’aquesta tesi. Els Capítols 1 i 2 descriuen, de forma general, un semigrup fortament continu induït per un semiflux en espais de Lebesgue i en espais de Sobolev, els quals són solució d’una equació diferencial lineal en derivades parcials de primer ordre. A més, algunes caracteritzacions de les principals propietats dinàmiques, incloent-hi hiperciclicitat, mesclant, dèbil mesclant, caos i estabilitat, s’obtenen al llarg d’aquests capítols. El Capítol 3 descrivís les propietats dinàmiques d’una equació en diferències basada en el model de naixement-mort i analitza les condicions prèviament provades per aquest model, millorant-les utilitzant una estratègia diferent.

La finalitat d’aquesta tesi és caracteritzar propietats dinàmiques d’aquest tipus de semigrups fortament continus de forma general, quan siga possible, i estendre aquests resultats a altres espais. Al llarg d’aquesta memòria, aquests resultats són comparats amb els resultats previs obtinguts per diversos autors en anys recents.

Summary

The purpose of the Ph.D. Thesis “Dynamics of strongly continuous semigroups associated to certain differential equations” is to analyse, from the point of view of functional analysis, the dynamics of solutions of linear evolution equations. These solutions can be represented by a strongly continuous semigroup on an infinite-dimensional Banach space. The aim of our research is to provide global conditions for chaos, in the sense of Devaney, and stability properties of strongly continuous semigroups which are solutions of linear evolution equations.

This work is composed of three principal chapters. Chapter 0 is introductory and defines basic terminology and notation used, besides presenting the basic results that we will use throughout this thesis. Chapters 1 and 2 describe, in general way, a strongly continuous semigroup induced by a semiflow in Lebesgue and Sobolev spaces which is a solution of a linear first order partial differential equation. Moreover, some characterizations of the main dynamical properties, including hypercyclicity, mixing, weakly mixing, chaos and stability are given along these chapters. Chapter 3 describes the dynamical properties of a difference equation based on the so-called birth-and-death model and analyses the conditions previously proven for this model improving them by employing a different strategy.

The goal of this thesis is to characterize dynamical properties of these kind of strongly continuous semigroups in a general way, whenever possible, and to extend these results to another spaces. Along this memory, these findings are compared with the previous ones given by many authors in recent years.

Contents

Introduction

Many authors have recently demonstrated an interest in dynamical properties of strongly continuous semigroups, in short C_0-semigroups, associated to dynamical linear systems. Several techniques have been used in different fields as Ergodic and Operator Theory, Strongly-mixing measures with full support or Distributional and Devaney chaos, see [1, 14, 16, 21, 22, 30, 38, 56, 57]. In practice, these dynamical linear systems are usually related to evolution equations which can be represented by first order partial differential equations or difference equations.

The origin of these studies lies in Operator Theory, particularly in investigations of the behaviour of the powers of a single operator. Birkhoff's Transitivity Theorem (1922, [24]) gives an equivalence between topological transitivity of an operator and the existence of hypercyclic vectors, defined by Beauzamy in 1987 ([18]). In 1989, Robert L. Devaney ([39]) introduced the notion of chaos for single operators, composed by the existence of a hypercyclic vector and of a dense set of periodic points and the well known "butterfly effect".

Desch, Schappacher and Webb ([38]) studied the notions of hypercyclicity and chaos for linear semigroups in a general way. These authors established the equivalence between hypercyclicity and topological transitivity in this context. They provided some criteria based on the infinitesimal generator of a semigroup and applied these results to some linear partial differential equations. Moreover, they characterized hypercyclicity and chaos for translation semigroups and discrete shifts. These semigroups were used, as a first step, to test certain criteria and dynamical properties, see [13, 28, 32, 33, 34, 52, 53, 54].

Chapter 0 introduces basic definitions of semigroup theory and dynamical properties, emphasizing different Banach spaces used in this memory and some useful properties and auxiliary results. In particular, we recall the notion of semiflow in order to describe a special kind of C_0-semigroups studied by Kalmes in [45, 46]. Semigroups as the solution of Lasota equation, analysed in depth by Dawidowicz, Brzeźniak and Poskrobko ([27, 36]), are particular cases of the semigroups used by

Kalmes. This type of C_0-semigroups have an admissible weight directly related to Desch et al. results.

Motivated by the work of Dawidowicz, et al. about dynamical properties of Lasota equation on Lebesgue and Hoelder spaces we try to characterize, in Chapters 1 and 2, when these properties occur. Lasota equation, introduced by Lasota and Mackey in [50], is related to the dynamical behaviour of blood cell populations. Dawidowicz, et al. study this equation assuming different hypotheses on its coefficients. A simple version called von Foerster-Lasota equation is the following:

$$\frac{\partial}{\partial t}u(t,x) + x\frac{\partial}{\partial x}u(t,x) = \gamma\, u(t,x), \quad t \geq 0,\, 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x), \quad 0 < x < 1,$$

where v is a given function.

Particularly, this equation is interesting if we observe the “complementary” behaviour of the solution $u(t,x) = e^{\gamma t}v(xe^{-t})$, which describes a C_0-semigroup in the Lebesgue space $L^p(0,1)$, $1 \leq p < \infty$. To be specific, this semigroup is chaotic if and only if it is not stable if and only if $\gamma > -1/p$ or, analogously, it is stable if and only if it is not chaotic (in particular, not hypercyclic) if and only if $\gamma \leq -1/p$. A generalized version of this equation was introduced by Dawidowicz, et al., improving the model and conserving the “complementary” behaviour, for example replacing γ by a suitable function.

Developing this idea still further, in Chapters 1 and 2 we characterize the principal dynamical properties as chaos and stability for Kalmes’s C_0-semigroups. These results show, as particular case, the previous ones given by Dawidowicz et al. In parallel, they obtain new results improving their own conditions in [26, 37]. We can observe in these findings similarities with our research or with the results given by Kalmes in [45] for the multidimensional case. Specifically, we study on Chapter 1 the chaotic behaviour of these C_0-semigroups on Lebesgue and Sobolev spaces. We compare our conditions with the previous ones provided by Dawidowicz et al. and Desch et al. On Chapter 2 we focus on stability properties following the same lines of the previous chapter. We try to show if the “complementary” behaviour always happens as Kalmes suggested us.

The contents of Chapter 1 have been published in [3] and the contents of Chapter 2 have been included in [4].

In the last chapter, we change the first order linear partial differential equations by models based on a difference equations. Singularly, we study the birth-and-death models which describe evolution families of cells populations like cancer cells. In

1992 Azmy and Protopopescu ([58]) considered this models and gave conditions under which the solution semigroup is chaotic or stable. Banasiak et al. in [7, 9, 11] improved these models, first separating this model in their birth and death parts and later with the joined model case. The joined model case can be represented by,

$$\begin{aligned} \frac{df_1}{dt} &= af_1 + df_2, \\ \frac{df_n}{dt} &= bf_{n-1} + af_n + df_{n+1}, \qquad n \geq 2, \end{aligned}$$

where the coefficients are real constants.

In this case the authors show that if $0 < |b| < |d|$ and $|a| < |b+d|$ then the solution C_0-semigroup in ℓ^p is chaotic. In this case, if we assume $|d| < |b|$ we lose the chaotic property. As in the previous chapters, the key is on the assumptions of the coefficients. We generalize this model assuming non-constant coefficients and improving, in part, the results given by Banasiak, Lachowicz and Moszyński. We focus in a special Banach space that assures the existence of a solution C_0-semigroup of this model, based on the spectral radius of its generator and considering two general cases: sequences of bounded or unbounded coefficients.

The contents of Chapter 3 were published in [5].

Chapter 0

Preliminaries

This chapter is dedicated to establish the basis of the thesis, with the main definitions and notation that we will use throughout it. We refer as basic reference on functional analysis, measure theory and complex analysis to [25, 59].

0.1 General framework and notation

0.1.1 Reminder of spectral theory

Let X be a Banach space on $\mathbb{C}$ and let $A : D(A) \subset X \to X$ be a closed, linear operator on X, denoting by $D(A)$ its domain of definition.

We call $\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\}$ *the resolvent set* of A. Its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A and its *spectral radius* is defined by $r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|$.

The *point spectrum* of A is $P_\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective }\}$, where $\lambda \in P_\sigma(A)$ is called an *eigenvalue*. Each $0 \neq x \in D(A)$ satisfying $(\lambda - A)x = 0$ is an *eigenvector* of A associated to λ.

0.1.2 Functional spaces

We denote by λ the Lebesgue measure on $\mathbb{R}$ and by λ^d the Lebesgue measure on $\mathbb{R}^d$ for every $d \in \mathbb{N}$, $d \geq 2$. If Ω is an open subset of $\mathbb{R}^d$, $1 \leq p < \infty$, $\rho : \Omega \to (0, +\infty)$ is a λ^d-measurable function, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we set

$$L^p_\rho(\Omega, \mathbb{K}) := \left\{ u : \Omega \to \mathbb{K} \mid \text{u measurable}, \int_\Omega |u(x)|^p \rho(x) dx < \infty \right\}.$$

For $p = \infty$ we set

$$\begin{aligned} L^\infty(\Omega, \mathbb{K}) &= \left\{ u : \Omega \to \mathbb{K} \mid \exists K > 0 \ |u| \leq K \ \lambda^d\text{-a.e.} \right\} \\ &= \left\{ u : \Omega \to \mathbb{K} \mid \exists K > 0 \ |u| \leq K \ \rho \cdot \lambda^d\text{-a.e.} \right\}. \end{aligned}$$

Endowed with the norm

$$\|u\|_{p,\rho} = \left(\int_\Omega |u(x)|^p \rho(x) dx \right)^{\frac{1}{p}},$$

respectively,

$$\|u\|_\infty = \inf \{ K > 0 \ : \ |u| \leq K \},$$

$L^p_\rho(\Omega, \mathbb{K})$ is a Banach space on $\mathbb{K}$. If $\rho = 1$, we write simply $L^p(\Omega, \mathbb{K})$. Observe that for $1 < p < \infty$ the topological dual of $L^p_\rho(\Omega, \mathbb{K})$ is $L^q_\rho(\Omega, \mathbb{K})$, where q is the conjugate exponent of p, while the dual of $L^1_\rho(\Omega, \mathbb{K})$ is $L^\infty(\Omega, \mathbb{K})$.

We will write $L^p_\rho(\Omega)$ if there is no need to specify the field $\mathbb{K}$.

Let $I = (a, b)$ be a bounded open interval of $\mathbb{R}$. For $1 \leq p < \infty$ we denote as usual by $W^{1,p}(I)$ the first order Sobolev space of p-integrable functions on I, i.e.

$$W^{1,p}(I) = \{ u \in L^p(I); u' \in L^p(I) \},$$

where u' denotes the distributional derivative of u. Endowed with the norm

$$\|u\|_{1,p} = \|u\|_p + \|u'\|_p,$$

$W^{1,p}(I)$ is a Banach space. It holds that $W^{1,p}(I) \subseteq C[a, b]$ and that for any $x \in [a, b]$ the point evaluation δ_x in x is a continuous linear form on $W^{1,p}(I)$. We are interested in the following closed subspace of $W^{1,p}(I)$,

$$W^{1,p}_*(I) := \ker \delta_a.$$

From the boundedness of I we have the topological direct sum

$$W^{1,p}(I) = W^{1,p}_*(I) \oplus \operatorname{span}\{\mathbb{1}\},$$

where $\mathbb{1}$ denotes the constant function with value 1.

First, we show a result that appears on [63, Lemma 1.2.5] originally proved on [55, pp. 313-315]. This is a tool, called mollifier, to approximate a function which is not smooth by sufficient smooth ones.

Let φ be a function, belonging to $C_0^\infty(\mathbb{R}^n)$, such that $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. Put $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. For $u \in L^1_{loc}(\mathbb{R}^n)$, that is, for u which is absolutely integrable on every compact set of $\mathbb{R}^n$, we define

$$(\varphi_\delta * u)(x) = \int_{\mathbb{R}^n} \varphi_\delta(x-y)u(y)dy.$$

If $u \in C(\mathbb{R}^n)$ is uniformly continuous, then as $\delta \to 0$ the function $\varphi_\delta * u$ converges uniformly to u. If $u \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, then $\varphi_\delta * u$ converges strongly to u in $L^p(\mathbb{R}^n)$ as $\delta \to 0$. Furthermore, the following lemma holds.

Let $n \in \mathbb{N}$, we say that $g \in B^1(\mathbb{R}^n)$ if g is bounded with continuous bounded derivative on $\mathbb{R}^n$.

Lemma 0.1.1 *For $1 \leq p < \infty$, $g \in B^1(\mathbb{R}^n)$, i.e., bounded with continuous bounded derivative on $\mathbb{R}^n$ and $u \in L^p(\mathbb{R}^n)$, we define*

$$C_\delta u = \varphi_\delta * \left(g\frac{\partial u}{\partial x_j}\right) - g\left(\varphi_\delta * \frac{\partial u}{\partial x_j}\right) = \left[\varphi_\delta * g\frac{\partial}{\partial x_j}\right]u,$$

where the differentiation $\frac{\partial}{\partial x_j}$ is to be understood in the sense of distribution. Then

(i) $C_\delta u \in L^p(\mathbb{R}^n)$ and there exists a constant C, independent of δ and u, such that $\|C_\delta u\|_p \leq C\|u\|_p$,

(ii) $C_\delta u$ converges strongly to 0 in $L^p(\mathbb{R}^n)$ as $\delta \to 0$.

Ending with Banach spaces, we denote by $\ell^p := \ell^p(Y)$ the space of sequences $y := (y_n)_{n\in\mathbb{N}}$ such that $y_n \in Y$ and its norm $\|y\|_p := \left(\sum_{n\in\mathbb{N}} |y_n|^p\right)^p$ are bounded. Observe that we can take Y any Banach space or vector space depending of its application.

0.1.3 Vector valued holomorphic functions

Let X be a Banach space on $\mathbb{C}$ and let E be a vector space over $\mathbb{C}$. A map $f : U \to X$ on an open set $U \subset \mathbb{C}$ is *weakly holomorphic* if the map $\lambda \mapsto \langle f(\lambda),\ \phi\rangle$ is holomorphic for every functional $\phi \in X^*$, where $\langle \cdot, \cdot\rangle$ denotes the inner product.

A subset U of E is said to be *finitely open* if $U \cap F$ is open in the Euclidean topology of F for each finite-dimensional subspace F of E.

Let U be a finitely open subset of E and let F be a locally convex space. We call $f : U \subset E \to F$ *Gâteaux* or *$\mathcal{G}$-holomorphic* if for each $\xi \in U$, $\eta \in E$ and $\phi \in F'$ the complex valued function of one complex variable

$$\nu \to \phi \circ f(\xi + \nu\eta)$$

is holomorphic on some neighbourhood of 0 in $\mathbb{C}$.

Let U and E as above and let F be a locally convex space over $\mathbb{C}$. We call f *Gâteaux differentiable* if for each $\xi \in U$ and each v in E and being $\nu \in \mathbb{C}$

$$df(\xi)(v) := \lim_{\nu\to 0} \frac{f(\xi + \nu v) - f(\xi)}{\nu}$$

exists in the completion of F.

If U is a finitely open subset of a vector space E and F is a locally convex space then the notions of $f : U \to F$ Gâteaux differentiable and $\mathcal{G}$-holomorphic are equivalent.

0.2 Strongly continuous semigroups

We recall in this section some basic definitions and properties of C_0-semigroups, referring to [42] for further information and notation.

Throughout this section, let X be a Banach space.

Definition 0.2.1 A C_0*-semigroup* is a one-parameter family $\mathcal{T} := (T(t))_{t\geq 0}$ of bounded linear operators on X satisfying the following conditions

(i) $T(0) = I$,

(ii) $T(t)T(s) = T(t + s)$ for all $t, s \geq 0$,

(iii) $\lim_{s\to t} T(s)x = T(t)x$ for all $x \in X$ and $t \geq 0$.

Every C_0-semigroup $\mathcal{T}$ is exponentially bounded, indeed there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0.$$

Definition 0.2.2 The *generator* $A : D(A) \subseteq X \to X$ of a C_0-semigroup $\mathcal{T}$ on X is the operator defined by

$$Ax := \lim_{t \to 0} \frac{1}{t}(T(t)x - x)$$

for every x in its maximal *domain*

$$D(A) = \{x \in X \ : \ \lim_{t \to 0} \frac{1}{t}(T(t)x - x) \text{ exists}\}.$$

Proposition 0.2.3 *For the generator $(A, D(A))$ of a strongly continuous semigroup $\mathcal{T}$, the following properties hold.*

(i) $A : D(A) \subseteq X \to X$ *is a closed and densely defined linear operator.*

(ii) *If $x \in D(A)$, then $T(t)x \in D(A)$ and*

$$\frac{d}{dx}T(t)x = T(t)Ax = AT(t)x, \text{ for all } t \geq 0.$$

The Hille-Yosida theorem characterizes the operators $(A, D(A))$ that are generators of C_0-semigroups. In the case that A is a bounded linear operator on X, in short $A \in \mathcal{L}(X)$, then A generates an uniformly continuous semigroup (i.e., continuous with respect to the norm in $\mathcal{L}(X)$) given by

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \ t \geq 0.$$

By the point spectral mapping theorem for semigroups with X complex Banach space, if $(T(t))_{t \geq 0}$ is a C_0-semigroup with generator $(A, D(A))$ and $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, $x \in X$, then $T(t)x = e^{\lambda t}x$.

In order to describe the elements of $D(A)$, we will use the following concept.

Definition 0.2.4 A subspace D of the domain $D(A)$ of a linear operator $A : D(A) \subseteq X \to X$ is called a *core* for A if D is dense in $D(A)$ for the graph norm

$$\|x\|_A := \|x\| + \|Ax\|.$$

Clearly, if $(A, D(A))$ is closed and D is a core for A then the closure of $(A|_D, D)$ is $(A, D(A))$.

We recall that a subspace $Y \subset X$ is $\mathcal{T}$*-invariant* or *invariant under* $\mathcal{T}$ if $T(t)Y \subseteq Y$ for all $t \geq 0$.

Proposition 0.2.5 *Let $(A, D(A))$ be the generator of a C_0-semigroup $\mathcal{T}$ on X. If a subspace D of $D(A)$ is $\|\cdot\|$-dense in X and invariant under $\mathcal{T}$, it is always a core for A.*

Given the generator of a C_0-semigroup, we can ask under which conditions an additive perturbation generates a new semigroup. The Bounded Perturbation Theorem provides an answer. For its proof see, for example, [42, Chapter III].

Theorem 0.2.6 (Bounded Perturbation Theorem) *Let $(A, D(A))$ be the generator of a C_0-semigroup $\mathcal{T} := (T(t))_{t\geq 0}$ on a Banach space X satisfying*

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0 \text{ and some } \omega \in \mathbb{R},\ M \geq 1.$$

If B is a bounded linear operator on X, then $C := A + B$ with $D(C) := D(A)$ generates a C_0-semigroup $\mathcal{S} := (S(t))_{t\geq 0}$ satisfying

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t} \text{ for all } t \geq 0.$$

Moreover, for every $t \geq 0$

$$S(t) = \sum_{n=0}^{\infty} S_n(t),$$

where $S_0(t) := T(t)$ and

$$S_{n+1}(t) := \int_0^t T(t-s)BS_n(s)ds,\ n \in \mathbb{N}.$$

0.3 Topological dynamics and strongly continuous semigroups

We recall in this section the notions of hypercyclicity, transitivity, chaos, mixing and weakly mixing properties for C_0-semigroups. All these properties where originally defined for the behaviour of the powers of a single operator, and, since the seminal paper [38], they have been widely studied also for semigroups. We refer to the recent monographs [17, 44] for a complete introduction to this topic.

Throughout this section let X be an infinite-dimensional separable Banach space. Separability is necessary because hypercyclicity implies that the space has to be separable.

Definition 0.3.1 Let $\mathcal{T}$ be a C_0-semigroup on X.

(i) $\mathcal{T}$ is called *topologically transitive* if, for any pair $U,\ V$ of nonempty open subsets of X, there exists some $t_0 \geq 0$ such that $T(t_0)U \cap V \neq \emptyset$.

(ii) $\mathcal{T}$ is said to be *mixing* if, for any pair $U,\ V$ of nonempty open subsets of X, there exists some $t_0 \geq 0$ such that $T(t)U \cap V \neq \emptyset$ for all $t \geq t_0$.

(iii) $\mathcal{T}$ is called *weakly mixing* if $(T(t) \oplus T(t))_{t\geq 0}$ is topologically transitive on $X \oplus X$.

(iv) $\mathcal{T}$ is *hypercyclic* if there exist $x \in X$, called hypercyclic vector, such that its orbit $\{T(t)x\ :\ t \geq 0\}$ is dense in X.

(v) $\mathcal{T}$ is *chaotic* or *Devaney chaotic* if it is hypercyclic and the set of its periodic points $Per(\mathcal{T}) = \{x \in X;\ \exists t > 0\ :\ T(t)x = x\}$ is dense in X.

On a linear dynamical system, like ours, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (iv) $\Leftarrow$ (v).

Remark 0.3.2 The definition of chaos by Robert L. Devaney was given in terms of the "butterfly effect" or sensitive dependence on initial conditions, density of periodic points and transitivity. Later, Banks, Brooks, et al. ([12]) proved that for a continuous function on a metric space, topological transitivity and the density of the set of periodic points imply sensitive dependence on initial conditions. On the other hand, by adapting the proof of Birkhoff's Transitivity Theorem, it follows that transitivity and hypercyclicity are equivalent for a C_0-semigroup (see [44], reference). This justify the definition, originally given by Godefroy and Shapiro [43] for a single operator, that appears in Definition 0.3.1-(v).

Another notion related to chaos was established by Banasiak and Moszyński ([10]) used primarily for invariant and closed subspaces.

Definition 0.3.3 Let $\mathcal{T}$ be a C_0-semigroup on X. We say that $\mathcal{T}$ is *sub-chaotic* if there exists a non trivial closed subspace $Y \subset X$ invariant under $\mathcal{T}$, such that $\mathcal{S} = (T(t)_{|Y})_{t\geq 0}$ is a chaotic C_0-semigroup in Y.

Remark 0.3.4 ([10]) Each chaotic C_0-semigroup is also sub-chaotic. Any subspace with chaotic behaviour has infinite dimension.

Finally we recall the definition of stable C_0-semigroup.

Definition 0.3.5 Let $\mathcal{T}$ be a C_0-semigroup on X.

(i) $\mathcal{T}$ is *stable* or *strongly stable* if $\lim_{t\to+\infty} \|T(t)x\| = 0$, for all $x \in X$.

(ii) $\mathcal{T}$ is said to be *exponentially stable* if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t} \|T(t)\| = 0.$$

By the definition, it is clear that stable C_0-semigroup cannot be hypercyclic.

Quasi-conjugate were introduced in order to transfer dynamical properties of a dynamical system to another (see [44, Definition 1.5]).

Definition 0.3.6 Let $\mathcal{T}$ and $\mathcal{S}$ be a C_0-semigroups on a Banach spaces Y and X respectively. We say that $\mathcal{S}$ is quasi-conjugate to $\mathcal{T}$ if there is a continuous mapping $\phi : Y \to X$ with dense range such that $\phi \circ T(t) = S(t) \circ \phi$ for every $t \geq 0$.

$\mathcal{T}$ and $\mathcal{S}$ are said to be *conjugate*, if the above ϕ is a homeomorphism.

It is immediate that all the dynamical properties are preserved under conjugacy, while stability is preserved under quasi-conjugacy for bounded semigroups. Moreover, we can assure that all the rest of dynamical properties are preserved under quasi-conjugacy, see for example [44, Proposition 7.7] for the proof of all the dynamical properties except stability. We will only prove this result for stability property of bounded semigroups to complete this result ([4]).

Proposition 0.3.7 (Comparison test) *The properties hypercyclicity (transitivity), (weakly) mixing and (sub-)chaos are preserved under quasi-conjugacy. Stability is preserved under quasi-conjugacy for bounded semigroups.*

Proof. Let $\mathcal{T}$ and $\mathcal{S}$ be quasi-conjugate bounded C_0-semigroups on Banach spaces Y and X respectively, via $\Phi : Y \to X$, and assume that $\mathcal{T}$ is stable on Y. We will prove that $\mathcal{S}$ is stable on X. Let $M = \sup_{t\geq 0} ||S(t)||$.

Let $g \in X$ and let $\varepsilon > 0$. Since Φ is continuous with dense range, there exists $f \in Y$ such that $|g - \Phi(f)| < \frac{\varepsilon}{2M}$. Being $\mathcal{T}$ stable, there exists $\bar{t} > 0$ such that

$$\forall\, t \geq \bar{t} \qquad ||T(t)f|| < \frac{\varepsilon}{2||\Phi||}.$$

Then for every $t \geq \bar{t}$:

$$
\begin{aligned}
\|S(t)(g)\| &= \|S(t)(g - \Phi(f)) + (S(t) \circ \Phi)(f)\| \\
&\leq \|S(t)\| \ \|g - \Phi(f)\| + \|(S(t) \circ \Phi)(f)\| \\
&\leq \frac{\varepsilon}{2} + \|\Phi \circ T(t)(f)\| \leq \varepsilon.
\end{aligned}
$$

□

In recent years, studies of the main dynamical properties have been of interest for several authors using many different techniques (see, e.g., [19, 31, 38, 41]), some of them are based on the corresponding discrete counterparts [10, 15, 20, 23, 43].

Some of the most commonly used criteria for chaos are the following. For the details we refer the reader to original versions in [10] and [38]. A general vision of most common criteria for C_0-semigroups are given in [44, Chapter 7, Section 4]. We denote by Id the *identity operator* on X.

Proposition 0.3.8 *Let X be a complex separable infinite-dimensional Banach space and let $(A, D(A))$ be the generator of a C_0-semigroup $\mathcal{T}$ on X. Assume that there exists an open connected subset $\mathcal{U}$ and a weakly holomorphic map $f : \mathcal{U} \to X$ such that:*

(i) $\mathcal{U} \cap i\mathbb{R} \neq \emptyset$,

(ii) $f(\lambda) \in \ker(\lambda Id - A)$ *for every* $\lambda \in \mathcal{U}$,

(iii) if for some $\phi \in X^*$ the function $h(\lambda) = \langle f(\lambda), \ \phi \rangle$ is identically zero on $\mathcal{U}$, then $\phi = 0$.

Then $\mathcal{T}$ is chaotic and mixing.

Proposition 0.3.9 *Let X be a complex separable infinite-dimensional Banach space and let $(A, D(A))$ be the generator of a C_0-semigroup $\mathcal{T}$ on X. Assume that there exists an open connected subset $\mathcal{U}$ and a non-zero weakly holomorphic function $f : \mathcal{U} \to X$ such that:*

(i) $\mathcal{U} \cap i\mathbb{R} \neq \emptyset$,

(ii) $f(\lambda) \in \ker(\lambda Id - A)$ *for every* $\lambda \in \mathcal{U}$.

Then the restriction of $\mathcal{T}$ to the invariant subspace

$$X_{\mathcal{U}} := \overline{\operatorname{span}\{f(\lambda) \ : \ \lambda \in \mathcal{U}\}}$$

is chaotic and mixing. In particular, $\mathcal{T}$ is sub-chaotic.

0.4 Semiflows

In this section we review some basic results on semiflows, referring to [2] for further information.

Definition 0.4.1 Let Ω be a topological space. A function $\varphi : [0,\infty[\times\Omega \to \Omega$ is said to be a *semiflow* if it is a continuous function such that $\varphi(0,\cdot) = id_\Omega$, $\varphi(t,\cdot)\circ\varphi(s,\cdot) = \varphi(t+s,\cdot)$ for all $t,s \geq 0$ and such that $\varphi(t,\cdot)$ is injective for all $t \geq 0$. If we let $t \in \mathbb{R}$ we call φ a *flow*.

Typical examples of semiflows are those associated with autonomous first order differential systems.

If $\Omega \subseteq \mathbb{R}^d$ is open and $F \in C^1(\Omega)$, for every $x_0 \in \Omega$ there is a unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x),\ x(0) = x_0. \tag{0.1}$$

Denoting its maximal domain of definition by $J(x_0)$ it is well-known that $J(x_0)$ is an open interval containing 0. Throughout this section we make the general assumption that $[0,\infty) \subset J(x_0)$ for every $x_0 \in \Omega$, i.e. $\varphi : [0,\infty) \to \Omega$.

From the uniqueness of the solution it follows that $\varphi(t,\cdot)$ is injective for every $t \geq 0$ and $\varphi(t+s,x) = \varphi(t,\varphi(s,x))$ for all $x \in \Omega$ and $s,t \in J(x)$ with $s+t \in J(x)$. Moreover, $\varphi(t,\cdot) : \Omega \to \varphi(t,\Omega)$ is bijective for $t \geq 0$ and for every $t \geq 0$, $x \in \varphi(t,\Omega)$ and for all $s \in [0,t]$ we have $\varphi(-s,x) = \varphi(s,\cdot)^{-1}(x)$. Thus φ is a semiflow according to Definition 0.4.1.

Since F is a C^1-function, the same is true for $\varphi(t,\cdot)$ on Ω and $\varphi(t,\cdot)^{-1}$ on $\varphi(t,\Omega)$ for every $t \geq 0$. With a slight abuse of notation we will denote by $\varphi(-t,\cdot)$ the inverse of $\varphi(t,\cdot)$.

Remark 0.4.2 If we set:

$$\Omega_0 := \{x \in \Omega;\, F(x) = 0\} \ \text{ and } \ \Omega_1 := \Omega \setminus \Omega_0.$$

The following properties are satisfied:

(i) Let $y \in \Omega_0$, then $\varphi(t,y) = y$ for all $t \geq 0$.

(ii) The uniqueness for the solutions of the initial value problem (0.1) immediately implies for Ω_0 that

$$\forall\, t \geq 0 :\ \varphi(t, \Omega_1) = \varphi(t, \Omega) \setminus \Omega_0 \subseteq \Omega_1.$$

Definition 0.4.3 A subset $M \subseteq \Omega$ is called *positively invariant under* φ if for every $x \in M$ and for every $t \in J(x_0) \cap]0, \infty[$ it holds that $\varphi(t, x) \in M$.

To assess when this property is obtained we recall [2, Theorem 16.9 and Corollary 16.10]. These results will be used to give a description of some generators, in Chapter 1.

Theorem 0.4.4 *Let $X = \mathbb{R}^m$ and let $\psi \in C^1(\Omega, \mathbb{R})$ be such that $\nabla\psi(x) \neq 0$ for all $x \in \psi^{-1}(0)$, that is, assume that 0 is a regular value of ψ. Then $M := \psi^{-1}(-\infty, 0]$ is positive invariant if and only if*

$$\langle \nabla\psi(x), F(x) \rangle \leq 0, \qquad \forall x \in \partial M = \psi^{-1}(0).$$

Corollary 0.4.5 *Assume that $X = \mathbb{R}^m$ and let $\psi_1, \ldots, \psi_k \in C^1(\Omega, \mathbb{R})$. Moreover, assume that 0 is a regular value of each ψ_j, $j = 1, \ldots, k$ and let*

$$M := \bigcap_{j=1}^{k} \psi_j^{-1}(-\infty, 0].$$

If

$$\langle \nabla\psi_j(x), F(x) \rangle \leq 0, \qquad \forall x \in \psi_j^{-1}(0),\ j = 1, \ldots, k$$

then M is positively invariant.

We refer to the monograph of Amann [2] for further results on this topic, in particular to Chapter II.10 and Chapter IV.16 where(semi)flows and positive invariance are studied.

Consider the following general assumptions.

General assumptions 0.4.6 As commented and to abbreviate notation, we establish:

(H1) $\Omega \subseteq \mathbb{R}$;

(H2) $F \in C^1(\Omega, \mathbb{R})$ and φ is the flow associated with F; i.e. for every $x_0 \in \Omega$, $\varphi(\cdot, x_0) : J(x_0) \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{x} = F(x),\ x(0) = x_0,$$

where the $J(x_0) \subseteq \mathbb{R}$ is the maximal domain of $\varphi(\cdot, x_0)$; It is known that $J(x_0)$ is an open interval such that $0 \in J(x_0)$.

(H3) $[0, \infty) \subset J(x_0)$ for every $x_0 \in \Omega$.

In the following consideration let $\Omega \subseteq \mathbb{R}$ be an open set and let $\partial_2\varphi(t, \cdot)$ denote the partial derivative of φ with respect to the second variable.

Remark 0.4.7 (i) For $x_0 \in \Omega$ we define $Z(x_0)$ to be the connected component of Ω_1 containing x_0 if $F(x_0) \neq 0$ and $Z(x_0) := \{x_0\}$ if $F(x_0) = 0$. It is well-known that $\varphi(t, Z(x_0)) \subseteq Z(x_0)$ for every $t \geq 0$, more precisely

$$\forall\, x_0 \in \Omega :\ \varphi([0, \infty), x_0) = \begin{cases} Z(x_0) \cap [x_0, \infty) & \text{if } F(x_0) \geq 0 \\ Z(x_0) \cap (-\infty, x_0] & \text{if } F(x_0) \leq 0. \end{cases}$$

(ii) By (i) and from the injectivity, it follows easily that $\varphi(t, \cdot) : \Omega \to \Omega$ is strictly increasing for all $t \geq 0$ and thus $\varphi(-t, \cdot) : \varphi(t, \Omega) \to \Omega$ is strictly increasing, too. Since

$$\forall\, t \geq 0, x \in \Omega :\ x = \varphi(-t, \varphi(t, x))$$

we obtain

$$\forall\, t \geq 0, x \in \Omega :\ 1 = \partial_2\varphi(-t, \varphi(t, x))\partial_2\varphi(t, x)$$

so that

$$\forall\, t \geq 0, x \in \Omega :\ \partial_2\varphi(t, x) > 0$$

as well as

$$\forall\, t \geq 0, x \in \varphi(t, \Omega) :\ \partial_2\varphi(-t, x) > 0.$$

We will take advantage of these properties. Moreover, we have the following representation of $\partial_2\varphi(t, x)$ at our disposal which will make easier to evaluate the characterizations of the different dynamical properties of a C_0-semigroup on $L^p_\rho(\Omega)$. This result was published in [3].

Proposition 0.4.8 *Assume (H1)-(H3). Then for every $x \in \Omega$ we have*

$$\forall\, t \geq 0: \quad \partial_2\varphi(t,x) = \exp\left(\int_0^t F'(\varphi(s,x))\,ds\right).$$

Moreover, for every $x \in \varphi(r,\Omega), r \geq 0$,

$$\forall\, t \in [0,r]: \quad \partial_2\varphi(-t,x) = \exp\left(-\int_{-t}^0 F'(\varphi(s,x))\,ds\right).$$

Proof. For $x \in \Omega$ we have $\partial_2\varphi(t,x) > 0$ by hypothesis on F and Remark 0.4.7-(ii). Since F is C^1 it is well-known that $\partial_1\partial_2\varphi$ exists and is continuous and $\partial_2\varphi(0,x) = 1$ for all $x \in \Omega$. Hence,

$$\begin{aligned}
\forall\, t \geq 0: \int_0^t F'(\varphi(s,x))\,ds &= \int_0^t \frac{F'(\varphi(s,x))\partial_2\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds = \int_0^t \frac{\frac{\partial}{\partial x}(F\circ\varphi(s,\cdot))(x)}{\partial_2\varphi(s,x)}ds \\
&= \int_0^t \frac{\frac{\partial}{\partial x}\frac{\partial}{\partial s}\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds = \int_0^t \frac{\frac{\partial}{\partial s}\partial_2\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds \\
&= \ln\partial_2\varphi(t,x) - \ln\partial_2\varphi(0,x) = \ln\partial_2\varphi(t,x).
\end{aligned}$$

Therefore, $\exp\left(\int_0^t F'(\varphi(s,x))\,ds\right) = \partial_2\varphi(t,x)$ for each $t \geq 0$. Now, if $x \in \varphi(r,\Omega)$ with $r \geq 0$ it follows as above for $t \in [0,r]$

$$-\int_{-t}^0 F'(\varphi(s,x))\,ds = -(\ln\partial_2\varphi(0,x) - \ln\partial_2\varphi(-t,x)) = \ln\partial_2\varphi(-t,x).$$

□

0.5 Some auxiliary results

In this section we show some useful results associated with some techniques used throughout this thesis.

In order to characterize the stability property on Chapter 2 for $\Omega \subseteq \mathbb{R}^d$, where $d > 1$, we show an essential known lemma about w^*-convergence in $L^\infty_\rho(\Omega)$ that we state and by sake more of completeness. The interested reader is referred to [25, Chapter 3] for further information.

Definition 0.5.1 Let E be a Banach space and let $f \in E^*$, where E^* denotes the topological dual space of E. For every $x \in E$ consider the linear functional

$\varphi_x : E^* \to \mathbb{R}$ defined by $\varphi_x(f) = \langle f, x\rangle$. As x runs through E we obtain a collection $(\varphi_x)_{x\in E}$ of maps from E^* into $\mathbb{R}$.

The *weak** *topology*, $\sigma(E^*, E)$, is the coarsest topology on E^* such that the maps φ_x are continuous for every $x \in E$.

The weak* topology turns to be a Hausdorff topology $\sigma(E^*, E)$. If a sequence (f_n) in E^* converges to f we shall write $w^*\text{-}\lim_{n\to\infty} f_n = f$. By the definition, $w^*\text{-}\lim_{n\to\infty} f_n = f$ in $\sigma(E^*, E)$ if and only if $\lim_{n\to\infty} \langle f_n, x\rangle = \langle f, x\rangle$, for all $x \in E$.

Lemma 0.5.2 *Let (Ω, μ) be a σ-finite measure space and let ψ be a locally bounded function, where $\psi : [0, +\infty[\to L^\infty(\Omega, \mu)$. Then $w^*\text{-}\lim_{t\to\infty} \psi(t) = 0$ if and only if*

(1) $\sup_{t\geq 0} ||\psi(t)||_\infty < +\infty$*;*

(2) for every $Q \subseteq \Omega$ with $\mu(Q) < +\infty$, it holds

$$\lim_{t\to+\infty} \int_Q \psi(t) d\mu = 0.$$

Proof. "⇒": (2) follows since for every $f \in L^1(\Omega, \mu)$

$$\int_\Omega \psi(t) f d\mu \to 0 \text{ when } t \to +\infty,$$

and this holds in particular if f is the characteristic function of a set Q with finite measure. Moreover, if $w^*\text{-}\lim_{t\to\infty} \psi(t) = 0$, then for every $f \in L^1(\mu)$ there exists $\bar{t} > 0$ such that

$$\sup_{t\geq\bar{t}} \left| \int_\Omega \psi(t) f d\mu \right| < +\infty.$$

Since ψ is locally bounded, it follows that

$$\sup_{t\geq 0} \left| \int_\Omega \psi(t) f d\mu \right| < +\infty.$$

Then, (1) follows by the Banach-Steinhaus theorem.

"⇐": The assertion follows by approximating any $f \in L^1(\Omega, \mu)$ with suitable functions. □

Finally, the following lemma is a generalization of the so-called *Jury test* for quadratic polynomials, originally formulated only for $w \in \mathbb{R}$. A proof of this result for $w \in \mathbb{C}$ can be found in [11]. For the sake of completeness, we give an alternative proof of this result published in [5].

Lemma 0.5.3 *For each* $r \in \mathbb{R}$*, let* $E_r := \{w \in \mathbb{C} \ : \ |z| < 1$ *whenever* $z^2+wz+r = 0\}$*. If* $|r| < 1$*, then*

$$E_r = W_r := \left\{ w \in \mathbb{C} \ : \ \left(\frac{Re\ w}{1+r}\right)^2 + \left(\frac{Im\ w}{1-r}\right)^2 < 1 \right\}.$$

Proof. Let $r \in \mathbb{R}$ with $|r| < 1$ and $z_1, z_2 \in \mathbb{C}$ such that $z^2+wz+r = (z-z_1)(z-z_2)$.

If $r = 0$, the equation is $z^2 + wz = 0$ and, w.l.o.g., the roots are $z_1 = 0$ and $z_2 = -w$. Thus $|w|^2 = (Re\ w)^2 + (Im\ w)^2 = |z_2|^2$, and $E_0 = W_0$.

If $r \neq 0$, we have $z_1 + z_2 = -w$ and $z_1 z_2 = r$. Then there exist $r_i \in \mathbb{R} \setminus \{0\}$, $i = 1, 2$, and $\theta \in [0,\ 2\pi[$, such that $z_1 = r_1 e^{i\theta}$ and $z_2 = r_2 e^{-i\theta}$. We consider the following cases:

Case 1: $r \in]0, 1[$. Without loss of generality we can assume that r_1 and r_2 are real positive numbers, otherwise we can select another θ. Note that

$$(|z_i| < 1,\ \ i = 1, 2) \text{ if, and only if, } (r_i \in]r, 1[,\ \ i = 1, 2).$$

If $r_i \in]r,\ 1[,\ i = 1, 2$, since $-w = z_1+z_2 = (r_1+r_2)\cos(\theta)+i(r_1-r_2)\sin(\theta)$, and the inequality $1 + r = 1 + r_1 r_2 = r_1 + [(1 - r_1) + r_1 r_2] > r_1 + r_2$ hold, we obtain that

$$\left(\frac{Re\ w}{1+r}\right)^2 + \left(\frac{Im\ w}{1-r}\right)^2 = \left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1-r_2}{1-r}\right)^2 \sin^2(\theta) < 1.$$

Conversely, if $\left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1-r_2}{1-r}\right)^2 \sin^2(\theta) < 1$ holds, without loss of generality we can suppose $r_1 \leq r_2$. If $r_2 \geq 1$, then $r_1 \leq r < 1$. This implies that $r_2 - r_1 \geq 1 - r$, and also $1+r = r_1+[(1-r_1)+r_1 r_2] \leq r_1+r_2$. This is a contradiction. So $r_2 < 1$, and thus $w \in E_r$.

Case 2: If $r \in]-1, 0[$, then this situation can be reduced to the first case by taking into account the equalities $iW_r = W_{-r}$ and $iE_r = E_{-r}$, which are easy to compute.

□

Chapter 1

Chaos for semiflow semigroups in Lebesgue spaces

In this chapter we give characterizations of chaos for C_0-semigroups induced by semiflows on $L^p_\rho(\Omega)$, in line with the characterizations of hypercyclicity and mixing of such C_0-semigroups proved in [46]. Moreover, we characterize hypercyclicity, mixing, and chaos for these classes of C_0-semigroups on $W^{1,p}_*(I)$ for a bounded interval $I := (a, b) \subset \mathbb{R}$ and prove that these C_0-semigroups are never hypercyclic on $W^{1,p}(I)$. We apply our results to some first order partial differential equations, such as the von Foerster-Lasota equation.

Given the present state of art, we introduce some equations which motivated us to give the first results. These mentioned results grew out of some fruitful conversations with Thomas Kalmes, who was also involved.

Such C_0-semigroups appear in a natural way when dealing with initial value problems for linear first order partial differential operators. While characterizations of hypercyclicity, (weak) mixing, and chaos of such C_0-semigroups where obtained for open $\Omega \subseteq \mathbb{R}^d$ for arbitrary dimension d in [45], evaluation of these conditions in concrete examples is sometimes rather involved. In contrast to general dimension the case $d = 1$ allows significantly simplified characterizations. In [46] these were given for hypercyclicity and mixing. In section 1.2, we give a simplified characterization of chaos for such C_0-semigroups. Moreover, we further evaluate and extend the conditions obtained in [46].

In section 1.3 we investigate the above kind of C_0-semigroups on the Sobolev spaces $W^{1,p}(I)$ and we characterize hypercyclicity, weakly mixing and mixing properties, and chaos on the closed subspace $W_*^{1,p}(I)$.

The contents of this chapter have been published in [3].

1.1 State of the art

Briefly we introduce the known results about some similar C_0-semigroups in order to show the improvement achieved with the characterizations commented previously.

Consider for example the results obtained in [26, 27, 36, 37]. The references [26, 27, 36, 37] clearly show the improvement on the conditions in chronological way, and motivate the principal results of this chapter. We will return to these references in the next chapter to compare stability properties.

In the quoted references it is studied the linear von Foerster-Lasota equation

$$\frac{\partial u}{\partial t}(t,x) + x\frac{\partial u}{\partial x}(t,x) = h(x)u(t,x) \quad t \geq 0, x \in [0,1] \tag{1.1}$$

with initial condition

$$u(0,x) = v(x) \quad x \in [0,1],$$

where v belongs to a suitable function space.

The equation (1.1) is a particular case of the equation

$$\frac{\partial u}{\partial t}(t,x) + c(x)\frac{\partial u}{\partial x}(t,x) = f(x,u(t,x)) \qquad t \geq 0,\ x \in [0,1]$$

that was introduced in [50] to describe the reproduction of a population of red blood cells, mainly in connection with studies about anemia. After the paper [49], this problem has already been studied in different function spaces by several authors either with an ergodic theoretical approach (see [60] and the references quoted therein) or by explicitly constructing hypercyclic and periodic solutions (see [27, 35, 36]), or by investigating spectral properties of the differential operator associated to the equation (1.1) (see [61, 62] and the applications in [29]).

In turns that, if $h : [0,1] \to \mathbb{C}$ is a continuous function, defining

$$T(t)v(x) = \exp\left(\int_{-t}^{0} h(xe^s)ds\right)v(xe^{-t}), \qquad t \geq 0,\ x \in [0,1], \tag{1.2}$$

the family $\mathcal{T} = (T(t))_{t\geq 0}$ is a C_0-semigroup in $L^p([0,1])$ with $1 \leq p < \infty$, and it gives a solution of the equation (1.1) with initial value $v \in L^p([0,1])$.

In [26, 27] the authors consider also a different space, namely, for every $\alpha \in]0,1]$

$$V_\alpha = \{h \in h^\alpha([0,1]) \mid h(0) = 0\}.$$

Here $h^\alpha([0,1])$ is the little Hoelder space of order α, that is the space of the functions $h : [0,1] \to \mathbb{R}$ such that

$$\|h\|_\alpha := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|^\alpha} < \infty$$

and

$$\lim_{r\to 0} \sup_{\substack{x,y\in[0,1]\\ 0<|x-y|<r}} \frac{|h(x)-h(y)|}{|x-y|^\alpha} = 0.$$

Observe that h^α is a separable Banach space endowed with the norm $\|h\|_\alpha$. Under suitable assumptions on the function h, $\mathcal{T}$ is a C_0-semigroup on V_α, too, and in [26, 27] the authors prove the following result.

Theorem 1.1.1 *Let $h : [0,1] \to \mathbb{C}$ be a continuous function and let $\mathcal{T}$ be the semigroup defined in* (1.2).

(i) If $\operatorname{Re} h(0) > -\frac{1}{p}$, $\mathcal{T}$ is chaotic on $L^p(0,1)$.

(ii) If $h(0) \in \mathbb{R}$, $\frac{\operatorname{Re} h(x)-h(0)}{x} \in L^1([0,1])$ and $h(0) > 1-\frac{1}{p}$, $\mathcal{T}$ is chaotic on V_α.

We will recover these result from our results for more general C_0-semigroups.

We mention that, more in general, in [37] the authors consider the following equation on the multidimensional case on certain compact set D of $\mathbb{R}^d$

$$\frac{\partial u}{\partial t}(t,x) + \sum_{i=1}^{d} c_i(x)\frac{\partial u}{\partial x_i}(t,x) = \gamma u(t,x) \quad t \geq 0, x \in D \subset \mathbb{R}^d_+ \cup \{0\} \tag{1.3}$$

with initial condition

$$u(0,x) = v(x) \quad x \in D,$$

where v belongs to some normed vector space of functions defined on D, $\gamma \in \mathbb{R}$, and $c := (c_i)_{i=1}^d : D \to \mathbb{R}^d$ with $c_i(0) = 0$ and $c_i(x) > 0$ for $x \in D \setminus \{0\}$, $i = 1, \dots, d$.

Theorem 1.1.2 ([37]) *Let $1 \le p < \infty$. Then there exist a C_0-semigroup $\mathcal{T}$ on $L^p(D)$, which is solution of the equation (1.3). Moreover, if $\gamma > -\frac{1}{p} \liminf_{x\to 0} (\nabla \cdot c)(x)$ then $\mathcal{T}$ is chaotic on $L^p(D)$.*

A similar problem was studied also by Kalmes in [45]. Observe that $0 < p \le 1$ is possible since the authors gives $\mu(D) < \infty$, for more details we refer the reader to the introduction of [37].

1.2 Chaotic dynamics on Lebesgue spaces

Our first aim in this section is to give a characterization of chaos for a C_0-semigroup on $L^p_\rho(\Omega)$ which is not present in [46]. The proof follows the idea of the results obtained in [46] for hypercyclicity and mixing for those C_0-semigroups. For this, consider the notation of section 0.1 and section 0.4.

Throughout this section let $\Omega \subseteq \mathbb{R}$, $\rho : \Omega \to (0, +\infty)$ λ-measurable, $F \in C^1(\Omega)$ satisfying the assumptions (H1), (H2) and (H3) of section 0.4 and φ the flow associated with F. Moreover let $h \in C(\Omega, \mathbb{K})$ and let

$$h_t(x) = \exp\left(\int_0^t h(\varphi(s, x))\, ds\right), \qquad \text{for } t \ge 0, x \in \Omega.$$

Let $1 \le p < \infty$ and define for every $f \in L^p_\rho(\Omega)$ and every $t \ge 0$

$$T_{F,h}(t)f(x) := h_t(x) f(\varphi(t, x)), \qquad x \in \Omega.$$

The next theorem gives a characterization of when $(T_{F,h}(t))_{t\ge 0}$ defines a C_0-semigroup in $L^p_\rho(\Omega)$. For its proof see [45, Theorem 4.7 and Proposition 4.12]. Although in [45] h is assumed to be real valued the proofs of [45, Theorem 4.7 and Proposition 4.12] are valid for complex valued h, too. Observe that for real valued h we have $h_t = |h_t|$ for all $t \ge 0$.

Theorem 1.2.1 *Let $p \in [1, \infty)$. Then the following are equivalent.*

i) The family $\mathcal{T}_{F,h} = (T_{F,h}(t))_{t\ge 0}$ is a C_0-semigroup on $L^p_\rho(\Omega)$.

ii) There exist $M \geq 1, \omega \in \mathbb{R}$ such that for every $t \geq 0$

$$|h_t(x)|^p \rho(x) \leq M e^{\omega t} \rho(\varphi(t,x)) |\partial_2 \varphi(t,x)| \text{ holds } \lambda\text{-a.e. on } \Omega.$$

Moreover, if ii) holds the generator of $\mathcal{T}_{F,h}$ is an extension of the operator

$$C_c^1(\Omega) \to L_\rho^p(\Omega), \qquad f \mapsto Ff' + hf$$

in $L_\rho^p(\Omega)$, where $C_c^1(\Omega)$ denotes the space of compactly supported, continuously differentiable functions on Ω. Additionally, if h is bounded and F is such that for every $x_0 \in \Omega$ the maximal domain of $\varphi(\cdot, x_0)$ equals $\mathbb{R}$, then $C_c^1(\Omega)$ is a core for the generator of the C_0-semigroup $\mathcal{T}_{F,h}$.

Definition 1.2.2 In what follows, we call a measurable function $\rho : \Omega \to (0,\infty)$ *p-admissible for F and h* ($p \in [1,\infty)$) if the conditions (H1)-(H3) are fulfilled and there are constants $M \geq 1, \omega \in \mathbb{R}$ such that

$$\forall\, t \geq 0, x \in \Omega : \; |h_t(x)|^p \rho(x) \leq M e^{\omega t} \rho(\varphi(t,x)) |\partial_2 \varphi(t,x)|.$$

Since $|h_t(x)| = \exp(\int_0^t \operatorname{Re} h(\varphi(s,\cdot)) ds)$, p-admissibility of ρ only depends on F and $\operatorname{Re} h$. By the above theorem, we have for a p-admissible ρ for F and h the well-defined C_0-semigroup $\mathcal{T}_{F,h}$ on $L_\rho^p(\Omega)$.

In order to formulate our results in a convenient way we introduce the following notions.

Definition 1.2.3 If ρ is p-admissible for F and h we define for $t \geq 0$

$$\rho_{t,p} : \Omega \to [0,\infty), \qquad \rho_{t,p}(x) := \chi_{\varphi(t,\Omega)}(x) |h_t(\varphi(-t,x))|^p \rho(\varphi(-t,x)) \partial_2 \varphi(-t,x)$$

as well as

$$\rho_{-t,p} : \Omega \to [0,\infty), \qquad \rho_{-t,p}(x) := |h_t(x)|^{-p} \rho(\varphi(t,x)) \partial_2 \varphi(t,x).$$

Obviously, $\rho_{t,p}$ and $\rho_{-t,p}$ depend on F and h but in order to keep notation simple we will not take this into account notationally as there will be no danger of confusion. Observe that $\rho_{0,p} = \rho$.

Remark 1.2.4 We can consider also the multidimensional case, that is $\Omega \subseteq \mathbb{R}^d$ ($d > 1$), F a locally Lipschitz continuous vector field, ρ a measurable locally integrable function and replace $\partial_2 \varphi(t,x)$ by $|\det D\varphi(-t,\cdot)|$, where $D\varphi(t,\cdot)$ denotes the Jacobian of $\varphi(t,\cdot)$.

We will consider this case in Chapter 2.

The next lemma will be a crucial tool (originally proved in [46, Lemma 7]).

Lemma 1.2.5 *Let ρ be p-admissible for F and h and let $[a,b] \subset \Omega_1$. Setting $\alpha := a, \beta := b$ if $F_{|[a,b]} > 0$, respectively $\alpha := b, \beta := a$ if $F_{|[a,b]} < 0$, there is a constant $C > 0$ such that*

$$\forall\, x \in [a,b]:\ \frac{1}{C} \leq \rho(x) \leq C$$

as well as

$$\forall\, t \geq 0, x \in [a,b]:\ \frac{1}{C}\rho_{t,p}(\alpha) \leq \rho_{t,p}(x) \leq C\rho_{t,p}(\beta)$$

and

$$\forall\, t \geq 0, x \in [a,b]:\ \frac{1}{C}\rho_{-t,p}(\alpha) \leq \rho_{-t,p}(x) \leq C\rho_{-t,p}(\beta).$$

We will see that the dynamical properties of $\mathcal{T}_{F,h}$ on $L^p_\rho(\Omega)$ are determined by the asymptotic behavior of the functions

$$t \mapsto \rho_{t,p}(x), \quad t \mapsto \rho_{-t,p}(x)$$

where $x \in \Omega$ is fixed.

The alternative representation of $\partial_2\varphi(t,x)$ in Proposition 0.4.8 gives the next result.

Corollary 1.2.6 *Let ρ be p-admissible for F and h. Then we have for all $t \geq 0$ and $x \in \Omega$*

$$\begin{aligned}
\rho_{t,p}(x) &= \chi_{\varphi(t,\Omega)}(x)\exp\left(p\int_{-t}^{0}\left[\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))\right]ds\right)\rho(\varphi(-t,x)) \\
&= \begin{cases} \exp\left(pt[\operatorname{Re} h(x) - \frac{1}{p}F'(x)]\right)\rho(x), & x \in \Omega_0, \\ \chi_{\varphi(t,\Omega)}(x)\exp\left(p\int_{\varphi(-t,x)}^{x}\frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\right)\rho(\varphi(-t,x)), & x \in \Omega_1 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\rho_{-t,p}(x) &= \exp\left(-p\int_{0}^{t}\left[\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))\right]ds\right)\rho(\varphi(t,x)) \\
&= \begin{cases} \exp\left(-pt[\operatorname{Re} h(x) - \frac{1}{p}F'(x)]\right)\rho(x), & x \in \Omega_0, \\ \exp\left(-p\int_{x}^{\varphi(t,x)}\frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\right)\rho(\varphi(t,x)), & x \in \Omega_1 \end{cases}
\end{aligned}$$

where $\Omega_0 = \{x : \; F(x) = 0\}$ *and* $\Omega_1 = \{x : \; F(x) \neq 0\}$.

Proof. While a straightforward calculation gives

$$\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x) \exp\left(p \int_{-t}^{0} \left[\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p} F'(\varphi(s,x))\right] ds\right) \rho(\varphi(-t,x))$$

and

$$\rho_{-t,p}(x) = \exp\left(-p \int_{0}^{t} \left[\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p} F'(\varphi(s,x))\right] ds\right) \rho(\varphi(t,x))$$

we observe that for $x \in \Omega_0$ we have $\varphi(t,x) = x$ for each t so that in Ω_0

$$\rho_{t,p}(x) = \exp\left(pt\left(\operatorname{Re} h(x) - \frac{1}{p} F'(x)\right)\right) \rho(x)$$

as well as

$$\rho_{-t,p}(x) = \exp\left(-pt\left(\operatorname{Re} h(x) - \frac{1}{p} F'(x)\right)\right) \rho(x).$$

For $x \in \Omega_1$ it is well-known that $\varphi(t,x) \in \Omega_1$ so that in Ω_1

$$\begin{aligned} \rho_{-t,p}(x) &= \exp\left(-p \int_{0}^{t} \frac{\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p} F'(\varphi(s,x))}{F(\varphi(s,x))} \partial_1 \varphi(s,x) ds\right) \rho(\varphi(t,x)) \\ &= \exp\left(-p \int_{x}^{\varphi(t,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p} F'(y)}{F(y)} dy\right) \rho(\varphi(t,x)) \end{aligned}$$

and similarly

$$\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x) \exp\left(p \int_{\varphi(-t,x)}^{x} \frac{\operatorname{Re} h(y) - \frac{1}{p} F'(y)}{F(y)} dy\right) \rho(\varphi(-t,x)).$$

□

Remark 1.2.7 Observe that if $\lambda(\Omega_0) > 0$ then

$$\rho_{\pm t,p}(x) = e^{\pm pt \operatorname{Re} h(x)} \rho(x), \qquad \text{a.e. } x \in \Omega_0.$$

In fact, we can rewrite $\Omega_0 = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$, where $a_n < b_n$ being $F \in C^1$ with $F(x) = 0$ on Ω_0 then $F' = 0$ in Ω_0.

The next result, published in [45, Theorem 5.3], show a characterization of chaos for arbitrary dimension $d \geq 1$.

Theorem 1.2.8 ([45]) *Under the general hypothesis on $\Omega \subseteq \mathbb{R}^d$, let φ be such that for every compact subset K of Ω there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ for every $t > t_K$. Then, the following are equivalent.*

(i) The C_0-semigroup $\mathcal{T}_{F,h}$ is chaotic on $L^p_\rho(\Omega)$.

(ii) $Per(\mathcal{T}_{F,h})$ is dense in $L^p_\rho(\Omega)$.

(iii) For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} \mu(K \setminus L_n) = 0$, and

$$\lim_{n\to\infty} \Big(\sum_{l=1}^{\infty} \int_{L_n} \rho_{lt_n,p}\, d\lambda^d + \int_{L_n} \rho_{-lt_n,p}\, d\lambda^d \Big) = 0.$$

Now, we are able to show our result in the case $d = 1$, it has been of interest by its simplified conditions.

Theorem 1.2.9 *Let $\Omega \subseteq \mathbb{R}$ be open, F satisfying (H1)-(H3), $h \in C(\Omega, \mathbb{R})$, and $\rho : \Omega \to (0, \infty)$ be a measurable function which is p-admissible for F and h. Then the following are equivalent.*

i) $\mathcal{T}_{F,h}$ is chaotic on $L^p_\rho(\Omega)$.

ii) $\lambda(\Omega_0) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \dots, C_m$ of Ω_1, for λ^m-almost all choices of $(x_1, \dots, x_m) \in \Pi_{j=1}^m C_j$ there is $t > 0$ such that

$$\sum_{j=1}^{m} \sum_{l\in\mathbb{Z}} \rho_{lt,p}(x_j) < \infty.$$

Proof. We first show that i) implies ii). Since for all $x \in \Omega_0$ and any $t \geq 0$ we have $\varphi(t, x) = x$ it follows

$$\forall\, f \in L^p_\rho(\Omega) : \big(T_{F,h}(t)f\big)_{|\Omega_0} = \big(\exp(t\, h)f\big)_{|\Omega_0}$$

so that $(T_{F,h}(t)f)_{t\geq 0}$ cannot be dense in $L^p_\rho(\Omega)$ for any $f \in L^p_\rho(\Omega)$ if $\lambda(\Omega_0) > 0$. Hence, since $\mathcal{T}_{F,h}$ is chaotic, we conclude $\lambda(\Omega_0) = 0$. As described in Remark 0.4.2-(ii) we are therefore actually dealing with $T_{F,h}$ on $L^p_\rho(\Omega_1)$. Now, if $K \subset \Omega_1$ is compact there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, recalling that μ denotes the Borel measure on Ω with Lebesgue density ρ we can apply Theorem 1.2.8 saying that, because $\mathcal{T}_{F,h}$ is chaotic on $L^p_\rho(\Omega_1)$ for every compact $K \subset \Omega_1$ there are a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ and a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K such that $\lim_{n\to\infty}\mu(L_n) = \mu(K)$ and

$$\lim_{n\to\infty}\Big(\sum_{l=1}^{\infty}\int_{L_n}\rho_{lt_n,p}\,d\lambda + \int_{L_n}\rho_{-lt_n,p}\,d\lambda\Big) = 0.$$

We will apply this condition to special compact sets in order to derive ii).

Let $x_1,\ldots,x_m$ be each from different connected components of Ω_1. As Ω_1 is open, there is $r<0$ such that $\varphi(t,x_j)$ is well-defined for all $t\in[r,\infty)$ and every $1\le j\le m$. For each $1\le j\le m$ we set $K_j := \{\varphi(t,x_j);\, t\in[0,1]\}$ if $F(x_j)>0$, respectively $K_j := \{\varphi(t,x_j);\, t\in[r,0]\}$ if $F(x_j)<0$. It follows that $K_j = [x_j,\varphi(1,x_j)]$ if $F(x_j)>0$, respectively $K_j = [\varphi(r,x_j),x_j]$ if $F(x_j)<0$. In particular $\lambda(K_j)>0$ and thus $\mu(K_j)>0$ for every j.

$K := \cup_{j=1}^m K_j \subset \Omega_1$ is compact so that there are $(t_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ as above. Let $L_{n,j} := L_n\cap K_j$ for $1\le j\le m$ and $n\in\mathbb{N}$. Applying Lemma 1.2.5 to K_j it follows that there are $C_j>0\,(1\le j\le m)$ such that for all $n\in\mathbb{N}$

$$\int_{L_{n,j}}\rho_{lt_n,p}(y)\,d\lambda(y) = \int_{L_{n,j}}\frac{\rho_{lt_n,p}(y)}{\rho(y)}\,d\mu(y) \ge C_j\rho_{lt_n,p}(x_j)\mu(L_{n,j})$$

and analogously

$$\int_{L_{n,j}}\rho_{-lt_n,p}(y)\,d\lambda(y) \ge C_j\rho_{-lt_n,p}(x_j)\mu(L_{n,j}).$$

Since for n large enough

$$\begin{aligned}\infty &> \sum_{l=1}^{\infty}\Big(\int_{L_n}\rho_{lt_n,p}\,d\lambda + \int_{L_n}\rho_{-lt_n,p}\,d\lambda\Big)\\ &= \sum_{j=1}^{m}\sum_{l=1}^{\infty}\Big(\int_{L_{n,j}}\rho_{lt_n,p}\,d\lambda + \int_{L_{n,j}}\rho_{-lt_n,p}\,d\lambda\Big)\\ &\ge \sum_{j=1}^{m}C_j\mu(L_{n,j})\sum_{l=1}^{\infty}\Big(\rho_{lt_n,p}(x_j)+\rho_{-lt_n,p}(x_j)\Big)\end{aligned}$$

and $\lim_{n\to\infty}\mu(L_{n,j}) = \mu(K_j)>0$ we deduce for n large enough

$$\sum_{j=1}^{m}\sum_{l\in\mathbb{Z}}\rho_{lt_n,p}(x_j) < \infty.$$

Since for $t = 0$ we have $\rho_{t,p} = \rho > 0$ the above t_n has to be strictly positive. Thus, ii) is proved.

It remains to show that ii) implies i). Since $\lambda(\Omega_0) = 0$ we consider $\mathcal{T}_{F,h}$ on $L^p_\rho(\Omega_1)$, as explained in Remark 0.4.2-(ii). If $K \subset \Omega_1$ is compact there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, we can again use Theorem 1.2.8. For fixed compact $K \subset \Omega_1$ there are finitely many intervals $[a_j, b_j] \subset \Omega_1$ such that each $[a_j, b_j]$ is contained in a different connected component of Ω_1 and $K \subseteq \cup_{j=1}^m [a_j, b_j]$. We define $x_j := b_j$ if $F_{|[a_j,b_j]} > 0$, respectively $x_j := a_j$ if $F_{|[a_j,b_j]} < 0$. Moreover, without loss of generality, we can assume by ii) that there is $t > 0$ with

$$\sum_{j=1}^{m} \sum_{l \in \mathbb{Z}} \rho_{lt,p}(x_j) < \infty.$$

Now it follows from Lemma 1.2.5 that for some constants $C_j > 0\,(1 \leq j \leq m)$ with $t_n := nt$, $n \in \mathbb{N}$

$$\begin{aligned}
\sum_{l=1}^{\infty} \Big(\int_K \rho_{lt_n,p}\, d\lambda + \int_K \rho_{-lt_n,p}\, d\lambda \Big) &\leq \sum_{j=1}^{m} \sum_{l=1}^{\infty} \Big(\int_{[a_j,b_j]} \rho_{lt_n,p}\, d\lambda + \int_{[a_j,b_j]} \rho_{-lt_n,p}\, d\lambda \Big) \\
&= \sum_{j=1}^{m} \sum_{l=1}^{\infty} \Big(\int_{[a_j,b_j]} \frac{\rho_{lt_n,p}}{\rho}\, d\mu + \int_{[a_j,b_j]} \frac{\rho_{-lt_n,p}}{\rho}\, d\mu \Big) \\
&\leq \sum_{j=1}^{m} C_j \mu([a_j,b_j]) \sum_{l=1}^{\infty} \Big(\rho_{lnt,p}(x_j) + \rho_{-lnt,p}(x_j) \Big)
\end{aligned}$$

so that

$$\lim_{n \to \infty} \sum_{l=1}^{\infty} \Big(\int_K \rho_{lt_n,p}\, d\lambda + \int_K \rho_{-lt_n,p}\, d\lambda \Big) = 0.$$

With $L_n := K$, $n \in \mathbb{N}$ it follows from Theorem 1.2.8 that $\mathcal{T}_{F,h}$ is chaotic on $L^p_\rho(\Omega_1)$. This proves the theorem. $\square$

Characterizations of hypercyclicity and mixing for $\mathcal{T}_{F,h}$ on $L^p_\rho(\Omega)$ for some p-admissible ρ for F and real valued h were proved in [46], although results involving chaos are not proved in the same reference. Since we will use them in sequel, we include them here for the reader's convenience. For the proofs see [46, Theorem 9 and Remark 12].

Theorem 1.2.10 *Let $\Omega \subseteq \mathbb{R}$ be open, F satisfying (H1)-(H3), $h \in C(\Omega, \mathbb{R})$ and $\rho : \Omega \to (0, \infty)$ be a measurable function which is p-admissible for F and h.*

a) For the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p_\rho(\Omega)$ the following are equivalent.

i) $\mathcal{T}_{F,h}$ *is hypercyclic.*

ii) $\mathcal{T}_{F,h}$ *is weakly mixing.*

iii) $\lambda(\Omega_0) = 0$ *and for every* $m \in \mathbb{N}$ *for which there are* m *different connected components* $C_1, \ldots, C_m$ *of* Ω_1*, for* λ^m*-almost all choices of* $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ *there is a sequence of positive numbers* $(t_n)_{n\in\mathbb{N}}$ *tending to infinity such that*

$$\forall\, 1 \leq j \leq m : \lim_{n\to\infty} \rho_{t_n,p}(x_j) = \lim_{n\to\infty} \rho_{-t_n,p}(x_j) = 0.$$

b) For the C_0*-semigroup* $\mathcal{T}_{F,h}$ *on* $L^p_\rho(\Omega)$ *the following are equivalent.*

i) $\mathcal{T}_{F,h}$ *is mixing.*

ii) $\lambda(\Omega_0) = 0$ *and for* λ*-almost every* $x \in \Omega$ *one has*

$$\lim_{t\to\infty} \rho_{t,p}(x) = \lim_{t\to\infty} \rho_{-t,p}(x) = 0.$$

Example 1.2.11 (Left translation semigroup) Let $\Omega = \mathbb{R}, F = 1$, and $h = 0$ so that $\varphi(t,x) = x + t$ and $h_t(x) = 1$. Then a measurable function $\rho : \mathbb{R} \to (0,\infty)$ is p-admissible for F and h for some $p \in [1,\infty)$ if the same holds for every $p \in [1,\infty)$ and the corresponding C_0-semigroup $\mathcal{T}_{F,h}$ is the *(bilateral) left translation semigroup* on $L^p_\rho(\mathbb{R})$ and given by $(T(t)f)(x) = f(x+t)$, its generator being an extension of

$$C^1_c(\mathbb{R}) \to L^p_\rho(\mathbb{R}), f \mapsto f'.$$

Moreover, we have

$$\rho_{t,p}(x) = \rho(x-t) \text{ and } \rho_{-t,p} = \rho(x+t).$$

Since $\Omega_0 = \emptyset$ we have only a single connected component of $\mathbb{R} \setminus \Omega_0$ so that by Theorem 1.2.9 the left translation semigroup on $L^p_\rho(\mathbb{R})$ is chaotic if and only if for λ-a.e. $x \in \mathbb{R}$ there is $t > 0$ such that

$$\sum_{l\in\mathbb{Z}} \rho(x + lt) < \infty.$$

This weight condition is originally due to Matsui et al. [53, 54, Theorem 2] (see also Chapter 7 and related exercises, in [44]).
Note that chaos is independent of $p \in [1,\infty)$ in this case.

It is worth observing that, by applying Theorem 1.2.10 to the left translation semigroup as stated in Example 1.2.11, one recovers the characterizations of hypercyclicity and mixing originally proved in [38] and [19] respectively.

Example 1.2.12 Let again $\Omega = \mathbb{R}$. Moreover, $F(x) := 1 - x, h(x) = 0\, (x \in \mathbb{R})$ so that $\varphi(t,x) = 1 + (x-1)e^{-t}$, $h_t(x) = 1$, and $\partial_2\varphi(t,x) = e^{-t}$. Then a measurable function $\rho : \mathbb{R} \to (0,\infty)$ is again p-admissible for F and h for some $p \in [1,\infty)$ if the same holds for every $p \in [1,\infty)$ and the corresponding C_0-semigroup $\mathcal{T}_{F,h}$ is given by $(T(t)f)(x) = f(1 + (x-1)e^{-t})$ with generator being an extension of

$$C_c^1(\mathbb{R}) \to L^p_\rho(\mathbb{R}), f \mapsto \big(x \mapsto (1-x)f'(x)\big).$$

Furthermore, $\mathbb{R} \setminus \Omega_0 = (-\infty, 1) \cup (1,\infty)$ has two connected components and

$$\forall\, t \in \mathbb{R} :\ \rho_{t,p}(x) = \rho(1 + (x-1)e^t)e^t,$$

so that $\mathcal{T}_{F,h}$ is chaotic on $L^p_\rho(\mathbb{R})$ if and only if for λ^2-a.e. $(x_1, x_2) \in (-\infty,1) \times (1,\infty)$ there is $t > 0$ such that

$$\sum_{j=1}^{2} \sum_{l\in\mathbb{Z}} \rho(1 + (x_j - 1)e^{lt})e^{lt} < \infty.$$

Again, the occurrence of chaos is independent of $p \in [1,\infty)$ as $h = 0$.

So far, we have only characterizations of hypercyclicity, mixing, and chaos of $\mathcal{T}_{F,h}$ in case of real valued h. In order to obtain some results for complex valued h we use the *comparison test* which was defined in Definition 0.3.6 and also we use Proposition 0.3.7.

Let F satisfying (H1)-(H3), $h \in C(\Omega,\mathbb{C})$ and let $g \in C(\Omega,\mathbb{R})$. Moreover, let ρ be p-admissible for F and h as well as for F and g. Our aim is to find some measurable function $\psi : \Omega \to \mathbb{C}$ such that $\exp\circ\psi$ induces a continuous, invertible multiplication operator M on $L^p_\rho(\Omega)$ for which the C_0-semigroups $\mathcal{T}_{F,h}$ and $\mathcal{T}_{F,g}$ are conjugate via M, i.e.

$$\forall\, t \geq 0, f \in L^p_\rho(\Omega) :\ \exp(\psi)\, T_{F,h}(t)(f) = T_{F,g}(t)(\exp(\psi)f) \tag{1.4}$$

Being $T_{F,g}(t)(Mf) = \exp(\psi(\varphi(t,\cdot)))\frac{g_t}{h_t}\, T_{F,h}(t)(f)$, (1.4) is satisfied, if for each $x \in \Omega$

$$\forall\, t \geq 0 :\ \psi(\varphi(t,x)) - \int_0^t (h(\varphi(s,x)) - g(\varphi(s,x)))\, ds = \psi(x). \tag{1.5}$$

Now, if $x \in \Omega_0$ it follows from $\varphi(t,x) = x$ that the above expression reduces to

$$\forall\, t \geq 0 : \psi(x) - t(h(x) - g(x)) = \psi(x).$$

Thus, it is necessary that h and g coincide on Ω_0.

Moreover, for $x \in \Omega_1$ we also have $F(\varphi(t,x)) \neq 0$ for every $t \geq 0$ so

$$\begin{aligned}
\forall\, t \geq 0 : \qquad & \psi(\varphi(t,x)) - \int_0^t h(\varphi(s,x)) - g(\varphi(s,x))ds \\
= \quad & \psi(\varphi(t,x)) - \int_0^t \frac{h(\varphi(s,x)) - g(\varphi(s,x))}{F(\varphi(s,x))}\partial_1\varphi(s,x)ds \\
= \quad & \begin{cases} \psi(\varphi(t,x)) - \int_x^{\varphi(t,x)} \frac{h(y)-g(y)}{F(y)}dy & , \text{ if } F(x) > 0 \\ \psi(\varphi(t,x)) + \int_{\varphi(t,x)}^x \frac{h(y)-g(y)}{F(y)}dy & , \text{ if } F(x) < 0. \end{cases}
\end{aligned}$$

Hence, if Ω is an interval, $\alpha := \inf\Omega, \omega := \sup\Omega$, and the function $\Omega \to \mathbb{R}, y \mapsto \frac{h(y)-g(y)}{F(y)}$ belongs to $L^1((\alpha,\beta))$ for every $\beta \in \Omega$, or to $L^1((\beta,\omega))$ for every $\beta \in \Omega$ we can set

$$\psi : \Omega \to \mathbb{R}, \psi(x) = \int_\alpha^x \frac{h(y) - g(y)}{F(y)}dy,$$

or

$$\psi : \Omega \to \mathbb{R}, \psi(x) = -\int_x^\omega \frac{h(y) - g(y)}{F(y)}dy,$$

respectively, and it follows from the above calculation that (1.5) holds on Ω_1 for this ψ.

Proposition 1.2.13 *Let $\Omega \subseteq \mathbb{R}$ be an open interval, F satisfying (H1)-(H3), $h \in C(\Omega,\mathbb{K})$. Moreover, let ρ be p-admissible for F and h and set $\alpha := \inf\Omega, \omega := \sup\Omega$. Consider the following conditions.*

1) $\forall\, x \in \Omega_0 : h(x) \in \mathbb{R}$.

2a) The function $\Omega \to \mathbb{R}, y \mapsto \frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\alpha,\beta))$ for all $\beta \in \Omega$.

2b) The function $\Omega \to \mathbb{R}, y \mapsto -\frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\beta,\omega))$ for all $\beta \in \Omega$.

If 1) and 2a) are satisfied, set $\psi : \Omega \to \mathbb{R}, \psi(x) = i\int_\alpha^x \frac{\operatorname{Im} h(y)}{F(y)}dy$, while if 1) and 2b) hold, set $\psi : \Omega \to \mathbb{R}, \psi(x) = -i\int_x^\omega \frac{\operatorname{Im} h(y)}{F(y)}dy$. Then $\exp\circ\psi$ defines a continuous, invertible multiplication operator M on $L^p_\rho(\Omega)$ such that the C_0-semigroups $\mathcal{T}_{F,h}$ and $\mathcal{T}_{F,\operatorname{Re} h}$ on $L^p_\rho(\Omega)$ are conjugate via M.

Proof. By the observation preceding the proposition applied to h and $g = \operatorname{Re} h$ we only have to show that $\exp \circ \psi$ defines a continuous, invertible multiplication operator on $L^p_\rho(\Omega)$. But this is obvious because $|\exp(\psi(x))| = 1$ for all $x \in \Omega$. □

Remark 1.2.14 From the above proposition and the comparison test it follows immediately, that in Theorems 1.2.10 and 1.2.9 we can replace the hypothesis of h being real valued by the weaker conditions

1) $\forall\, x \in \Omega_0 : h(x) \in \mathbb{R}$.

2) With $\alpha := \inf \Omega$ and $\omega := \sup \Omega$ the function $\Omega \to \mathbb{R}, y \mapsto \frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\alpha, \beta))$ for all $\beta \in \Omega$ or to $L^1((\beta, \omega))$ for all $\beta \in \Omega$.

We finish this section by taking a closer look at the case $\rho = 1$ for some special cases which are of particular interest for the examples of linear differential equations in section 1.4 as is the case of von Foerster-Lasota equation.

Theorem 1.2.15 *Let $I \subseteq \mathbb{R}$ be an open interval, F satisfying (H2)-(H3) on I with $F(x) < 0$ for each $x \in I$ and such that $\varphi(t, I) \neq I$ for some $t > 0$. Moreover, let $h \in C(I, \mathbb{C})$ be such that with $\alpha := \inf I$ and $\omega := \sup I$ the function*

$$I \to \mathbb{R}, \qquad x \mapsto \frac{\operatorname{Im} h(x)}{F(x)}$$

belongs to $L^1((\alpha, \beta))$ for every $\beta \in I$ or to $L^1((\beta, \omega))$ for every $\beta \in I$. Furthermore, let $\rho = 1$ be p-admissible for F and h for some $1 \leq p < \infty$.

a) For the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p(I)$ the following are equivalent.

i) $\mathcal{T}_{F,h}$ is hypercyclic.

ii) $\mathcal{T}_{F,h}$ is weakly mixing.

iii) There is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in I converging to α such that for some $x_0 \in I$

$$\lim_{n \to \infty} \int_{\alpha_n}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p} F'(y)}{F(y)} dy = -\infty.$$

b) For the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p(I)$ the following are equivalent.

i) $\mathcal{T}_{F,h}$ *is mixing.*

ii) *For some* $x_0 \in I$

$$\int_\alpha^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy = -\infty.$$

c) *For the* C_0*-semigroup* $\mathcal{T}_{F,h}$ *on* $L^p(I)$ *the following are equivalent.*

i) $\mathcal{T}_{F,h}$ *is chaotic.*

ii) *There is some* $x_0 \in I$ *such that for every* $x \in I$ *there is* $t > 0$ *such that*

$$\sum_{l=1}^{\infty} \exp\big(p \int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy\big) < \infty.$$

Proof. From Corollary 1.2.6 it follows

$$\forall\, t \geq 0, x \in I: \quad \rho_{t,p}(x) = \chi_{\varphi(t,I)}(x) \exp\left(p \int_{\varphi(-t,x)}^{x} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy\right)$$

and

$$\forall\, t \geq 0, x \in I: \quad \rho_{-t,p}(x) = \exp\left(-p \int_{x}^{\varphi(t,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy\right).$$

For each $x \in I$ the trajectory $\{\varphi(t,x);\, t \geq 0\}$ is either an open subinterval of I or equals $\{x\}$. Since the later occurs if and only if $F(x) = 0$ it follows from $F(x) < 0$ that $\inf\{\varphi(t,x);\, t \geq 0\} = \alpha$ for every $x \in I$. Moreover, the assumption $\varphi(t,I) \neq I$ for some $t > 0$ implies that for every $x \in I$ there is $t_0 > 0$ such that $\chi_{\varphi(t,I)}(x) = 0$ whenever $t > t_0$. In particular, for all $x \in I$ we have $\rho_{t,p}(x) = 0$ for sufficiently large t.

Proof of part a). It follows from Remark 1.2.14 and Theorem 1.2.10 that i) and ii) in a) are equivalent and by the preceding observation they hold if and only if for λ-a.e. $x \in I$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ tending to infinity such that

$$0 = \lim_{n\to\infty} \rho_{-t_n,p}(x) = \exp\left(p \int_{\varphi(t_n,x)}^{x} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy\right).$$

Being $\varphi(\cdot,x)$ strictly decreasing and $\alpha = \inf\{\varphi(t,x);\, t \geq 0\}$ for every $x \in I$, the above relation obviously holds if and only if for some sequence $(\alpha_n)_{n\in\mathbb{N}}$ in I

converging to α we have

$$\lim_{n\to\infty}\int_{\alpha_n}^{x_0}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy=-\infty$$

for some (and then any) $x_0\in I$. Thus a) is proved.

The proof of parts b) and c) go along the same lines as the one of part a) by applying Theorem 1.2.10 b) and Theorem 1.2.9, respectively, instead of Theorem 1.2.10 a), so that we omit them. □

Remark 1.2.16 i) If under the hypotheses of the above theorem we have for some $t_0>0$, $\varphi(t_0,I)=I$ it is easily seen that the same holds for every $t>0$ so that

$$\rho_{t,p}(x)=\exp\left(p\int_{\varphi(-t,x)}^{x}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy\right).$$

It follows by the same kind of arguments as in the above proof that $\mathcal{T}_{F,h}$ is hypercyclic on $L^p(I)$ if and only if $\mathcal{T}_{F,h}$ is weakly mixing if and only if there are sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\omega_n)_{n\in\mathbb{N}}$ in I converging to α and ω respectively, such that for some $x_0\in I$

$$\lim_{n\to\infty}\int_{\alpha_n}^{x_0}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy=-\infty,\quad \lim_{n\to\infty}\int_{x_0}^{\omega_n}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy=+\infty.$$

Mixing of $\mathcal{T}_{F,h}$ then occurs if and only if

$$\int_{\alpha}^{x_0}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy=-\infty,\quad \int_{x_0}^{\omega}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy=+\infty,$$

while $\mathcal{T}_{F,h}$ is chaotic if and only if there is some $x_0\in I$ such that for every $x\in I$ there is $t>0$ such that

$$\sum_{l=1}^{\infty}\exp\left(p\int_{\varphi(lt,x)}^{x_0}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy\right)$$
$$+\sum_{l=1}^{\infty}\exp\left(-p\int_{x_0}^{\varphi(-lt,x)}\frac{\operatorname{Re}h(y)-\frac{1}{p}F'(y)}{F(y)}dy\right)<\infty.$$

ii) Of course it is possible to characterize hypercyclicity, mixing, and chaos of $\mathcal{T}_{F,h}$ in case of F being strictly positive on I, too. The other hypotheses of Corollary 1.2.6 unchanged, it follows for example that $\mathcal{T}_{F,h}$ is hypercyclic if and only if there is a sequence $(\omega_n)_{n\in\mathbb{N}}$ in I converging to $\omega = \sup I$ such that for some $x_0 \in I$

$$\lim_{n\to\infty} \int_{x_0}^{\omega_n} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)} dy = \infty.$$

The conditions characterizing mixing property and chaos change, respectively, along the same line.

To conclude this section we give a concrete description of the generator of $\mathcal{T}_{F,h}$ in case of $\rho = 1$, at least under some mild additional assumptions on F and h.

Theorem 1.2.17 *Let $F \in C^1(\Omega, \mathbb{R})$ and let $h \in C(\Omega, \mathbb{C})$ be as above, $1 \le p < \infty$ and assume that $\rho = 1$ is p-admissible for F and h. Assume that additionally $h \in L^\infty(\Omega)$ and that F can be extended as a C^1-function to $\mathbb{R}$ such that*

i) $F, F' \in L^\infty(\mathbb{R})$,

ii) $\overline{\Omega}$ *is positively invariant under the flow $\varphi(\cdot,\cdot)$ associated with the problem $\dot{x} = F(x), x(0) = x_0$ (defined on $\mathbb{R} \otimes \mathbb{R}$ by the assumptions on F).*

Then the generator $(A, D(A))$ of the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p(\Omega)$ is given by

$$D(A) = \{f \in L^p(\Omega);\ Ff' \in L^p(\Omega)\}$$

and

$$A : D(A) \to L^p(\Omega), \qquad Af = Ff' + hf,$$

where f' denotes the distributional derivative of f.

Proof. Let $D := \{f \in L^p(\Omega);\ Ff' \in L^p(\Omega)\}$ and

$$B : D \to L^p(\Omega), \qquad Bf := Ff'.$$

We first observe that the operator (B, D) is a closed operator. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} u_n = f$ and $\lim_{n\to\infty} Bu_n = g$ in $L^p(\Omega)$. Since $F \in C^1(\Omega)$ we obtain for the distributional derivative of Fu_n

$$(Fu_n)' = F'u_n + Fu_n' = F'u_n + Bu_n$$

which converges in $L^p(\Omega)$ to $F'f + g$ because $F' \in L^\infty(\Omega)$.

It follows $\lim_{n\to\infty} Fu_n' = g$ in $L^p(\Omega)$. On the other hand $(Fu_n')_{n\in\mathbb{N}}$ converges in the sense of distributions to Ff' and thus $Ff' = g \in L^p(\Omega)$. Hence $f \in D$ and $Bf = g$, so that (B, D) is a closed operator.

Next we show that

$$D_1 := \{f \in C^1(\overline{\Omega}) \cap L^p(\Omega);\ f' \in L^p(\Omega)\}.$$

is a core for (B, D). Being $F \in L^\infty(\Omega)$, we have $D_1 \subseteq D$. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi \geq 0$ and $\int_R \psi(x)\,dx = 1$ and set $\psi_n(x) := n\psi(nx), n \in \mathbb{N}$. In what follows we extend all functions from $L^p(\Omega)$ by 0 to all $\mathbb{R}$. Fix $f \in L^p(\Omega)$. Then $\psi_n * f \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$, thus its restriction to $\overline{\Omega}$ belongs to $C^1(\Omega)$. Moreover, as is well-known $(\psi_n * f)' = \psi_n' * f \in L^p(\mathbb{R})$ and therefore $(\psi_n * f) \in D_1$ with $\lim_{n\to\infty}(\psi_n * f)_{|\Omega} = f$ in $L^p(\Omega)$. Since we assumed F to be extendable to $\mathbb{R}$ such that $F \in C^1(\mathbb{R})$ with $F, F' \in L^\infty(\mathbb{R})$ it follows from Lemma 0.1.1 that $F(\psi_n*f)' - \psi_n*(Ff') \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} F(\psi_n*f)' - \psi_n*(Ff') = 0$ in $L^p(\mathbb{R})$. As $Ff' \in L^p(\mathbb{R})$ by $f \in D$ we have

$$F(\psi_n * f)' - Ff' = \Big(\psi_n * (Ff') - Ff'\Big) + \Big(F(\psi_n * f') - \psi_n * (Ff')\Big),$$

so $\lim_{n\to\infty} F(\psi_n * f)'_{|\Omega} = Ff'$ in $L^p(\Omega)$ implying that $(B(\psi_n * f)_{|\Omega})_{n\in\mathbb{N}}$ converges to Bf in $L^p(\Omega)$. Hence D_1 is dense in D equipped with the graph norm of B, i.e. D_1 is a core of (B, D).

Since we assume $h \in L^\infty(\Omega)$ it follows that $|h_t(x)| \geq e^{-t\|h\|_\infty}$ for all $x \in \Omega$. Being $\rho = 1$ is p-admissible for F and h we conclude that

$$\forall\, x \in \Omega : \qquad 1 \leq Me^{(\omega + p\|h\|_\infty)t}|\partial_2\varphi(t, x)|$$

so that $\rho = 1$ is p-admissible for F and 0, too. Denote the generator of the C_0-semigroup $\mathcal{T}_{F,0} = (T_{F,0}(t))_{t\geq 0}$ on $L^p(\Omega)$ by $(A_0, D(A_0))$. By using Lebesgue's dominated convergence theorem it is straightforward to verify that $D_1 \subseteq D(A_0)$ and $A_0 f = Bf$ for all $f \in D_1$.

Next we show that D_1 is also a core for $(A_0, D(A_0))$. Indeed, as $C_c^1(\Omega) \subseteq D_1$ it follows that D_1 is dense in $L^p(\Omega)$. Moreover, it follows immediately from $\mathcal{T}_{F,0}(t)f = f(\varphi(t, \cdot))$ that D_1 is invariant under $T_{F,0}$ because of the additional hypothesis ii). Applying Proposition 0.2.5 i) we conclude that D_1 is a core for $(A_0, D(A_0))$. As both operators (B, D) and $(A_0, D(A_0))$ are closed and coincide on the common core D_1, we obtain $(A_0, D(A_0)) = (B, D)$.

As $h \in L^\infty(\Omega)$, the operator

$$M_h : L^p(\Omega) \to L^p(\Omega), \qquad f \mapsto hf$$

is well-defined and continuous. Being a bounded perturbation of $(A_0, D(A_0))$

$$C : D(A_0) \to L^p(\Omega), \qquad f \mapsto A_0 f + M_h f = Bf + M_h$$

generates a C_0-semigroup $\mathcal{S}$ on $L^p(\Omega)$ (see e.g. Theorem 0.2.6) and D_1 is a core of $(C, D(A_0))$.

Now, let $(A, D(A))$ be the generator of the C_0-semigroup $\mathcal{T}_{F,h}$. As above for the special case $h = 0$ one shows that $D_1 \subseteq D(A)$ and that A and C coincide on D_1. Moreover, if $\alpha \in \rho(A) \cap \rho(C)$, it follows from D_1 being a core for $(C, D(A_0))$ that

$$\overline{(\alpha - A)(D_1)} = \overline{(\alpha - C)(D_1)} = L^p(\Omega).$$

From Proposition 0.2.5 ii) we derive that D_1 is a core for $(A, D(A))$. Since $(A, D(A))$ and $(C, D(A_0))$ are both closed operators coinciding on the common core D_1 we finally obtain the assertion. □

Remark 1.2.18 Conditions on F that ensure that ii) holds can be found by using Theorem 0.4.4 and Corollary 0.4.5. In particular if $\Omega = (a, b)$, then ii) holds if $F(a) \geq 0$ and $F(b) \leq 0$.

1.3 Chaotic dynamics on Sobolev spaces

In this section we will consider the semigroup $\mathcal{T}_{F,h}$ when acting on the Sobolev spaces $W^{1,p}(I)$ and $W_*^{1,p}(I)$, where I is a bounded open interval. We will use Lemma 1.3.2 and Proposition 1.3.3 to see how the dynamical properties can be transferred into these spaces from $L^p(I)$.

The following result is standard in the theory of Sobolev spaces. A proof can be found [25, Lemma 8.2].

Lemma 1.3.1 *Let $g \in L^1_{loc}(I)$; for y_0 fixed in I, set*

$$v(x) = \int_{y_0}^{x} g(t)dt, \ x \in I.$$

Then $v \in C(I)$ and

$$\int_I v\varphi' = -\int_I g\varphi, \ \forall \varphi \in C_c^1(I).$$

A straightforward consequence is the following lemma.

Lemma 1.3.2 *Let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$ and $1 \leq p < \infty$. Then*

$$\Phi : L^p(I) \to W_*^{1,p}(I), \qquad \Phi(f)(x) := \int_a^x f(y)\, dy$$

is a well-defined, linear and continuous bijection with continuous inverse

$$\Phi^{-1} : W_*^{1,p}(I) \to L^p(I), \qquad \Phi^{-1}(f) = f'.$$

Proof. A straightforward application of Jensen's inequality gives that

$$\begin{aligned}\int_a^b |\Phi(f)(x)|^p\, dx &\leq \int_a^b (x-a)^{p-1}\, dx \int_a^b |f(y)|^p\, dy \\ &= \frac{1}{p}(b-a)^p \int_a^b |f(y)|^p\, dy. \qquad (1.6)\end{aligned}$$

Moreover, by Lemma 1.3.1 it follows that the distributional derivative of $\Phi(f)$ equals f, so that Φ is in fact well-defined, obviously linear, and by (1.6) and $\Phi(f)' = f$ continuous. Injectivity of Φ follows from $\Phi(f)' = f$. Additionally, from $\Phi(u') = u$ for every $u \in W_*^{1,p}(I)$ we obtain the surjectivity of Φ. Obviously, $\Phi^{-1}(f) = f'$ for all $W_*^{1,p}(I)$. □

The comparison test and Lemma 1.3.2 imply the next result.

Proposition 1.3.3 *Let $I = (a,b)$ be a bounded open interval, $1 \leq p < \infty$, and let Φ be as in Lemma 1.3.2. Moreover, let $\mathcal{T}$ be a C_0-semigroup of $L^p(I)$. Then $\mathcal{S} := (\Phi \circ T(t) \circ \Phi^{-1})_{t\geq 0}$ is a C_0-semigroup on $W_*^{1,p}(I)$ which is hypercyclic, (weakly) mixing, chaotic, or stable respectively, if and only if the same holds for $\mathcal{T}$.*

Proof. Clearly, $\mathcal{S}$ is a C_0-semigroup on $W_*^{1,p}(I)$ by Lemma 1.3.2. The rest of the assertion follows immediately from the comparison test. □

The first part of this section is devoted to find conditions that ensure the restriction of the semigroup $\mathcal{T}_{F,h}$ to $W^{1,p}(I)$ is a C_0-semigroup.

Lemma 1.3.4 *Let $I = (a,b)$ be a bounded interval and $F : [a,b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3). Then the function $\rho = 1$ is p-admissible for F and F' for every $1 \leq p < \infty$, i.e. the semigroup $\mathcal{T}_{F,F'} = (T_{F,F'}(t))_{t\geq 0}$ defined by*

$$\forall\, t \geq 0, f \in L^p(I) :\ (T_{F,F'}(t)f)(x) := \exp\left(\int_0^t F'(\varphi(s,x))\, ds\right) f(\varphi(t,x))$$

is a C_0-semigroup on $L^p(I)$.

Proof. Since F is C^1 on $[a,b]$ there is $\omega \in \mathbb{R}$ such that $F'(x) \leq \omega$ for all $x \in [a,b]$. Hence, for $t \geq 0$ and $x \in (a,b)$ we have

$$\begin{aligned} 0 &\leq \partial_2\varphi(t,x) = 1 + \int_0^t \frac{\partial}{\partial t}\partial_2\varphi(s,x)\,ds = 1 + \int_0^t \frac{\partial}{\partial x}\frac{\partial}{\partial t}\varphi(s,x)\,ds \\ &= 1 + \int_0^t F'(\varphi(s,x))\partial_2\varphi(s,x)\,ds \leq 1 + \omega\int_0^t \partial_2\varphi(s,x)\,ds. \end{aligned}$$

An application of Gronwall's lemma yields

$$\forall\, x \in (a,b), t \geq 0: \quad \partial_2\varphi(t,x) \leq e^{\omega t}.$$

For $1 \leq p < \infty$ it follows from the above inequality, the hypothesis, and Proposition 0.4.8 that for every $x \in (a,b)$ we have

$$\begin{aligned} \forall\, t \geq 0: \left(\exp\left(\int_0^t F'(\varphi(s,x))\,ds\right)\right)^p &= \big(\partial_2\varphi(t,x)\big)^{p-1}|\partial_2\varphi(t,x)| \\ &\leq e^{(p-1)\omega t}|\partial_2\varphi(t,x)|. \end{aligned}$$

We finish the proof by using of Theorem 1.2.1 with $h = F'$ and $\rho = 1$. □

Theorem 1.3.5 *Let $I = (a,b)$ be a bounded interval and $F : [a,b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3) with $F(a) = 0$. Then for every $1 \leq p < \infty$ and $\gamma \in \mathbb{R}$, setting*

$$\forall\, t \geq 0, f \in W^{1,p}(I): \big(S_{F,\gamma}(t)f\big)(x) := e^{\gamma t} f(\varphi(t,x)),$$

the family $\mathcal{S}_{F,\gamma} = (S_{F,\gamma}(t))_{t\geq 0}$ is a C_0-semigroup on $W^{1,p}(I)$. Moreover $W^{1,p}_*(I)$ is $S_{F,\gamma}$-invariant.

The generator of $\mathcal{S}_{F,\gamma}$ in $W^{1,p}(I)$ is given by

$$B: \{f \in W^{1,p}(I);\, Ff'' \in L^p(I)\} \to W^{1,p}(I), \qquad Bf = Ff' + \gamma f,$$

while its generator in $W^{1,p}_*(I)$ is

$$B_*: \{f \in W^{1,p}_*(I);\, Ff'' \in L^p(I)\} \to W^{1,p}_*(I), \qquad B_*f = Ff' + \gamma f.$$

Proof. By the preceding lemma $\mathcal{T}_{F,F'} = (T_{F,F'}(t))_{t\geq 0}$ is a well-defined strongly continuous semigroup on $L^p(I)$ for every $1 \leq p < \infty$. As $F(a) = 0$ we have $\varphi(t,a) = a$ for all $t \geq 0$ so that for all $f \in L^p(I)$ and $t \geq 0$ with Φ as in Lemma 1.3.2 and the fact that $\varphi(t,\cdot)$ is increasing by Proposition 0.4.8

$$\Phi(T(t)f)(x) = \int_a^x f(\varphi(t,y))|\partial_2\varphi(t,y)|\,dy = \int_a^{\varphi(t,x)} f(y)\,dy = \Phi(f)(\varphi(t,x)).$$

Since Φ is bijective it follows that $\mathcal{S}_{F,\gamma} := (S_{F,\gamma}(t))_{t\geq 0} = \Big(\Phi \circ \big(e^{\gamma t}T_{F,F'}(t)\big) \circ \Phi^{-1}\Big)_{t\geq 0}$ defines a C_0-semigroup on $W^{1,p}_*(I)$. Clearly, for every $t \geq 0$ the mapping

$$S_{F,\gamma}(t) : \operatorname{span}\{\mathbb{1}\} \to \operatorname{span}\{\mathbb{1}\}, \qquad \alpha\mathbb{1} \mapsto e^{\gamma t}\,(\alpha\mathbb{1}) \circ \varphi(t,\cdot)$$

is well-defined, linear and continuous. It follows that $\mathcal{S}_{F,\gamma}$ is a well-defined C_0-semigroup on $W^{1,p}(I) = W^{1,p}_*(I) \oplus \operatorname{span}\{\mathbb{1}\}$ such that $W^{1,p}_*(I)$ is $\mathcal{S}_{F,\gamma}$-invariant.

The generator of $\mathcal{S}_{F,\gamma|W^{1,p}_*(I)}$ is given by $(\Phi \circ A \circ \Phi^{-1}, \Phi(D(A)))$, where $(A, D(A))$ is the generator of $(e^{\gamma t}T_{F,F'}(t))_{t\geq 0}$ in $L^p(I)$ which by Theorem 1.2.17 is

$$A : \{f \in L^p(I);\, Ff' \in L^p(I)\} \to L^p(I), \qquad Af = Ff' + F'f + \gamma f.$$

Therefore

$$\Phi\big(D(A)\big) = \{f \in W^{1,p}_*(I);\, Ff'' \in L^p(I)\}$$

and

$$\forall\, f \in \Phi\big(D(A)\big) :\ \Phi \circ A \circ \Phi^{-1}(f) = \Phi(Ff'' + F'f' + \gamma f') = Ff' + \gamma f.$$

Obviously, $\operatorname{span}\{\mathbb{1}\}$ is contained in the domain of the generator $(B, D(B))$ of $\mathcal{S}_{F,\gamma}$ with $B\mathbb{1} = \gamma\mathbb{1}$. Therefore,

$$D(B) = \Phi\big(D(A)\big) \oplus \operatorname{span}\{\mathbb{1}\} = \{f \in W^{1,p}(I);\, Ff'' \in L^p(I)\}$$

and

$$\forall\, f \in D(B) : \quad Bf = \Phi \circ A \circ \Phi^{-1}(f - f(a)\mathbb{1}) + \gamma f(a)\mathbb{1} = Ff' + \gamma f.$$

□

By combining the above theorem with the results stated in section 1.2, we are now able to prove the following theorem.

Theorem 1.3.6 *Let $I = (a, b)$ be a bounded interval, $1 \leq p < \infty, \gamma \in \mathbb{R}$, and $F : [a, b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3) with $F(a) = 0$. Then the following holds.*

a) The C_0-semigroup $\mathcal{S}_{F,\gamma}$ is not hypercyclic on $W^{1,p}(I)$.

b) On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $\mathcal{S}_{F,\gamma}$.

i) $\mathcal{S}_{F,\gamma}$ is weakly mixing on $W^{1,p}_*(I)$, *ii)* $\mathcal{S}_{F,\gamma}$ *is hypercyclic on* $W^{1,p}_*(I)$*,*

iii) $\lambda(\Omega_0) = 0$ *and for every* $m \in \mathbb{N}$ *for which there are* m *different connected components* $C_1, \ldots, C_m$ *of* $I \setminus \Omega_0$*, for* λ^m*-almost all choices of* $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ *there is a sequence of positive numbers* $(t_n)_{n\in\mathbb{N}}$ *tending to infinity such that for every* $1 \leq j \leq m$

$$\lim_{n\to\infty} \chi_{\varphi(t_n,I)}(x_j)\, e^{p\gamma t_n}\, \partial_2\varphi(-t_n, x_j)^{1-p} = 0$$

and

$$\lim_{n\to\infty} e^{-p\gamma t_n} \frac{\partial_2\varphi(t_n, x_j)^{1+p}}{\partial_2\varphi(2t_n, x_j)^p} = 0.$$

c) *On* $W^{1,p}_*(I)$ *the following are equivalent for the* C_0*-semigroup* $\mathcal{S}_{F,\gamma}$*.*

i) $\mathcal{S}_{F,\gamma}$ *is mixing on* $W^{1,p}_*(I)$*,*

ii) $\lambda(\Omega_0) = 0$ *and for* λ*-almost every* $x \in I$

$$\lim_{t\to\infty} \chi_{\varphi(t,I)}(x)\, e^{p\gamma t}\, \partial_2\varphi(-t, x)^{1-p} = \lim_{t\to\infty} e^{-p\gamma t} \frac{\partial_2\varphi(t, x)^{1+p}}{\partial_2\varphi(2t, x)^p} = 0.$$

d) *On* $W^{1,p}_*(I)$ *the following are equivalent for the* C_0*-semigroup* $\mathcal{S}_{F,\gamma}$*.*

i) $\mathcal{S}_{F,\gamma}$ *is chaotic on* $W^{1,p}_*(I)$*,*

ii) $\lambda(\Omega_0) = 0$ *and for every* $m \in \mathbb{N}$ *for which there are* m *different connected components* $C_1, \ldots, C_m$ *of* $I \setminus \Omega_0$*, for* λ^m*-almost all choices of* $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ *there is* $t > 0$ *such that*

$$\sum_{l=1}^\infty \chi_{\varphi(lt,I)}(x)\, e^{p\gamma lt}\, \partial_2\varphi(-lt, x)^{1-p} + \sum_{l=1}^\infty e^{-p\gamma lt} \frac{\partial_2\varphi(lt, x)^{1+p}}{\partial_2\varphi(2lt, x)^p} < \infty$$

for every $1 \leq j \leq m$*.*

Proof. Let $P_1 : W^{1,p}(I) \to \operatorname{span}\{\mathbb{1}\}, P_1(f) = f(a)\mathbb{1}$. As $F(a) = 0$ we have $\varphi(t, a) = a$ for all $t \geq 0$ which implies $P_1 \circ S_{F,\gamma}(t) = S_{F,\gamma}(t) \circ P_1$. Since there are no hypercyclic C_0-semigroups on finite-dimensional spaces part a) follows from the comparison test.

Now, let Φ be as in Lemma 1.3.2. Since $\big(\Phi^{-1} \circ S_{F,\gamma}(t) \circ \Phi\big)_{t\geq 0} = \mathcal{T}_{F,F'+\gamma}$ and $\Phi^{-1}\big(W^{1,p}_*(I)\big) = L^p(I)$ it follows from Theorem 1.2.10 a) for $\rho = 1$ and the comparison test that i) and ii) in b) are equivalent and that these are equivalent to hypercyclicity of $\mathcal{T}_{F,F'+\gamma}$ on $L^p(I)$. For $h(x)=F'(x)+\gamma$ and $\rho(x)=1$ it follows that for $\rho_{t,p}$ and $\rho_{-t,p}$ from definition 1.2.3 we have

$$\forall\, t\geq 0, x\in(a,b): \quad \rho_{t,p}(x)=\chi_{\varphi(t,I)}(x)h_t^p(\varphi(-t,x))\partial_2\varphi(-t,x)$$

as well as

$$\forall\, t\geq 0, x\in(a,b): \quad \rho_{-t,p}(x)=h_t^{-p}(x)\partial_2\varphi(t,x).$$

Observe that for $h(x)=F'(x)+\gamma$ we have by Proposition 0.4.8

$$\forall\, t\geq 0, x\in(a,b): \quad h_t(x)=\exp(\gamma t+\int_0^t F'(\varphi(s,x))\,ds)=e^{\gamma t}\partial_2\varphi(t,x).$$

Moreover, because $\varphi(s+t,x)=\varphi(s,\varphi(t,x))$ for all $s,t\in\mathbb{R}$ and each $x\in(a,b)$ for which the involved quantities are defined it follows

$$\forall\, t\geq 0, x\in(a,b): \quad \partial_2\varphi(2t,x)=\partial_2\varphi(t,\varphi(t,x))\partial_2\varphi(t,x)$$

and thus for every $x\in(a,b)$ we have

$$\forall\, t\geq 0: \quad \partial_2\varphi(t,\varphi(t,x))=\frac{\partial_2\varphi(2t,x)}{\partial_2\varphi(t,x)}$$

as well as

$$\forall\, t\geq 0: \quad 1=\partial_2\varphi(t,\varphi(-t,x))\partial_2\varphi(-t,x)$$

for every $x\in\varphi(t,I)$. Taking all this into account it follows

$$\begin{aligned}\forall\, t\geq 0, x\in(a,b): \quad \rho_{t,p}(x) &= \chi_{\varphi(t,I)}(x)\,e^{p\gamma t}\left(\partial_2\varphi(t,\varphi(-t,x))\right)^p\partial_2\varphi(-t,x)\\ &= \chi_{\varphi(t,I)}(x)\,e^{p\gamma t}\left(\partial_2\varphi(-t,x)\right)^{1-p}\end{aligned}$$

as well as

$$\begin{aligned}\forall\, t\geq 0, x\in(a,b): \quad \rho_{-t,p}(x) &= e^{-p\gamma t}\left(\partial_2\varphi(t,\varphi(t,x)\right)^{-p}\partial_2\varphi(t,x)\\ &= e^{-p\gamma t}\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p}.\end{aligned}$$

By Theorem 1.2.10 a) $\mathcal{T}_{F,F'+\gamma}$ is hypercyclic on $L^p(I)$ if and only if $\lambda(\Omega_0)=0$ and for every $m\in\mathbb{N}$ for which there are m different connected components $C_1,\ldots,C_m$ of $I\setminus\Omega_0$, for λ^m-almost all choices of $(x_1,\ldots,x_m)\in\Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that

$$\forall\, 1\leq j\leq m: \lim_{n\to\infty}\rho_{t_n,p}(x_j)=\lim_{n\to\infty}\rho_{-t_n,p}(x_j)=0,$$

so that by the above considerations b) follows.

The proofs of part c) and d) of the theorem follow with the analogous arguments by referring to Theorem 1.2.10 b) and Theorem 1.2.9, respectively. □

So far, we have only considered γ to be a real constant. If $h \in W^{1,\infty}(I)$ we would like to have results such as the above theorem for the C_0-semigroup on $W^{1,p}(I)$ generated by

$$f \mapsto Ff' + hf.$$

Since $h \in W^{1,\infty}(I)$ it follows that the corresponding multiplication operator

$$M_h : W^{1,p}(I) \to W^{1,p}(I), \qquad M_h(f) = hf$$

is well-defined and continuous. If we denote the generator of $\mathcal{S}_{F,0}$ in $W^{1,p}(I)$ by $(A, D(A))$ it follows that $(A + M_h, D(A))$ generates a C_0-semigroup $\mathcal{S}_{F,h}$ by Theorem 0.2.6 and this semigroup is given by

$$S_{F,h}(t)f = \sum_{n=0}^{\infty} T_n(t)f, \tag{1.7}$$

with $T_0(t) = S_{F,0}(t)$ and

$$\begin{aligned} T_{n+1}(t)f &= \int_0^t S_{F,0}(t-s)M_h T_n(s)f\,ds \\ &= \int_0^t h(\varphi(t-s,\cdot))\Big(T_n(s)f\Big)(\varphi(t-s,\cdot))ds, \end{aligned}$$

where the integrals are Riemann integrals in $W^{1,p}(I)$. In order to get an explicit description $\mathcal{S}_{F,h}$ we use the following result.

Proposition 1.3.7 *Let $v : [0,\infty) \to \mathbb{C}$ be continuous. Then*

$$\forall\, n \in \mathbb{N}_0, t \geq 0 : \qquad \int_0^t v(t-s)\Big(\int_0^s v(t-s+r)dr\Big)^n ds = \frac{1}{n+1}\Big(\int_0^t v(s)ds\Big)^{n+1}.$$

Proof. Setting $V(t) := \int_0^t v(s)ds$ it follows that

$$\begin{aligned}
\int_0^t v(t-s)\left(\int_0^s v(t-s+r)dr\right)^n ds &= \int_0^t V'(t-s)\Big(V(t)-V(t-s)\Big)^n ds \\
&= \sum_{k=0}^n \binom{n}{k} V^k(t)(-1)^{n-k}\int_0^t V'(u)V^{n-k}(u)du \\
&= \sum_{k=0}^n \binom{n}{k} V^k(t)(-1)^{n-k}\frac{1}{n+1-k}V^{n+1-k}(t) \\
&= \frac{-1}{n+1}V^{n+1}(t)\sum_{k=0}^n \binom{n+1}{k}(-1)^{n+1-k} \\
&= \frac{1}{n+1}\left(\int_0^t v(s)\,ds\right)^{n+1}.
\end{aligned}$$

□

By applying Proposition 1.3.7 to $t \mapsto h(\varphi(t,x))$ with fixed $x \in [a,b]$ and since point evaluations in $W^{1,p}(I)$ are continuous it follows by induction on n that the above $T_n(t)$ in equation (1.7) are given by

$$T_n(t)f(x) = \frac{1}{n!}\Big(\int_0^t h(\varphi(s,x))ds\Big)^n S_{F,0}(t)f(x)$$

and therefore

$$\forall\, t \geq 0, f \in W^{1,p}(I): \quad S_{F,h}(t)f(x) = h_t(x)f(\varphi(t,x)),$$

where, we recall, $h_t(x) = \exp(\int_0^t h(\varphi(s,x))ds)$. Thus we obtain the next proposition.

Proposition 1.3.8 *Let $I = (a,b)$ be a bounded interval, $1 \leq p < \infty$, $F : [a,b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3) with $F(a) = 0$ and $h \in W^{1,\infty}(I)$. Then*

$$A : \{f \in W^{1,p}(I); \quad Ff'' \in L^p(I)\}, f \mapsto Ff' + hf$$

generates the C_0-semigroup $\mathcal{S}_{F,h}$ on $W^{1,p}(I)$ which is given by

$$\forall\, t \geq 0 : S_{F,h}(t)f(x) = h_t(x)f(\varphi(t,x)).$$

Moreover, $W_*^{1,p}(I)$ is $\mathcal{S}_{F,h}$-invariant.

Our next aim is to generalize the content of Theorem 1.3.6 to the C_0-semigroups $S_{F,h}$, at least under some mild additional hypothesis. As in section 1.2 we need to find a measurable function $\psi : [a,b] \to \mathbb{R}$ such that $\exp \circ \psi$ induces a continuous, invertible multiplication operator M on $W^{1,p}(I)$ and for some $\gamma \in \mathbb{R}$, for which C_0-semigroups $\mathcal{S}_{F,\gamma}$ and $\mathcal{S}_{F,h}$ are conjugate via M, i.e.

$$\forall\, t \geq 0, f \in W^{1,p}(I): \quad \exp(\psi)\, S_{F,h}(t)(f) = S_{F,\gamma}(t)(\exp(\psi) f). \tag{1.8}$$

(1.8) is satisfied if, for all $x \in [a,b]$,

$$\forall\, t \geq 0: \quad \psi(\varphi(t,x)) - \int_0^t \big(h(\varphi(s,x)) - \gamma\big)\, ds = \psi(x). \tag{1.9}$$

Now, if $x \in \Omega_0$ it follows from $\varphi(t,x) = x$ for all $t \geq 0$

$$\forall\, t \geq 0: \quad \psi(x) - t(h(x) - \gamma) = \psi(x).$$

Thus, since $F(a) = 0$ it is necessary that $h(x) = h(a)$ for every $x \in \Omega_0$.

As in section 1.2, if the function $[a,b] \to \mathbb{R},\ y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^1(a,b)$ we can set

$$\psi : [a,b] \to \mathbb{R}, \quad \psi(x) = \int_a^x \frac{h(y) - h(a)}{F(y)} dy$$

and it follows that this function ψ satisfies (1.9) on Ω_1.

Proposition 1.3.9 *Let $I = (a,b)$ be a bounded interval, $1 \leq p < \infty, F : [a,b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3) with $F(a) = 0$, and $h \in W^{1,\infty}(I)$. Assume that the following conditions are satisfied.*

i) $\forall\, x \in \Omega_0 : \quad h(x) = h(a) \in \mathbb{R}$.

ii) The function $[a,b] \to \mathbb{R}, \quad y \mapsto \frac{h(y)-h(a)}{F(y)}$ *belongs to* $L^\infty(I)$.

Then $\exp \circ \psi$ *with* $\psi : [a,b] \to \mathbb{R},\ \psi(x) = \int_a^x \frac{h(y)-h(a)}{F(y)} dy$ *defines a continuous, invertible multiplication operator M such that the C_0-semigroups $\mathcal{S}_{F,h(a)}$ and $\mathcal{S}_{F,h}$ on $W^{1,p}(I)$ are conjugate via M.*

Proof. By the observation preceding the proposition we only have to show that $\exp \circ \psi$ defines a continuous, invertible multiplication operator on $W^{1,p}(I)$. First we note that $\psi \in W^{1,\infty}(I)$ and therefore $\exp \circ \psi \in W^{1,\infty}(I)$, too. Hence,

$$M : W^{1,p}(I) \to W^{1,p}(I), \qquad Mf = \exp(\psi) f$$

is a well-defined continuous multiplication operator on $W^{1,p}(I)$ with continuous inverse given by $\exp(-\psi) f$. □

Remark 1.3.10 In the discussion preceding the above proposition we only required that $[a,b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^1(a,b)$. Nevertheless, it is not hard to show that this function actually has to belong to $L^\infty(a,b)$ in order for $\exp(\psi)$ to induce a bounded multiplication operator on $W^{1,p}(I)$.

Obviously, the multiplication operator M in the above proposition satisfies

$$M(W_*^{1,p}(I)) = W_*^{1,p}(I)$$

so that the restrictions of the C_0-semigroups $\mathcal{S}_{F,h(a)}$ and $\mathcal{S}_{F,h}$ to $W_*^{1,p}(I)$ are conjugate via M, too. Combining Theorem 1.3.6 with Proposition 1.3.9 the next theorem follows directly from the comparison test.

Theorem 1.3.11 *Let $I = (a,b)$ be a bounded interval, $1 \leq p < \infty$, $F : [a,b] \to \mathbb{R}$ a C^1-function satisfying (H2)-(H3) with $F(a) = 0$ and $h \in W^{1,\infty}(I)$. Assume that*

1) $\forall\, x \in \Omega_0 : h(x) = h(a) \in \mathbb{R}$.

2) *The function* $[a,b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ *belongs to* $L^\infty(I)$.

Then the following holds.

a) *The C_0-semigroup $\mathcal{S}_{F,h}$ is not hypercyclic on $W^{1,p}(I)$.*

b) *On $W_*^{1,p}(I)$ the following are equivalent for the C_0-semigroup $\mathcal{S}_{F,h}$.*

   i) *$\mathcal{S}_{F,h}$ is weakly mixing on $W_*^{1,p}(I)$,*

   ii) *$\mathcal{S}_{F,h}$ is hypercyclic on $W_*^{1,p}(I)$,*

   iii) *$\lambda(\Omega_0) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \ldots, C_m$ of $I \setminus \Omega_0$, for λ^m-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that for every $1 \leq j \leq m$*

$$\lim_{n\to\infty} \chi_{\varphi(t_n,I)}(x_j)\, e^{ph(a)t_n}\, \partial_2\varphi(-t_n, x_j)^{1-p} = 0$$

   and

$$\lim_{n\to\infty} e^{-ph(a)t_n} \frac{\partial_2\varphi(t_n, x_j)^{1+p}}{\partial_2\varphi(2t_n, x_j)^p} = 0.$$

c) On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $\mathcal{S}_{F,h}$.

i) $\mathcal{S}_{F,h}$ is mixing on $W^{1,p}_*(I)$,

ii) $\lambda(\Omega_0) = 0$ and for λ-almost every $x \in I$

$$\lim_{t\to\infty} \chi_{\varphi(t,I)}(x)\, e^{ph(a)t}\, \partial_2\varphi(-t,x)^{1-p} = \lim_{t\to\infty} e^{-ph(a)t} \frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p} = 0.$$

d) On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $\mathcal{S}_{F,h}$.

i) $\mathcal{S}_{F,h}$ is chaotic on $W^{1,p}_*(I)$,

ii) $\lambda(\Omega_0) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \dots, C_m$ of $I \setminus \Omega_0$, for λ^m-almost all choices of $(x_1, \dots, x_m) \in \Pi_{j=1}^m C_j$ there is $t > 0$ such that

$$\sum_{l=1}^{\infty} \chi_{\varphi(lt,I)}(x)\, e^{ph(a)lt}\, \partial_2\varphi(-lt,x)^{1-p} + \sum_{l=1}^{\infty} e^{-ph(a)lt} \frac{\partial_2\varphi(lt,x)^{1+p}}{\partial_2\varphi(2lt,x)^p} < \infty$$

for every $1 \leq j \leq m$.

Instead of formulating analogues to Corollary 1.2.6 for $W^{1,p}_*(I)$ we apply our results to some concrete examples in the next section.

1.4 The Lasota equation

We close this chapter by analyzing the solution semigroup (1.2) of the Lasota equation (1.1) in the light of our result. We will cover the result mentioned in section 1.1.

1.4.1 von Foerster-Lasota equation

Consider for a complex valued function $h \in C(0,1) \cap L^\infty(0,1)$ the von Foerster-Lasota equation, that is the first order partial differential equation

$$\frac{\partial}{\partial t} u(t,x) + x \frac{\partial}{\partial x} u(t,x) = h(x)\, u(t,x), \quad t \geq 0,\ 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x), \ \ 0 < x < 1,$$

where v is a given function.

If we set $F(x) = -x$ and $Au = Fu' + hu$, for every $u \in D(A)$, where

$$D(A) = \{u \in L^p(0,1) \mid xu' \in L^p(0,1)\},$$

then by Theorem 1.2.17, $(A, D(A))$ generates in $L^p(0,1)$ the C_0-semigroup $\mathcal{T}_h = (T_h(t))_{t\geq 0}$ defined by

$$T_h u(x) = \exp\left(\int_{-t}^{0} h(xe^s)ds\right) u(xe^{-t}), \qquad t \geq 0, \ u \in L^p(0,1), \ x \in (0,1).$$

If $h \in W^{1,\infty}(0,1)$, in view of Proposition 1.3.8 it is again natural to apply also our results from section 1.3. We call the resulting C_0-semigroup the von Foerster-Lasota semigroup on $W^{1,p}(0,1)$, respectively $W_*^{1,p}(0,1)$, associated with h and denote it by $\mathcal{S}_h$. The dynamical properties of these semigroups are given in the next theorem.

Theorem 1.4.1

a) Let h be in $C(0,1) \cap L^\infty(0,1)$. Then the following properties of the associated von Foerster-Lasota semigroup $\mathcal{T}_h$ on $L^p(0,1)$ are equivalent.

i) $\mathcal{T}_h$ is hypercyclic.

ii) $\mathcal{T}_h$ is weakly mixing.

iii) There is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to zero such that for some $x_0 \in (0,1)$

$$\lim_{n\to\infty} \int_{\alpha_n}^{x_0} \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy = +\infty.$$

b) Assume that for $h \in C[0,1]$ the function

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - \operatorname{Re} h(0)}{x}$$

belongs to $L^1(0,1)$. Then the following properties of the associated von Foerster-Lasota semigroup $\mathcal{T}_h$ on $L^p(0,1)$ are equivalent.

i) $\mathcal{T}_h$ *is hypercyclic.*

ii) $\mathcal{T}_h$ *is weakly mixing.*

iii) $\mathcal{T}_h$ *is mixing.*

iv) $\mathcal{T}_h$ *is chaotic.*

v) $\operatorname{Re} h(0) > -\frac{1}{p}$.

c) Assume that for $h \in W^{1,\infty}(0,1)$ *the function*

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^\infty(0,1)$ *and that* $h(0) \in \mathbb{R}$. *Then the von Foerster-Lasota semigroup* $\mathcal{S}_h$ *is not hypercyclic on* $W^{1,p}(0,1)$. *For the restriction of* $\mathcal{S}_h$ *to* $W_*^{1,p}(0,1)$ *the following are equivalent.*

i) $\mathcal{S}_h$ *is hypercyclic on* $W_*^{1,p}(0,1)$.

ii) $\mathcal{S}_h$ *is weakly mixing on* $W_*^{1,p}(0,1)$.

iii) $\mathcal{S}_h$ *is mixing on* $W_*^{1,p}(0,1)$.

iv) $\mathcal{S}_h$ *is chaotic on* $W_*^{1,p}(0,1)$.

v) $h(0) > 1 - \frac{1}{p}$.

Proof. For $F(x) = -x$ we have $\varphi(t,x) = xe^{-t}$ and thus the hypotheses of Theorem 1.2.15 are satisfied.

Proof of part a). Since $(0,1) \to \mathbb{R}$, $x \mapsto \dfrac{\operatorname{Im} h(x)}{x}$ belongs to $L^1(\beta,1)$ for each $\beta \in (0,1)$ part a) is obviously a direct application of Theorem 1.2.15 a).

Proof of part b). Since the hypotheses of a) are satisfied, it follows that i) and ii) are equivalent and that because of $h \in C[0,1]$ i) holds if and only if

$$\lim_{n\to\infty} \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy = +\infty \tag{1.10}$$

for some $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 0. Since

$$\begin{aligned}\int_{\alpha_n}^1 \frac{\operatorname{Re} h(y)+\frac{1}{p}}{y}dy &= \int_{\alpha_n}^1 \frac{\operatorname{Re} h(0)+\frac{1}{p}}{y}dy + \int_{\alpha_n}^1 \frac{\operatorname{Re}(h(y)-h(0))}{y}dy \\ &= -\Big(\operatorname{Re} h(0)+\frac{1}{p}\Big)\ln(\alpha_n) + \int_{\alpha_n}^1 \frac{\operatorname{Re}(h(y)-h(0))}{y}dy\end{aligned}$$

it follows from the hypothesis on h that (1.10) holds if and only if $\operatorname{Re} h(0)+\frac{1}{p}>0$. Hence, i), ii), and v) in b) are equivalent. As iii) and iv) imply i), respectively, it remains to prove that v) implies iii) and iv).

Let us denote by c the $L^1(0,1)$-norm of the L^1-function $x\mapsto \frac{h(x)-\operatorname{Re} h(0)}{x}$. If v) holds, it follows that for each $x\in(0,1), t>0$, and $l\in\mathbb{N}$ we have

$$\begin{aligned}\exp\left(p\int_x^{xe^{-lt}} \frac{\operatorname{Re} h(y)+\frac{1}{p}}{y}dy\right) &= \exp\left(-p\int_{xe^{-lt}}^x \frac{\operatorname{Re} h(0)+\frac{1}{p}}{y}dy\right)\cdot \\ &\quad \cdot\exp\left(-p\int_{xe^{-lt}}^x \frac{\operatorname{Re}(h(y)-h(0))}{y}dy\right) \qquad (1.11)\\ &\le e^{-ptl\left(\operatorname{Re} h(0)+\frac{1}{p}\right)}\, e^{pc}.\end{aligned}$$

Because of v) it follows that for every $x\in(0,1)$ and $t>0$

$$\sum_{l=1}^{\infty}\exp\left(p\int_x^{xe^{-lt}} \frac{\operatorname{Re} h(y)+\frac{1}{p}}{y}dy\right)<\infty,$$

hence $\mathcal{T}_h$ is chaotic by part c) of Theorem 1.2.15.

In order to show that v) implies iii) we observe that as in the inequality (1.11) we obtain for each $x\in(0,1)$ and $t\ge 0$

$$\exp\left(p\int_x^{xe^{-t}} \frac{\operatorname{Re} h(y)+\frac{1}{p}}{y}dy\right)\le e^{-ptl\left(\operatorname{Re} h(0)+\frac{1}{p}\right)}\, e^{pc},$$

so that by part b) of Theorem 1.2.15 iii) holds. Hence, b) is proved.

Proof of part c). That $\mathcal{S}_h$ is not hypercyclic on $W^{1,p}(0,1)$ follows immediately from Theorem 1.3.11. As $[0,1]\setminus\Omega_0$ has only one connected component, $\chi_{\varphi(t,I)}(x)=0$ for every $x\in(0,1]$ for sufficiently large $t>0$, and

$$\forall\, x\in(0,1]:\quad e^{-ph(0)t}\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p}=e^{-t\left(1+(h(0)-1)p\right)}$$

the rest of c) now follows from Theorem 1.3.11 b),c), and d). □

Remark 1.4.2 i) Of course Theorem 1.2.15 also provides characterizations of mixing and chaos for $\mathcal{T}_h$ under the general hypotheses of part a) of the above theorem.

ii) It should be noted that the proof of part b) remains valid if we replace the hypothesis $h \in C[0,1]$ by $h \in C(0,1) \cap L^\infty(0,1), h$ continuously extendable into the origin.

iii) The hypothesis in c) is obviously satisfied if $h(0) \in \mathbb{R}$ and h is differentiable at the origin.

Remark 1.4.3 Observe that if $\alpha = 1 - \frac{1}{p}$, then $W^{1,p}(I) \hookrightarrow h^\alpha(0,1)$ continuously and thus $W_*^{1,p}(I) \hookrightarrow V_\alpha$. Moreover they are dense. As a consequence, by Theorem 1.4.1, if $h(0) > \alpha = 1 - 1/p$, then the C_0-semigroup $\mathcal{T}_h$ is mixing and chaotic on V_α and we obtain the result of Theorem 1.1.1.

1.4.2 Generalization of von Foerster-Lasota equation

Let us consider for $h \in C(0,1) \cap L^\infty(0,1)$ and $r > 1$ the first order partial differential equation

$$\frac{\partial}{\partial t} u(t,x) + x^r \frac{\partial}{\partial x} u(t,x) = h(x)\, u(t,x), \quad t \geq 0,\, 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x), \quad 0 < x < 1,$$

where v is a given function. As for $F(x) = -x^r$ we have

$$\varphi(t,x) = ((r-1)t + x^{1-r})^{\frac{1}{1-r}}$$

and therefore $\varphi([0,\infty),(0,1)) \subseteq (0,1)$, it is again natural to consider our results from section 1.2 for $I = (0,1)$ and h and the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p(0,1)$. We write $\mathcal{T}_{r,h}$ instead of $\mathcal{T}_{F,h}$.

Again, if $h \in W^{1,\infty}(0,1)$, in view of Proposition 1.3.8 it is natural to apply our results from section 1.3. The corresponding C_0-semigroup on $W^{1,p}(0,1)$, respectively $W_*^{1,p}(0,1)$ is denoted by $\mathcal{S}_{r,h}$. The next theorem summarizes the dynamical properties of these semigroups. Observe that contrary to the von Foerster-Lasota semigroup the dynamical properties of $\mathcal{S}_{r,h}$ are independent of p.

Theorem 1.4.4

a) *Let h belong to $C(0,1) \cap L^\infty(0,1)$. Then the following properties of the C_0-semigroup $\mathcal{T}_{r,h}$ on $L^p(0,1)$ are equivalent.*

i) *$\mathcal{T}_{r,h}$ is hypercyclic.*

ii) *$\mathcal{T}_{r,h}$ is weakly mixing.*

iii) *There is a sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to zero such that for some $x_0 \in (0,1)$*

$$\lim_{n\to\infty} \int_{r_n}^{x_0} \frac{\operatorname{Re} h(y) + \frac{r}{p} y^{r-1}}{y^r} dy = \infty.$$

b) *Assume that for $h \in C[0,1]$ the function*

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - x^{r-1}\operatorname{Re} h(0)}{x^r}$$

belongs to $L^1(0,1)$. Then the following properties of $\mathcal{T}_{r,h}$ on $L^p(0,1)$ are equivalent.

i) *$\mathcal{T}_{r,h}$ is hypercyclic.*

ii) *$\mathcal{T}_{r,h}$ is weakly mixing.*

iii) *$\mathcal{T}_{r,h}$ is mixing.*

iv) $\operatorname{Re} h(0) > \frac{-r}{p}$.

Moreover $\mathcal{T}_{r,h}$ is chaotic if and only if $Reh(0) > -\frac{1}{p}$.

c) *Assume that for $h \in W^{1,\infty}(0,1)$ the function*

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x^r}$$

belongs to $L^\infty(0,1)$ and that $h(0) \in \mathbb{R}$. Then the C_0-semigroup $\mathcal{S}_{r,h}$ is not hypercyclic on $W^{1,p}(0,1)$. For the restriction of $\mathcal{S}_{r,h}$ to $W^{1,p}_*(0,1)$ the following are equivalent.

i) *$\mathcal{S}_{r,h}$ is hypercyclic on $W^{1,p}_*(0,1)$.*

ii) *$\mathcal{S}_{r,h}$ is weakly mixing on $W^{1,p}_*(0,1)$.*

iii) $\mathcal{S}_{r,h}$ *is mixing on* $W^{1,p}_*(0,1)$.

iv) $\mathcal{S}_{r,h}$ *is chaotic on* $W^{1,p}_*(0,1)$.

v) $h(0) > 0$.

Proof. For $F(x) = -x^r$ we have $\varphi(t,x) = ((r-1)t + x^{1-r})^{\frac{1}{1-r}}$ and thus the hypotheses of Theorem 1.2.15 are satisfied. The proofs of parts a) and b) are mutatis mutandis the same as the corresponding parts of Theorem 1.4.1. In fact, the part a) is a direct consequence of Theorem 1.2.15 a) taking into account that $(0,1) \to \mathbb{R},\ x \mapsto \dfrac{\operatorname{Im} h(x)}{x}$ belongs to $L^1(\beta,1)$ for each $\beta \in (0,1)$.

In the following we show the main calculations behind of the proof of part b). It is based on the result of part a) and the fact that $h \in C[0,1]$ and

$$(0,1) \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - x^{r-1}\operatorname{Re} h(0)}{x^r}$$

belongs to $L^1(0,1)$, obviously i) and ii) are equivalent. Since $h \in C[0,1]$ i) holds if and only if

$$\lim_{n\to\infty} \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) + \frac{r}{p} y^{r-1}}{y^r} dy = +\infty \tag{1.12}$$

for some $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 0. Since

$$\begin{aligned}\int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) + \frac{r}{p} y^{r-1}}{y^r} dy &= \int_{\alpha_n}^1 \frac{\operatorname{Re} h(0) y^{r-1} + \frac{r}{p} y^{r-1}}{y^r} dy + \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) - y^{r-1}\operatorname{Re} h(0)}{y^r} dy \\ &= -\left(\operatorname{Re} h(0) + \frac{r}{p}\right) \ln(\alpha_n) + \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) - y^{r-1}\operatorname{Re} h(0)}{y^r} dy\end{aligned}$$

it follows from the hypothesis on h that (1.12) holds if and only if $\operatorname{Re} h(0) + \frac{r}{p} > 0$. Hence, i), ii), and iv) in b) are equivalent. As iii) and chaos imply i), respectively, it remains to prove that iv) implies iii) and chaos implies $\operatorname{Re} h(0) > -\frac{1}{p}$.

Let us denote by c the $L^1(0,1)$-norm of the L^1-function $x \mapsto \frac{h(x) - x^{r-1}\operatorname{Re} h(0)}{x^r}$. $\mathcal{T}_{r,h}$ is chaotic if and only if there exists $x_0 \in (0,1)$ such that for every $x \in (0,1)$ and $t > 0$

$$\sum_{l=1}^{\infty} exp\left(-p \int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(y) + \frac{r}{p} y^{r-1}}{y^r} dy\right) < \infty \tag{1.13}$$

For each $x \in (0,1), t > 0$, and $l \in \mathbb{N}$ we have

$$\begin{aligned} -p\int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(y) + \frac{r}{p}y^{r-1}}{y^r} dy &= p\int_{x_0}^{\varphi(lt,x)} \frac{\operatorname{Re} h(y) - y^{r-1}\operatorname{Re} h(0)}{y^r} dy + \\ &\quad + p\int_{x_0}^{\varphi(lt,x)} \frac{\operatorname{Re} h(0) + \frac{r}{p}}{y} dy \qquad (1.14) \\ &\leq pc + p\left(\operatorname{Re} h(0) + \frac{r}{p}\right)\log\left(\frac{\varphi(tl,x)}{x_0}\right). \end{aligned}$$

Hence iii) holds by iv) and part b) of Theorem 1.2.15 for $\alpha = 0$. Moreover,

$$\exp\left(-p\int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(0) + \frac{r}{p}}{y} dy\right) = \frac{1}{x_0^{p\operatorname{Re} h(0)+r}}\left[(r-1)lt + x^{1-r}\right]^{\frac{p\operatorname{Re} h(0)+r}{1-r}},$$

and therefore the series in (1.13) converges if and only if

$$\frac{p\operatorname{Re} h(0) + r}{1-r} < -1,$$

that is $\operatorname{Re} h(0) > \frac{-1}{p}$.

We just need to prove that part c). For this, observe that $\chi_{\varphi(t,I)}(x) = 0$ for every $x \in (0,1]$ for sufficiently large $t > 0$ and $[0,1] \setminus \Omega_0$ has only one connected component. As

$$\forall\, t \geq 0, x \in (0,1]: \qquad \partial_2\varphi(t,x) = x^{-r}((r-1)t + x^{1-r})^{\frac{r}{1-r}}$$

we obtain

$$\begin{aligned} \forall\, t \geq 0, x \in (0,1]: \qquad & \exp(-ph(0)t)\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p} \\ = \quad & \exp(-ph(0)t)t^{-\frac{r}{r-1}} \frac{(2(r-1) + \frac{1}{x^{r-1}t})^{\frac{pr}{r-1}}}{x^r((r-1) + \frac{1}{x^{r-1}t})^{\frac{(p+1)r}{r-1}}}. \end{aligned}$$

Now the proof follows again as the proof of part c) of Theorem 1.4.1. In fact, that $\mathcal{S}_{r,h}$ is not hypercyclic on $W^{1,p}(0,1)$ follows immediately from Theorem 1.3.11, the rest of c) now follows from Theorem 1.3.11 b),c), and d).

□

The interest in this generalization of the von Foerster-Lasota equation was originated in the result [36, Theorem 3.1] characterizing and comparing our results in this particular case. Moreover, the parameter $r > 1$ gives another point of view of this equation. In fact, the case $r = 1$ has properties like the equivalence between hypercyclicity and chaos, it is not the case for $r > 1$ on L^p if, for example,

$$\frac{-r}{p} < \operatorname{Re} h(0) \leq \frac{-1}{p}.$$

Open Problem 1.4.5 Recently Kalmes in [47, 48] studied the Frequent hypercyclic criterion (see [51]) for the C_0-semigroups considered in this chapter. Actually, Kalmes shows under mild assumptions on F and h an equivalence between satisfying the Frequent Hypercyclic Criterion and Devaney chaos on Lebesgue and Sobolev spaces. A natural progression of our studies would be to investigate whether Kalmes conditions can be furtherly relaxed.

Chapter 2

Stable semiflow semigroups in Lebesgue spaces

The aim of this chapter is to provide characterizations of stability for strongly continuous semigroups induced by semiflows on $L^p_\rho(\Omega)$, on $W^{1,p}_*(I)$ and on $W^{1,p}(I)$ for a bounded interval $I := (a, b) \subset \mathbb{R}$.

This chapter was motivated by a question of Thomas Kalmes, namely whether, for this kind of C_0-semigroups, stability with not hypercyclicity, are equivalent. In particular cases such as Lasota equation this holds but, as we will see, it is not true in general.

Many authors studied these kind of semigroups in this sense, for example, we can find results about stability applied on a different contexts given by Dawidowicz et al. in [27, 36, 37].

First, we will present these known results in order to compare them with ours. Even though we start with the multidimensional case, we will concentrate on the one-dimensional case and, in particular, we give characterizations for $\rho(x) = 1$ on $W^{1,p}_*(I)$ and on $W^{1,p}(I)$. To finish we will apply our results to the same examples of the previous chapter.

Without loss of generality, throughout this chapter, we consider $h \in C(\Omega)$ real valued. We will give an argument to assume h complex valued based on the previous case. We will write stable C_0-semigroup or simply stable if we consider strongly stability property of the Definition 0.3.5 and we will specify in the case of exponential stability property.

The contents of this chapter have been included in [4].

2.1 State of the art

We go back to the setting that has been described in section 0.1.

Keeping the same notation for the semigroups and the involved space, we recall the following two results about stability of von Foerster-Lasota semigroups and of its multidimensional generalization:

Theorem 2.1.1 ([36]) *Let $h : [0,1] \to \mathbb{R}$ be a continuous function and let $\mathcal{T}$ be the semigroup defined in* (1.2)*.*

(i) If exist $C, q > 0$ such that $|h(x) - h(0)| \leq Cx^q$ for all $x \in [0,1]$ and $h(0) \leq -\frac{1}{p}$, $\mathcal{T}$ is strongly stable on $L^p(0,1)$.

(ii) If exist $C > 0$ such that $|h(x) - h(0)| \leq Cx$ for all $x \in [0,1]$ and $h(0) \leq 1 - \frac{1}{p}$, $\mathcal{T}$ is strongly stable on V_α.

Moreover, we obtain the property exponentially stable for $\mathcal{T}$ if $h(0) < -\frac{1}{p}$ and $h(0) < 1 - \frac{1}{p}$, respectively.

For an argument of complexification we can extend these results for $h : [0,1] \to \mathbb{C}$ continuous function. All these results will be consequence of our results.

We finish with the equation (1.3) on the multidimensional case on certain compact set D of $\mathbb{R}^d$.

Theorem 2.1.2 ([37]) *Let $\mathcal{T}$ be the C_0-semigroup solution of the equation* (1.3) *on $L^p(D)$. If $\gamma \leq -\frac{1}{p} \limsup_{x \to 0} (\nabla \cdot c)(x)$ then $\mathcal{T}$ is strongly stable on $L^p(D)$.*

Moreover, for strict inequality we obtain the property exponentially stable for $\mathcal{T}$.

Remark 2.1.3 Comparing these results with the previous ones given in section 1.1, we obtain stability for the equation (1.1) if and only if it is not chaotic, in particular not hypercyclic. These properties have complementary behaviour for this equation.

It is not the case for equation (1.3), unless the limit superior and limit inferior coincide, i.e., if and only if $\lim_{x \to 0} (\nabla \cdot c)(x)$ exists.

2.2 Multidimensional case on Lebesgue spaces

In this section we will consider the multidimensional setting that we mentioned in Remark 1.2.4. We start by recalling some basic properties. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\varphi : [0, \infty[\times\Omega \to \Omega$ be a semiflow according to Definition 0.4.1. Moreover let $h : \Omega \to \mathbb{R}$ be a continuous function. Define for every $t \geq 0$:

$$h_t : \Omega \to (0, +\infty), \quad x \to \exp\left(\int_0^t h(\varphi(s,x))ds\right).$$

Let $\rho : \Omega \to (0, +\infty)$ be a λ^d-measurable locally integrable function and let $1 \leq p < \infty$. Define

$$(T(t)f)(x) := h_t(x)f(\varphi(t,x)), \qquad f \in L^p_\rho(\Omega),\ x \in \Omega,\ t \geq 0 \tag{2.1}$$

If φ is continuously differentiable, in [45, Proposition 4.12, Theorem 4.7], it has been proved that $\mathcal{T} = (T(t))_{t\geq 0}$ is a C_0-semigroup in $L^p_\rho(\Omega)$ if and only if there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $t \geq 0$ the following inequality holds

$$|h_t|^p\rho \leq Me^{\omega t}\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|,\ \lambda\text{-a.e. on } \Omega,$$

where λ denotes Lebesgue measure and $D\varphi(t,\cdot)$ is the Jacobian of $\varphi(t,\cdot)$, or equivalently

$$\chi_{\varphi(t,\Omega)}\frac{h_t^p(\varphi(-t,\cdot))\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|}{\rho} \leq Me^{t\omega} \qquad \lambda - \text{a.e.},$$

where $\varphi(-t,\cdot)$ is the inverse function of $\varphi(t,\cdot)$ and $D\varphi(-t,\cdot)$ is its Jacobian. In this case ρ is said to be a p-admissible weight and for every $t \geq 0$:

$$||T(t)|| = \left\|\frac{\rho_{t,p}}{\rho}\right\|_\infty, \tag{2.2}$$

where

$$\rho_{t,p} = \chi_{\varphi(t,\Omega)}|h_t(\varphi(-t,\cdot))|^p\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|.$$

Our aim is to characterize the stability of the C_0-semigroup $\mathcal{T}$ on $L^p_\rho(\Omega)$, namely when

$$\forall f \in L^p_\rho(\Omega) \qquad \lim_{t\to+\infty} \|T_t f\|_p = 0.$$

Theorem 2.2.1 *Let Ω be an open subset of $\mathbb{R}^d$, φ a continuously differentiable semiflow on Ω, $h \in C(\Omega)$, ρ a p-admissible function, and let $\mathcal{T}$ be the semigroup on $L^p_\rho(\Omega)$ defined by (2.1). Then the following conditions are equivalent:*

(i) $\mathcal{T}$ is stable on $L^p_\rho(\Omega)$,

(ii) It holds:

(1) $\mathcal{T}$ is bounded.

(2) for every bounded interval $Q \subseteq \Omega$, it holds

$$\lim_{t\to+\infty} \int_Q \rho_{t,p}(x)dx = 0.$$

Proof. Observe that, by (2.2), $\mathcal{T}$ is bounded if and only if $\sup_{t\geq 0} \left\| \frac{\rho_{t,p}}{\rho} \right\|_\infty < +\infty$.

Then $\mathcal{T}$ is stable on $L^p_\rho(\Omega)$ if and only if

$$\lim_{t\to+\infty} \int_\Omega |h_t(x) f(\varphi(t,x))|^p \rho(x)dx = 0, \qquad \forall f \in L^p_\rho(\Omega),$$

or, equivalently, with a change of variable (see also the argument in the proof of [45, Proposition 3.11]):

$$\lim_{t\to+\infty} \int_\Omega |f(y)|^p \rho_{t,p}(y)dy = 0 \qquad \forall f \in L^p_\rho(\Omega)$$

that is

$$\lim_{t\to+\infty} \int_\Omega |g(y)| \rho_{t,p}(y)dy = 0. \qquad \forall g \in L^1_\rho(\Omega).$$

By applying the previous to $g^+ = g \vee 0$ and $g^- = g \wedge 0$, we get that it is equivalent to

$$\lim_{t\to+\infty} \int_\Omega g(x) \frac{\rho_{t,p}(x)}{\rho(x)} \rho(x)dx = 0. \qquad \forall g \in L^1_\rho(\Omega).$$

namely, $\frac{\rho_{t,p}(x)}{\rho(x)} \to 0$ with respect to the w^*-topology on $L^\infty(\Omega)$ induced by $L^1_\rho(\Omega)$. Since the function $t \mapsto \rho_{t,p} \in L^\infty(\Omega)$ is locally bounded, the last assertion is equivalent to (ii) by Lemma 0.5.2. □

We will use this result to give simplified versions of condition (ii) and obtain characterizations for one-dimensional case and Sobolev spaces with $\rho = 1$.

We have considered until now the real-valued case. To finish this section, we consider an argument to assume h complex valued based on real valued case.

Remark 2.2.2 If we consider $L^p_\rho(\Omega, \mathbb{C})$, $h \in C(\Omega, \mathbb{C})$ and $\varphi(t, x)$ a semiflow satisfying (H2)-(H3), then we define for every $f \in L^p_\rho(\Omega, \mathbb{C})$ and every $t \geq 0$

$$T_h^{\mathbb{C}}(t)(f)(x) = \exp\left(\int_0^t h(\varphi(s,x))\right) f(\varphi(x,t)).$$

If ρ is admissible, then $\mathcal{T}_h^{\mathbb{C}} = (T_h^{\mathbb{C}}(t))_{t\geq 0}$ is a strongly continuous semigroup. We consider, for every $f \in L^p_\rho(\Omega, \mathbb{R})$ and $t \geq 0$

$$T_{\mathrm{Re}h}^{\mathbb{R}}(t)(f)(x) = \exp\left(\int_0^t \mathrm{Re}h(\varphi(s,x))\right) f(\varphi(x,t)).$$

It holds that $\mathcal{T}_h^{\mathbb{C}}$ is stable if and only if $\mathcal{T}_{\mathrm{Re}h}^{\mathbb{R}}$ is stable.

Indeed, if $f \in L^p_\rho(\Omega, \mathbb{R})$, then

$$|T_h^{\mathbb{C}}(t)(f)(x)| = |T_{\mathrm{Re}h}^{\mathbb{R}}(t)(f)(x)|$$

and we get immediately that if $\mathcal{T}_h^{\mathbb{C}}$ is stable then $\mathcal{T}_{\mathrm{Re}h}^{\mathbb{R}}$ is stable. Conversely, if $f \in L^p_\rho(\Omega, \mathbb{C})$, then $|f| \in L^p_\rho(\Omega, \mathbb{R})$ and

$$|T_h^{\mathbb{C}}(t)(f)(x)| = |T_{\mathrm{Re}h}^{\mathbb{R}}(t)(|f|)(x)|$$

and again we get that if $\mathcal{T}_{\mathrm{Re}h}^{\mathbb{R}}$ is stable, then $\mathcal{T}_h^{\mathbb{C}}$ is stable too.

2.3 One-dimensional case on Lebesgue spaces

We now consider the case $\Omega \subseteq \mathbb{R}$, with more details, referring to the assumptions (H1)-(H3) in section 0.1 and the notation in section 1.2.

A first consequence of Theorem 2.2.1 is a characterization of stability for the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p_\rho(\Omega)$ for real valued h.

Theorem 2.3.1 *Let $\Omega \subseteq \mathbb{R}$ be open, F satisfying (H1)-(H3) and let ρ be p-admissible measurable function for F and h. Then the following conditions are equivalent:*

i) $\mathcal{T}_{F,h}$ *is stable on* $L^p_\rho(\Omega)$*,*

ii) It holds:

(1) $\mathcal{T}_{F,h}$ *is bounded;*

(2) $\lim_{t\to+\infty} \rho_{t,p}(x) = 0$ *for* λ*-a.e.* $x \in \Omega_1$

(3) if $\lambda(\Omega_0) > 0$*,* $h(x) < 0$ λ*-a.e. in* Ω_0*.*

Proof. As usual, let us define the following subsets of Ω:

$$\Omega_0 := \{x \in \Omega \mid F(x) = 0\}, \qquad \Omega_1 = \Omega \setminus \Omega_0.$$

Moreover we define

$$X_i = \{f \in L^p_\rho(\Omega) \mid f = 0 \text{ a.e. in } \Omega_i\}, \qquad i = 0, 1.$$

Clearly we can identify X_i with $L^p_\rho(\Omega_i)$ and

$$L^p_\rho(\Omega) = X_0 \oplus X_1 = L^p_\rho(\Omega_0) \oplus L^p_\rho(\Omega_1).$$

If $\lambda(\Omega_0) = 0$, then X_0 reduces to $\{0\}$ and $L^p_\rho(\Omega)$ can be identified with $L^p_\rho(\Omega_1)$.

For every $x_0 \in \Omega_0$ we have that $\varphi(t, x_0) = x_0$ for every $t \geq 0$. By the uniqueness of the solutions of the initial value problems

$$\dot{x} = F(x),\ x(0) = x_0\ (x_0 \in \Omega),$$

$\varphi(t, \Omega_0) \subseteq \Omega_0$ and $\varphi(t, \Omega_1) \subseteq \Omega_1$. This implies that $L^p_\rho(\Omega_i)$ is invariant under $\mathcal{T}_{F,h}$ for $i = 0, 1$. Thus we can define $\mathcal{T}^i_{F,h} = \mathcal{T}_{F,h|L^p_\rho(\Omega_i)}$ and we can write

$$\mathcal{T}_{F,h} = \mathcal{T}^0_{F,h} \oplus \mathcal{T}^1_{F,h}.$$

Clearly $\mathcal{T}_{F,h}$ is stable on $L^p_\rho(\Omega)$ if and only if $\mathcal{T}^i_{F,h}$, $i = 0, 1$, are stable on $L^p_\rho(\Omega_i)$.

Observe moreover that $T^0_{F,h}(t)(f)(x) = \exp(th(x))f(x)$ for every $f \in L^p_\rho(\Omega_0)$ and $x \in \Omega_0$.

"$\Rightarrow$": Let $x \in \Omega_1$ such that $F(x) \neq 0$. If $F(x) > 0$ there exists $r > 0$ such that $[x, x + r] \subseteq \Omega_1$ with $F(s) > 0$ for $s \in [x, x + r]$. By Lemma 1.2.5, there exists $C > 0$ such that

$$\rho_{t,p}(x) \leq C\rho_{t,p}(s) \qquad \text{a.e. } s \in [x, x + r],$$

hence

$$\rho_{t,p}(x) = \frac{1}{r}\int_x^{x+r} \rho_{t,p}(x)ds \leq \frac{C}{r}\int_x^{x+r} \rho_{t,p}(s)ds,$$

and therefore, by assumption and Theorem 2.2.1, $\lim_{t\to\infty} \rho_{t,p}(x) = 0$. If $F(x) < 0$, we consider an interval $[x-r, x]$ with $F(s) < 0$ for $s \in [x-r, x]$ and we get the assertion again by Lemma 1.2.5 arguing as in the case $F(x) > 0$.

By the stability of $\mathcal{T}^0_{F,h}$, we get that

$$\lim_{t\to\infty} \int_{\Omega_0} e^{pth(x)}|f(x)|^p \rho(x)dx = 0$$

hence either $\lambda(\Omega_0) = 0$ or if $\lambda(\Omega_0) > 0$ we have $h(x) < 0$ λ -a.e. in Ω_0.

By the previous argument, in the case that $\lambda(\Omega_0) = 0$ the condition (3) is not necessary and the only restrictions over h are given by condition (1), in particular for $x \in \Omega_0$ it is equivalent to $h(x) \leq \frac{1}{p}F'(x)$.

$'' \Leftarrow''$: By (1) and (3), there exists $M > 0$ such that $\rho_{t,p}(s) \leq M\rho(s)$ a.e. in Ω. Then, for every $f \in L^p_\rho(\Omega_1)$, we have that being ρ locally integrable on Ω, we can apply the dominated convergence theorem to get that for any $f \in L^p_\rho(\Omega_1)$:

$$\lim_{t\to\infty} \int_{\Omega_1} |T^1_{F,h}(t)(f)(x)|^p \rho(x)dx = \lim_{t\to\infty} \int_{\Omega_1} |f(s)|^p \rho_{t,p}(s)ds = 0,$$

by using the change of variable $x = \varphi(-t, s)$, and the invariance of Ω_1 under $\varphi(-t, \cdot)$, that is $T^1_{F,h}$ is stable on $L^p_\rho(\Omega_1)$.

On the other hand, if $\lambda(\Omega_0) > 0$ and being $h < 0$ a.e. in Ω_0, we get, again by the dominated convergence theorem, that $T^0_{F,h}$ is stable, too. □

Example 2.3.2 (Left translation semigroup) Let $\Omega = \mathbb{R}, F = 1$, and $h = 0$ so that $\varphi(t, x) = x + t$ where $\varphi(t, \mathbb{R}) = \mathbb{R}$ and $h_t(x) = 1$. As we commented, ρ is p-admissible for F and h for some $p \in [1, \infty)$ if the same holds for every $p \in [1, \infty)$. Consider then, the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p_\rho(\mathbb{R})$ defined by $(T_{F,h}(t)f)(x) = f(x+t)$ and remember that its generator is an extension of

$$C^1_c(\mathbb{R}) \to L^p_\rho(\mathbb{R}), f \mapsto f'.$$

Moreover, we have $\Omega_0 = \emptyset$ and $\rho_{t,p}(x) = \rho(x - t)$.

By Theorem 2.3.1, $\mathcal{T}_{F,h}$ is stable on $L^p_\rho(\mathbb{R})$ if and only if for every $x \in \mathbb{R}$ exists a real finite constant $C \geq 0$ such that

$$\rho(x-t) \leq C\rho(x), \ \forall t \text{ and } \lim_{t\to\infty} \rho(x-t) = 0.$$

Or equivalently, for every $x \in \mathbb{R}$ exists a real finite constant $C \geq 0$ such that

$$\rho(x-t) \leq C\rho(x), \ \forall t \text{ and } \lim_{x\to-\infty} \rho(x) = 0.$$

This condition is independent of p.

From this example follows that if we consider an increasing p-admissible weight with the property $\lim_{t\to-\infty}\rho(t) \neq 0$, then the left translation semigroup is not stable and not hypercyclic.

Example 2.3.3 Let again $\Omega = \mathbb{R}$, $F(x) := 1-x, h(x) = 0$ so that $\varphi(t,x) = 1+(x-1)e^{-t}$, $h_t(x) = 1$, where $\varphi(t,\mathbb{R}) = \mathbb{R}$ and $\partial_2\varphi(t,x) = e^{-t}$. Recall that in this case $\mathcal{T}_{F,h}$ is given by $(T_{F,h}(t)f)(x) = f(1+(x-1)e^{-t})$ with generator being an extension of

$$C^1_c(\mathbb{R}) \to L^p_\rho(\mathbb{R}), f \mapsto \big(x \mapsto (1-x)f'(x)\big).$$

Furthermore, $\mathbb{R}\setminus\Omega_0 = (-\infty,1)\cup(1,\infty)$ with $\lambda(\Omega_0) = 0$ and

$$\forall\, t \in \mathbb{R}: \ \rho_{t,p}(x) = \rho(1+(x-1)e^t)e^t,$$

Arguing as in the previous example with the p-admissibility condition, Theorem 2.3.1 gives stable behaviour of $\mathcal{T}_{F,h}$ on $L^p_\rho(\mathbb{R})$ if and only if for every $x \in \mathbb{R}\setminus\{-1\}$ there exists a real finite constant $C \geq 0$ such that

$$\rho(1+(x-1)e^t)e^t \leq C\rho(x), \ \forall t \text{ and } \lim_{t\to\infty} \rho(1+(x-1)e^t)e^t = 0.$$

Or equivalently, there exist a real finite constant $C \geq 0$ such that

$$\rho(1+(x-1)e^t)e^t \leq C\rho(x), \ \forall t \geq 0, \ \forall x \in \mathbb{R}\setminus\{-1\} \text{ and } \lim_{|r|\to\infty} \rho(r)r = 0.$$

Now, if we distinguish between $\varphi(t,\Omega) = \Omega$ and $\varphi(t,\Omega) \subsetneq \Omega$ then we can avoid one of the above conditions.

Corollary 2.3.4 *Let $\Omega \subseteq \mathbb{R}$ be open, F satisfying (H1)-(H3) and let ρ be p-admissible measurable function for F and h. Assume that*

$$\forall x \in \Omega_1, \ \exists\, \bar{t} > 0 \ : \qquad x \notin \varphi(\bar{t},\Omega).$$

where φ is the semiflow associated with F. Then the following conditions are equivalent:

(i) $T_{F,h}$ *is stable on* $L^p_\rho(\Omega)$*,*

(ii) It holds:

(1) $T_{F,h}$ *is bounded,*

(2) if $\lambda(\Omega_0) > 0$*,* $h(x) < 0$ λ*-a.e. in* Ω_0*.*

Proof. Simply observe that the assumption implies that $x \notin \varphi(t, \Omega)$ for every $t > \bar{t}$ and for every $x \in \Omega_1$, and therefore $\lim_{t\to\infty} \rho_{t,p}(x) = 0$. □

Remark 2.3.5 A concrete example in which the assumption of the previous corollary are satisfied is the case in which $\Omega = (0, 1)$ and $F(x) = -x$, corresponding to the Lasota semigroup. Then $\varphi(t, x) = e^{-t}x$ and, for every $t > 0$,

$$\varphi(t, (0, 1)) = (0, e^{-t}).$$

Therefore, for every $x \in (0, 1)$, choosing $\bar{t} > -\log x$, we get that $x \notin \varphi(\bar{t}, \Omega)$.

In order to use this result on section 2.4, we assume $\rho = 1$ to simplify the conditions.

By applying Corollary 1.2.6, a straightforward calculation gives the following characterization.

Theorem 2.3.6 *Let* $\Omega \subseteq \mathbb{R}$ *be open,* F *satisfying (H1)-(H3) and assume that* $\rho = 1$ *is a* p*-admissible function for* F *and* h*. Assume that* $F(x) \neq 0$ *for every* $x \in \Omega$*.*

(1) If $\varphi(t, \Omega) = \Omega$ *for every* $t > 0$*, then t.f.a.e.:*

(i) $\mathcal{T}_{F,h}$ *is stable on* $L^p(\Omega)$*;*

(ii) It holds,

(a) there exists $C \in \mathbb{R}$ *such that*

$$\int_y^{\varphi(t,y)} \frac{h(s) - \frac{1}{p}F'(s)}{F(s)} ds \leq C \quad a.e.\ y \in \Omega,\ t \geq 0,$$

or, equivalently, $\mathcal{T}_{F,h}$ *is bounded;*

(b) for every $y \in \Omega$

$$\lim_{t\to+\infty}\int_y^{\varphi(t,y)} \frac{h(s)-\frac{1}{p}F'(s)}{F(s)}ds = -\infty.$$

(2) If

$$\forall x \in \Omega_1 \ \exists\, \bar{t} > 0 \qquad x \notin \varphi(\bar{t},\Omega).$$

then t.f.a.e.:

(i) $\mathcal{T}_{F,h}$ is stable on $L^p(\Omega)$;

(ii) there exists $C \in \mathbb{R}$ such that

$$\int_y^{\varphi(t,y)} \frac{h(s)-\frac{1}{p}F'(s)}{F(s)}ds \le C \quad a.e.\ y \in \Omega,\ t \ge 0,$$

or, equivalently, $\mathcal{T}_{F,h}$ is bounded.

We would like now to compare the previous stability conditions with hypercyclicity or mixing condition. Clearly if a semigroup is hypercyclic, then it cannot be stable and, in general, a semigroup can be not stable and not hypercyclic. Indeed, if we consider a semigroup $T_{F,h}$ such that $\lambda(\Omega_0) > 0$ and $h(x) > 0$ on Ω_0 then we get a semigroup which is not stable by Theorem 2.3.1 and not hypercyclic by Theorem 1.2.10.

Nevertheless, there are cases in which stability and not hypercyclicity are equivalent. We analyze with details the case $\rho = 1$ and $F < 0$.

Theorem 2.3.7 *Let $\Omega = (\alpha,\beta) \subseteq \mathbb{R}$ be a bounded interval, $F \in C^1([\alpha,\beta])$ satisfying (H2)-(H3). Assume F decreasing, $F(x) < 0$ for each $x \in (\alpha,\beta]$, $F(\alpha) = 0$ and such that*

$$\forall x \in \Omega_1,\ \exists\, \bar{t} > 0\ : \qquad x \notin \varphi(\bar{t},\Omega).$$

Moreover, let $h = -\lambda F'$ for some $\lambda \in \mathbb{R}$.

Then, for the C_0-semigroup $\mathcal{T}_{F,h}$ on $L^p(\Omega)$ the following are equivalent.

(i) $\mathcal{T}_{F,h}$ is stable.

(ii) $\mathcal{T}_{F,h}$ is not hypercyclic.

(iii) $\lambda \leq -\frac{1}{p}$

Proof. Observe that $\rho = 1$ is p-admissible for F and h by Lemma 1.3.4. If $\lambda = -\frac{1}{p}$, then clearly the semigroup is stable and not hypercyclic. Then assume that $\lambda \neq -\frac{1}{p}$.

First observe that for every $y, z \in (\alpha, \beta)$

$$\int_y^z \frac{h(s) - \frac{1}{p}F'(s)}{F(s)} ds = -\left(\lambda + \frac{1}{p}\right) \log \frac{F(z)}{F(y)}.$$

In particular, it follows that for every $z \in (\alpha, \beta)$

$$\exists \lim_{y \to \alpha} \int_y^z \frac{h(s) - \frac{1}{p}F'(s)}{F(s)} ds = -\left(\lambda + \frac{1}{p}\right)(+\infty)$$

thus $\mathcal{T}_{F,h}$ is hypercyclic if and only if $\lambda > -\frac{1}{p}$.

On the other hand, $\mathcal{T}_{F,h}$ is stable if and only if there exists $C > 0$ such that

$$\forall x \in (\alpha, \beta),\ \forall t > 0 : \quad \int_x^{\varphi(t,x)} \frac{h(s) - \frac{1}{p}F'(s)}{F(s)} ds \leq C,$$

that is

$$\forall x \in (\alpha, \beta),\ \forall t > 0 : \quad -\left(\lambda + \frac{1}{p}\right) \log \frac{F(\varphi(t,x))}{F(x)} \leq C.$$

Being $F < 0$ in (α, β), we have that $\varphi(\cdot, x)$ is strictly decreasing, hence $\varphi(t, x) < x$ for every $t > 0$. Hence, since F is decreasing,

$$\forall x \in (\alpha, \beta),\ t > 0 : \quad \frac{F(\varphi(t,x))}{F(x)} \leq 1.$$

Thus, if $\lambda < -\frac{1}{p}$, then

$$\forall x \in (\alpha, \beta),\ \forall t > 0 : \quad -\left(\lambda + \frac{1}{p}\right) \log \frac{F(\varphi(t,x))}{F(x)} \leq 0,$$

and so $\mathcal{T}_{F,h}$ is stable. □

Remark 2.3.8 Of course it is possible to characterize stability for $\mathcal{T}_{F,h}$ in case of F being strictly positive, taking into account that φ would be increasing. In this case, we only need to change condition (iii) by $\lambda \geq -\frac{1}{p}$.

2.4 Stability on Sobolev spaces

From now on, we are able to use our results in order to show similar characterization of stability on Sobolev spaces.

Let $I = (a, b)$ be a bounded interval in $\mathbb{R}$ and, as we saw in section 0.1, let $W^{1,p}(I)$ the first order Sobolev space of p-integrable functions on I, where $1 \leq p < \infty$.

We will use several results following the argumentation part of Sobolev spaces on Chapter 1. More precisely, we need to use Lemma 1.3.2 and Proposition 1.3.3 to transfer the dynamical properties for the standard described C_0-semigroup from $L^p(I)$ on $W^{1,p}_*(I)$. Finally we use Propositions 1.3.8 and 1.3.9 to see that actually we need to prove stability only for the C_0-semigroup $\mathcal{S}_{F,h(a)}$.

As commented, we only need to prove stability behaviour of the C_0-semigroup $\mathcal{S}_{F,h(a)}$ and assume the previous results to generalize this ones for $\mathcal{S}_{F,h}$ by using the comparison test.

Theorem 2.4.1 *Let $I = (a, b)$ be a bounded interval, $1 \leq p < \infty$, F with $F(a) = 0$ and $F(x) \neq 0$ in $]a, b[$ and satisfying (H2), (H3), $h \in W^{1,\infty}(I)$. Assume that the function $[a, b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^\infty(I)$. The following hold*

(1) If $\varphi(t, I) = I$ for every $t > 0$, then t.f.a.e.:

(i) $\mathcal{S}_{F,h}$ is stable on $W^{1,p}_*(I)$;

(ii) It hold,

(a) there exists $C \in \mathbb{R}$ such that

$$\int_y^{\varphi(t,y)} \frac{h(a) - \left(\frac{1}{p} - 1\right) F'(s)}{F(s)} ds \leq C \quad a.e.\ y \in I,\ t \geq 0;$$

(b) for every $y \in I$

$$\lim_{t\to+\infty} \int_y^{\varphi(t,y)} \frac{h(a) - \left(\frac{1}{p} - 1\right) F'(s)}{F(s)} ds = -\infty.$$

(2) If

$$\forall x \in I\ \exists\ \bar{t} > 0 \qquad x \notin \varphi(\bar{t}, I)$$

then t.f.a.e.:

(i) $\mathcal{S}_{F,h}$ *is stable on* $W^{1,p}_*(I)$*;*

(ii) there exists $C \in \mathbb{R}$ *such that*

$$\int_y^{\varphi(t,y)} \frac{h(a) - \left(\frac{1}{p} - 1\right) F'(s)}{F(s)} ds \le C \quad a.e.\ y \in I,\ t \ge 0.$$

Moreover, $\mathcal{S}_{F,h}$ *is stable on* $W^{1,p}(I)$ *if and only if it is stable on* $W^{1,p}_*(I)$ *and* $h(a) < 0$*.*

Proof. As we have previously discussed and by Proposition 1.3.9 we know that the C_0-semigroups $\mathcal{S}_{F,h(a)}$ and $\mathcal{S}_{F,h}$ are conjugate.

Moreover the C_0-semigroup $\mathcal{S}_{F,h(a)}$ on $W^{1,p}_*(I)$ is conjugate to $\mathcal{T}_{F,F'+h(a)}$ on $L^p(I)$ by Proposition 1.3.3.

Thus, $\mathcal{S}_{F,h}$ is stable on $W^{1,p}_*(I)$ if and only if $\mathcal{T}_{F,F'+h(a)}$ is stable on $L^p(I)$.

A short calculation shows that $\mathcal{T}_{F,F'+h(a)}$ is stable on $L^p(I)$ if and only if the condition (ii) holds. In fact, for $\mathcal{T}_{F,F'+h(a)}$ observe that $h_t(x) = e^{th(a)}\partial_2\varphi(t,x)$ and $\rho(x) = 1$ is p-admissible for F and $F' + h(a)$ with $M = 1$ and $w = ph(a) + (p-1)\|F'\|_\infty$. Depending on the behavior of $\varphi(t,\Omega)$ for $t \ge 0$, the proof on $W^{1,p}_*(I)$ is over if we replace h by $F' + h(a)$ in the conditions of Theorem 2.3.6.

Observe that $W^{1,p}(I) = W^{1,p}_*(I) \oplus \operatorname{span}\{\mathbb{1}\}$ and that

$$\mathcal{S}_{F,h(a)} = \mathcal{S}_{F,h(a)_{|W^{1,p}_*(I)}} \oplus \mathcal{S}_{F,h(a)_{|\operatorname{span}\{\mathbb{1}\}}}$$

Thus $\mathcal{S}_{F,h(a)}$ is stable if and only if $\mathcal{S}_{F,h(a)}$ is stable on $W^{1,p}_*(I)$ and on $\operatorname{span}\{\mathbb{1}\}$, i.e., $h(a) < 0$. In fact, for $\lambda \neq 0$ we have

$$\mathcal{S}_{F,h(a)}(\lambda\mathbb{1}) = e^{h(a)t}\lambda \overset{t\to\infty}{\to} 0 \text{ if and only if } h(a) < 0.$$

□

2.5 The Lasota equation

As in the previous chapter we analyze the Lasota equation showing that our results cover the results described in section 2.1.

2.5.1 von Foerster-Lasota equation

Consider von Foerster-Lasota equation with $I = (0,1)$, $F(x) = -x$ and $h \in C(0,1) \cap L^\infty(0,1)$, described by

$$\frac{\partial}{\partial t} u(t,x) + x \frac{\partial}{\partial x} u(t,x) = h(x)\, u(t,x), \quad t \geq 0,\, 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x), \quad 0 < x < 1,$$

where v is a given function. Denote by $\mathcal{T}_h$ the associated C_0-semigroup on $L^p(0,1)$. For Sobolev spaces we consider $h \in W^{1,\infty}(0,1)$ and denote by $\mathcal{S}_h$ the resulting C_0-semigroup on $W^{1,p}(0,1)$, respectively $W^{1,p}_*(0,1)$.

Theorem 2.5.1

a) Assume that for $h \in C[0,1]$ real valued the function

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^1(0,1)$. Then the following properties of the associated von Foerster-Lasota semigroup $\mathcal{T}_h$ on $L^p(0,1)$ are equivalent.

i) $\mathcal{T}_h$ is stable on $L^p(0,1)$.

ii) $h(0) \leq -\frac{1}{p}$.

b) Assume that for $h \in W^{1,\infty}(0,1)$ real valued the function

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^\infty(0,1)$. Then the von Foerster-Lasota semigroup $\mathcal{S}_h$ is not stable on $W^{1,p}(0,1)$ and for the restriction of $\mathcal{S}_h$ to $W^{1,p}_*(0,1)$ the following are equivalent.

i) $\mathcal{S}_h$ *is stable on* $W_*^{1,p}(0,1)$.

ii) $h(0) \leq 1 - \frac{1}{p}$.

Proof. For $F(x) = -x$ we have $\varphi(t,x) = xe^{-t}$ thus, following Remark 2.3.5, for every $x \in (0,1)$ we get for $t > -\log x$ that $x \notin \varphi(t,\Omega)$. We will use an alternative proof without using the comparison test between $\mathcal{T}_h$ and $\mathcal{T}_{h(0)}$.

Proof of part a). We can consider for λ- a.e. $x \in I$ and for all $t \geq 0$,

$$\int_x^{xe^{-t}} \frac{h(y) + \frac{1}{p}}{-y} dy = \int_{xe^{-t}}^x \frac{h(y) - h(0) + h(0) + \frac{1}{p}}{y} dy.$$

Since

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^1(0,1)$, we obtain for some constant $K \geq 0$ that

$$\int_{xe^{-t}}^x \frac{h(y) - h(0) + h(0) + \frac{1}{p}}{y} dy \leq K + \int_{xe^{-t}}^x \frac{h(0) + \frac{1}{p}}{y} dy.$$

By Theorem 2.3.6-(2), $\mathcal{T}_h$ is stable if and only if there exists $C \in \mathbb{R}$ such that for λ- a.e. $x \in I$ and for all $t \geq 0$

$$\int_{xe^{-t}}^x \frac{h(0) + \frac{1}{p}}{y} dy \leq C.$$

Observe that

$$\int_{xe^{-t}}^x \frac{h(0) + \frac{1}{p}}{y} dy = \left(h(0) + \frac{1}{p} \right) t.$$

Then $\mathcal{T}_h$ is stable if and only if $h(0) \leq -\frac{1}{p}$.

Proof of part b). It is a direct consequence of Theorem 2.3.7 if we replace h by $F' + h(0)$, if we use Theorem 2.4.1-(2) and the fact that $\mathcal{S}_{F,h}$ is stable on $W_*^{1,p}(I)$ if and only if $\mathcal{T}_{F,F'+h(a)}$ is stable on $L^p(I)$. Arguing as the beginning of case a) we obtain stability for $\mathcal{S}_h$ if and only if

$$h(0) \leq -\frac{1-p}{p} = 1 - \frac{1}{p}.$$

□

Remark 2.5.2 The previous result remains valid for h complex valued if we use Remark 2.2.2. In fact, for $L^p(0,1)$ we only have to replace $h(0)$ by $\operatorname{Re} h(0)$ and for the Sobolev spaces we only need to assume $h(0) \in \mathbb{R}$.

2.5.2 Generalized von Foerster-Lasota equation

Let us consider $I = (0,1)$, $h \in C(0,1) \cap L^\infty(0,1)$ and $r > 1$ for the first order partial differential equation

$$\frac{\partial}{\partial t} u(t,x) + x^r \frac{\partial}{\partial x} u(t,x) = h(x)\, u(t,x), \quad t \geq 0,\ 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x), \quad 0 < x < 1,$$

where v is a given function. As for $F(x) = -x^r$ recall that

$$\varphi(t,x) = ((r-1)t + x^{1-r})^{\frac{1}{1-r}}$$

Denote by $\mathcal{T}_{r,h}$ the C_0-semigroup on $L^p(0,1)$. For Sobolev spaces we consider $h \in W^{1,\infty}(0,1)$ and let $\mathcal{S}_{r,h}$ be the C_0-semigroup on $W^{1,p}(0,1)$ and on $W^{1,p}_*(0,1)$.

Theorem 2.5.3

a) Assume that for $h \in C[0,1]$ real valued the function

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - x^{r-1} h(0)}{x^r}$$

belongs to $L^1(0,1)$. Then the following properties of $\mathcal{T}_{r,h}$ on $L^p(0,1)$ are equivalent.

i) $\mathcal{T}_{r,h}$ is stable.

ii) $h(0) \leq \frac{-r}{p}$.

b) Assume that for $h \in W^{1,\infty}(0,1)$ real valued the function

$$[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x) - h(0)}{x^r}$$

belongs to $L^\infty(0,1)$*. Then the* C_0*-semigroup* $\mathcal{S}_{r,h}$ *is not stable on* $W^{1,p}(0,1)$*. For the restriction of* $\mathcal{S}_{r,h}$ *to* $W^{1,p}_*(0,1)$ *the following are equivalent.*

i) $\mathcal{S}_{r,h}$ *is stable on* $W^{1,p}_*(0,1)$*.*

ii) $h(0) \leq 0$*.*

Proof. Recall that $F(x) = -x^r$ thus, $\varphi(t,x) = ((r-1)t + x^{1-r})^{\frac{1}{1-r}}$ and

$$\forall\, t \geq 0, x \in (0,1]: \qquad \partial_2\varphi(t,x) = x^{-r}((r-1)t + x^{1-r})^{\frac{r}{1-r}}$$

The proofs are similar to those of the previous example. In fact, observe that for all $x \in I$ choosing $\bar{t} > \frac{x^{1-r}-1}{r-1}$ we have $x \notin \varphi(\bar{t},(0,1)) = (0,x)$. By using Theorem 2.3.6-(2) and Theorem 2.4.1-(2) we finish the proofs of a) and b) respectively. In fact, for the proof of part a) since $[0,1] \to \mathbb{R}, \qquad x \mapsto \frac{h(x)-x^{r-1}h(0)}{x^r}$ belongs to $L^1(0,1)$, if we denote by K its norm, we obtain for $y \in (0,1)$ and $t \geq 0$

$$\begin{aligned}
\int_y^{\varphi(t,y)} \frac{h(s) - \frac{1}{p}F'(s)}{F(s)} ds &= \int_{\varphi(t,y)}^y \frac{h(s) + \frac{r}{p}s^{r-1}}{s^r} ds \\
&= \int_{\varphi(t,y)}^y \frac{h(s) + s^{r-1}h(0)}{s^r} ds + \int_{\varphi(t,y)}^y \frac{h(0) + \frac{r}{p}}{s} ds \\
&\leq K + \int_{\varphi(t,y)}^y \frac{h(0) + \frac{r}{p}}{s} ds \\
&= K + \left(h(0) + \frac{r}{p}\right) \log\left(\frac{y}{\varphi(t,y)}\right) \\
&= K + \frac{1}{r-1}\left(h(0) + \frac{r}{p}\right) \log\left(1 + (r-1)ty^{r-1}\right)
\end{aligned}$$

Then $\mathcal{T}_{r,h}$ is stable on $L^p(0,1)$if and only if $\left(h(0) + \frac{r}{p}\right) \leq 0$, by Theorem 2.3.6-(2).

It only remains to prove part b) using Theorem 2.4.1-(2).

$$\begin{aligned}\int_y^{\varphi(t,y)} \frac{h(0)-\left(\frac{1}{p}-1\right)F'(s)}{F(s)}ds =& \int_{\varphi(t,y)}^{y} \frac{h(0)+\left(\frac{1}{p}-1\right)rs^{r-1}}{s^r}ds \\ =& \int_{\varphi(t,y)}^{y} \frac{h(0)}{s^r}ds + \int_{\varphi(t,y)}^{y} \frac{\left(\frac{1}{p}-1\right)r}{s}ds \\ =& -\frac{h(0)}{r-1}\left(y^{1-r}-\varphi(t,y)^{1-r}\right)+ \\ &+\left(\frac{1}{p}-1\right)r\log\left(\frac{y}{\varphi(t,y)}\right).\end{aligned}$$

Since $p \geq 1$, $r > 1$, and $\frac{y}{\varphi(t,y)} = (1+(r-1)ty^{r-1})^{\frac{1}{r-1}}$ we have for some $C > 0$

$$\left(\frac{1}{p}-1\right)r\log\left(\frac{y}{\varphi(t,y)}\right) \leq 0 < C,$$

then

$$\begin{aligned}\int_y^{\varphi(t,y)} \frac{h(0)-\left(\frac{1}{p}-1\right)F'(s)}{F(s)}ds \leq& -\frac{h(0)}{r-1}\left(y^{1-r}-\varphi(t,y)^{1-r}\right)+C \\ =& h(0)t + C.\end{aligned}$$

Finally, $S_{r,h}$ is stable on $W^{1,p}_*(0,1)$ if and only if $h(0) \leq 0$, by Theorem 2.4.1-(2). Of course, $S_{r,h}$ is stable in $W^{1,p}$ if and only if $h(0) < 0$ by the same Theorem.

□

As the previous chapter this equation is of interest to compare our results by the previous one given by Dawidowicz and Poskrobko in [36]. As expected, we obtain a complementary behaviour between chaos and stability as is the case of the the equation for $r = 1$. In contrast, there are no difference like chaotic and hypercyclic properties.

Open Problem 2.5.4 We mentioned the duality about stability and chaotic behaviour of the semigroups $\mathcal{T}_{F,h}$ in this Chapter, for example Lasota equation on (0,1). Are there other semiflow semigroups for which this duality hold?

Chapter 3

Chaotic behaviour of birth-and-death models

In the last chapter of the thesis we will focus on the models of kinetic theory as is the case of the birth-and-death models. Azmy and Protopopescu studied these processes for the first time [58]. Later, the subject was intensively studied by Banasiak, Lachowicz and Moszyński.

During recent years, the birth-and-death models with proliferation have been of interest for their applications in biomedicine, for instance the behaviour of evolution of cancer cells and the drug-resistance of these cells [7, 9, 11].

We will show some new results for the joined birth-and-death model. More precisely, we will study this model for the case of non-constant coefficients, and we give conditions under which there is chaotic behaviour on an infinite-dimensional subspace. These results generalize in part previous ones obtained by Banasiak and Moszyński in [11].

As in previous chapters, we will show an evolution of the main known results related with these models in order to show the improvement achieved. Obviously these models are not governed by the same kind of semigroups used in the previous chapters.

The contents of this chapter were published in [5].

3.1 State of the art

The main purpose of this section is to present some results related to chaos for the evolution of birth-and-death type models with proliferation discussed in [7, 11, 44, 58].

Following the description in [11], we consider a population of cells divided into subpopulations characterized by a different number of copies of a certain gene in a cell. The integer n can be viewed as the number of copies of a drug-resistant gene in the nth subpopulation of cells. The bird-and-death model is characterized by two components described by the coefficients b_n and d_n, the first one describes the number of shifts by mutations from the nth subpopulation to the $n+1$th one, in unit time, per one cell of the nth subpopulation, the second one describes the same fact but from the nth subpopulation to the $n-1$th one. There is another coefficient in this model that we denote by a_n and it depends on the previous two ones and on the cell proliferation process (rate of cell division). We suppose that the coefficients of this model are time independent. The problem for time-dependent coefficients is currently an open problem but there are results in this sense, see for example [6].

In this section, we denote $\mathbf{f} = (f_n)_{n\geq 0} \in \ell^p$, $1 \leq p < \infty$, with its norm $\|\mathbf{f}\|_p = \left(\sum_{n=0}^{\infty} |f_n|^p\right)^{1/p}$.

$f_n(t)$ represents the number of cells in the nth subpopulation at the time $t \geq 0$.

Before beginning we must emphasize that the birth-and-death model can be separated in death and birth part of the process. If $b_n = 0$ we have the death part and, as expected, the birth part is obtained if $d_n = 0$. We are interested in the whole process with death and birth parts, but we will give a description of death part. The birth part is not of our interest since no chaotic properties are possible in this case, see for example [7].

3.1.1 Death model with variable coefficients

In [58], the authors studied the death part of the birth-and-death process. Following the same line, but allowing variable coefficients, Banasiak and Lachowicz [7] considered the following system of equations:

$$\frac{df_n}{dt} = (\mathcal{L}f)_n = -\alpha_n f_n + \beta_n f_{n+1}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where the coefficients α_n and β_n satisfy $0 < \alpha_n < \beta_n$ for any $n \in \mathbb{N}_0$ and there are parameters $(a'_n)_{n\in\mathbb{N}_0}$ such that

$$\begin{aligned} &(A1) \quad \alpha_n = \alpha + a'_n, \text{ for some } \alpha \geq 0 \text{ and with } \lim_{n\to\infty} a'_n = 0; \\ &(A2) \quad \beta_n = \beta b_n, \text{ for some } \beta > \alpha \text{ and with } \lim_{n\to\infty} b_n = 1. \end{aligned} \tag{3.2}$$

The main result in [7] about chaotic behaviour of this model is the following theorem.

Theorem 3.1.1 *Suppose that the sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ satisfy the assumptions* (3.2)*, and exists $q < 1$ such that $|a'_k/\beta| \leq q^{k+1}$ for $k \in \mathbb{N}_0$. Then the C_0-semigroup generated by the operator associated with $\mathcal{L}$ in* (3.1) *is chaotic on any ℓ^p, $1 \leq p < \infty$, and on c_0.*

In [44, Chapter 7], the above result was improved:

Proposition 3.1.2 *Let $\alpha_n, \beta_n \in]0, +\infty[$, $n \in \mathbb{N}$, be bounded sequences such that*

$$\alpha := \sup_k \alpha_k < \beta := \liminf_k \beta_k. \tag{3.3}$$

Then the solution C_0-semigroup generated by the operator associated with $\mathcal{L}$ in (3.1) *is chaotic and topologically mixing on ℓ^p.*

3.1.2 Birth-and-death model with constant coefficients

A first approach of the whole process was studied by Banasiak and Moszyński in [11], in this case the model was presented with constant coefficients as follows:

$$\begin{aligned} \frac{df_1}{dt} &= (\mathcal{L}f)_1 = && af_1 + df_2, \\ \frac{df_n}{dt} &= (\mathcal{L}f)_n = && bf_{n-1} + af_n + df_{n+1}, \qquad n \geq 2. \end{aligned} \tag{3.4}$$

The main result concerning chaos in [11] is the following theorem.

Theorem 3.1.3 *We suppose $a, b, d \in \mathbb{R}$. If $0 < |b| < |d|$ and $|a| < |b + d|$, then the C_0-semigroup generated by the operator associated with $\mathcal{L}$ in* (3.4) *is chaotic on ℓ^p.*

Remark 3.1.4 As shown by Banasiak et al. [11], with a slight change in the coefficients we lose ownership of chaos. In fact, the model is not chaotic if $|d| < |b|$ or if $p = 1$ and $|d| \leq |b|$.

Stability properties of this model were demonstrated for a dense subset of ℓ^1 provided that $a, b, d \in \mathbb{R}$ with $b, d \neq 0$ and one of the following two conditions:

(1) $a < -(|b| + |d|)$, and

(2) $-(|b| + |d|) \leq a < -2\sqrt{|b||d|}$ and $a < -2|b|$.

In particular the assertion holds if

$$a < -2\sqrt{|b||d|} \text{ and } 0 < |b| \leq |d|.$$

3.1.3 Birth-and-death model with variable coefficients

Let us consider the previous problem with variable coefficients.

$$\begin{aligned} \frac{df_1}{dt} &= a_1 f_1 + d_1 f_2, \\ \frac{df_n}{dt} &= b_n f_{n-1} + a_n f_n + d_n f_{n+1}, \quad n \geq 2, \end{aligned} \tag{3.5}$$

with $a_n,\ b_n,\ d_n \in \mathbb{R}$ and the infinite matrix

$$\mathcal{L} = \begin{pmatrix} a_1 & d_1 & & & \\ b_2 & a_2 & d_2 & & \\ & b_3 & a_3 & d_3 & \\ & & b_4 & a_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{3.6}$$

In [8], Banasiak, Lachowicz and Moszyński gave conditions under which the maximal operator L_{max} on ℓ^p associated with $\mathcal{L}$ generates a C_0-semigroup, and in [9] they obtained conditions for sub-chaos by proving the following theorem.

Theorem 3.1.5 *Suppose that there exists $N_0 \geq 2$ such that*

$$\begin{aligned} &a_n = a(n-1) + \alpha,\ d_n = d(n-1) + \delta,\ b_n = b(n-1) + \beta, \text{ for } n \geq N_0, \\ &\text{with } a = -(b+d),\ b, d \geq 0,\ \alpha, \beta, \delta \in \mathbb{R} \end{aligned} \tag{3.7}$$

holds with $d > b$ and $\alpha + \beta + \delta - \frac{d-b}{p} > 0$. Then the C_0-semigroup generated by the maximal operator L_{max} on ℓ^p associated with $\mathcal{L}$ in (3.6) *is sub-chaotic.*

Remark 3.1.6 Consider the coefficients with the property (3.7). If either of the following two cases hold, $b > d$ or $d_{m_0} = 0$ for some $m_0 \geq 1$, then the C_0-semigroup generated by the maximal operator L_{max} on ℓ^p is not chaotic. We refer to [8, Theorem 3.6] for its proof.

3.2 Birth-and-death models with variable coefficients: general approach

In this section we intend to obtain sub-chaos results for birth-and-death type models with proliferation in a wide range of variable coefficients. We consider the model given in (3.5), and we assume that $d_n \neq 0$ for each $n \in \mathbb{N}$.

The first aspect to consider is the Banach space on which the operator associated with $\mathcal{L}$ according to (3.6) generates a C_0-semigroup. Given $1 \leq p < \infty$ and $\gamma > 0$ we consider the space $X(\gamma)$ defined as follows:

$$X(\gamma) := \{f \in \ell^p \ : \ \mathcal{L}^n f \in \ell^p, \ \forall n \in \mathbb{N}, \ \text{and} \ \|f\| := \sum_{n=0}^{\infty} \|\mathcal{L}^n f\|_p \gamma^{-n} < \infty\},$$

The space $X(\gamma)$ might seem a bit artificial but it can be continuously embedded in ℓ^p. At this point, it is important to clarify that, if the sequences $(a_n)_n$, $(b_n)_n$ and $(d_n)_n$ are bounded, $\mathcal{L}$ has an associated bounded operator S_p on ℓ^p, with spectral radius $r(S_p) < \infty$, and $X(\gamma) = \ell^p$ for $\gamma > r(S_p)$. If any of the sequences $(a_n)_n$, $(b_n)_n$ or $(d_n)_n$ are unbounded, we have that the operator $S_{X(\gamma)}$ associated with $\mathcal{L}$ is a bounded operator on $X(\gamma)$ and, therefore, it generates a C_0-semigroup $\mathcal{T}_{X(\gamma)}$ on $X(\gamma)$. As we will see later, a suitable selection of γ will allow us to obtain that the formal eigenvectors of $\mathcal{L}$, associated to their corresponding eigenvalues in a certain open subset of $\mathbb{C}$, belong to $X(\gamma)$.

Lemma 3.2.1 *Let $1 \leq p < \infty$ and $\gamma > 0$. Then $X(\gamma)$ is a Banach space.*

Proof. Indeed, if $(f(k))_k$ is a Cauchy sequence in $X(\gamma)$, given an arbitrary $\varepsilon > 0$ we find $k_0 \in \mathbb{N}$ with $\|f(k) - f(l)\| < \varepsilon/2$ for all $k, l \geq k_0$.

Let $f \in \ell^p$ be such that $f = \lim_n f(n)$. Since $(\|f(k)\|)_k$ is a bounded sequence, we get by the definition of $\|\cdot\|$ that $\mathcal{L}^n f \in \ell^p$ for all $n \in \mathbb{N}$ and that $\sum_{n=0}^{\infty} \|\mathcal{L}^n f\|_p \gamma^{-n} < \infty$. That is, $f \in X(\gamma)$.

Moreover, by passing to the limit, $\|f_k - f\| \leq \varepsilon/2 < \varepsilon$ for all $k \geq k_0$. $\square$

To prove our main result for this model, we need several technical results. A holomorphic selection of eigenvectors of $S_{X(\gamma)}$ will be found. Suppose, in this section, that $\lambda \in \mathbb{C}$ denotes an eigenvalue of $S_{X(\gamma)}$ and $\mathbf{f} = (f_n)_{n\geq 1} \in X(\gamma)$ is an associated eigenvector. Then it satisfies

$$f_2 = \frac{\lambda - a_1}{d_1} f_1,\ b_n f_{n-1} + (a_n - \lambda) f_n + d_n f_{n+1} = 0, \quad n \geq 2. \tag{3.8}$$

We fix $f_1 \neq 0$, $f_2 = ((\lambda - a_1)/d_1) f_1$, and

$$f_{n+1} = \frac{\lambda - a_n}{d_n} f_n - \frac{b_n}{d_n} f_{n-1}, \quad n \geq 2. \tag{3.9}$$

We rewrite the above equation as

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = A_n \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}, \tag{3.10}$$

where

$$A_n = \begin{pmatrix} 0 & 1 \\ -\frac{b_n}{d_n} & \frac{\lambda - a_n}{d_n} \end{pmatrix}.$$

Observe that equation (3.10) is equivalent to

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = [A_n \dots A_2] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{3.11}$$

We want to find conditions under which $\sum_{n\geq 2} \|A_n \dots A_2\| < \infty$. To do this, we consider the characteristic polynomial $p_{n,\lambda}(z) = d_n z^2 + (a_n - \lambda) z + b_n$ of A_n. Let $z_n^+ : \mathbb{C} \to \mathbb{C}$ and $z_n^- : \mathbb{C} \to \mathbb{C}$ be two (non-necessarily continuous) maps such that $p_{n,\lambda}(z) = (z - z_n^+(\lambda)) \cdot (z - z_n^-(\lambda))$ for each $\lambda \in \mathbb{C}$. That is, $z_n^+(\lambda)$ and $z_n^-(\lambda)$ are the (non-necessarily different) roots of $p_{n,\lambda}(z) = 0$. If the discriminant is non-zero, i.e. if $(a_n - \lambda)^2 \neq 4 d_n b_n$, then the roots are simple and we can decompose A_n as $A_n = P_n D_n P_n^{-1}$ where,

$$P_n = \begin{pmatrix} 1 & 1 \\ z_n^+(\lambda) & z_n^-(\lambda) \end{pmatrix} \text{ and } D_n = \begin{pmatrix} z_n^+(\lambda) & 0 \\ 0 & z_n^-(\lambda) \end{pmatrix}. \tag{3.12}$$

A basic reasonable assumption is that $\lim_{n\to\infty} z_n^{\pm}(\lambda) = z^{\pm}(\lambda)$ with $z^+(\lambda) \neq z^-(\lambda)$. That is, $\lim_{n\to\infty} P_n = P$ and $\lim_{n\to\infty} P_n^{-1} = P^{-1}$ with

$$P = \begin{pmatrix} 1 & 1 \\ z^+(\lambda) & z^-(\lambda) \end{pmatrix} \text{ and } P^{-1} = \frac{1}{z^-(\lambda) - z^+(\lambda)} \begin{pmatrix} z^-(\lambda) & -1 \\ -z^+(\lambda) & 1 \end{pmatrix}. \quad (3.13)$$

We are interested in the case when $0 < |z^{\pm}(\lambda)| < 1$.

Lemma 3.2.2 *Assume* $\lim_{n\to\infty} z_n^{\pm}(\lambda) = z^{\pm}(\lambda)$ *with* $|z^{\pm}(\lambda)| < 1$ *and* $z^-(\lambda) \neq z^+(\lambda)$. *Then*

$$\sum_{n\geq 2} \|A_n \cdots A_2\| < +\infty.$$

Consequently, if moreover $\mathcal{L}\boldsymbol{f} = \lambda \boldsymbol{f}$ *and* $|\lambda| < \gamma$, *then* $\boldsymbol{f} \in X(\gamma)$.

Proof. We fix $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $0 < |z_n^{\pm}(\lambda)| < \delta < 1$ and $|z_n^+(\lambda) - z_n^-(\lambda)| > 1 - \delta$ for all $n \geq n_0$. By assumption, $(P_n)_n$ and $(P_n^{-1})_n$ converge to P and P^{-1}, respectively. This implies that there exists $M > 0$ such that $\|P_m\| < M$ and $\|P_m^{-1}\| < M$, for all $m \geq n_0$. Let $R > \delta^2/(1-\delta)$ and $n_1 \geq n_0$ such that $\|P_{m+1}^{-1} P_m - I\| < \delta/R$ for every $m \geq n_1$. Thus $\|P_{m+1}^{-1} P_m\| < 1 + (\delta/R) < (1/\delta)$ for $m \geq n_1$. Finally, if we denote by $N = \|A_{n_1 - 1} \cdots A_2\|$ and $\delta_1 := (\delta + \delta^2/R) < 1$, then we get

$$\begin{aligned} \|A_n \cdots A_2\| &\leq N \|P_n D_n (P_n^{-1} P_{n-1}) D_{n-1} \cdots (P_{n_1+1}^{-1} P_{n_1}) D_{n_1} P_{n_1}^{-1}\| \\ &\leq N \|P_n\| \|P_{n_1}^{-1}\| \left(\prod_{m=n_1}^{n} \|D_m\| \right) \left(\prod_{m=n_1}^{n-1} \|P_{m+1}^{-1} P_m\| \right) \\ &< M^2 N \delta^{n-n_1} \left(1 + \frac{\delta}{R}\right)^{n-n_1} \\ &= M^2 N \delta_1^{n-n_1}, \end{aligned}$$

for any $n > n_1$, which yields the result. Finally, if $f \in \ell^p$ satisfies $\mathcal{L}f = \lambda f$ with $|\lambda| < \gamma$, then $\mathcal{L}^n f = \lambda^n f \in \ell^p$ for all $n \in \mathbb{N}$, and $\|f\| < \infty$. That is, $f \in X(\gamma)$. □

We are in condition to prove our main results. First, we present a version for bounded coefficients.

Theorem 3.2.3 *Let* (a_n), $(b_n)_n$ *and* $(d_n)_n$ *be sequences of real numbers such that* $d_n \neq 0$ *for all* $n \in \mathbb{N}$, $1 \leq p < \infty$, *and* $\gamma > 0$. *Assume that,* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} d_n = d \neq 0$, *with* $|b| < |d|$ *and* $|a| < |b + d|$.

Then the C_0-semigroup $\mathcal{T}_{X(\gamma)}$ is sub-chaotic on $X(\gamma)$. Moreover, S_p generates a sub-chaotic C_0-semigroup $\mathcal{T}_p$ on ℓ^p.

Proof. We fix a holomorphic branch ξ of the square root defined on the open set $\mathcal{V} := \mathbb{C} \setminus i[0, +\infty[$. Provided that $(a-\lambda)^2 - 4db \in \mathcal{V}$ and $(a_n - \lambda)^2 - 4d_n b_n \in \mathcal{V}$, we set $z^+(\lambda) = (-a+\lambda+\xi((a-\lambda)^2-4db))/2d$, $z^-(\lambda) = (-a+\lambda-\xi((a-\lambda)^2-4db))/2d$, $z_n^+(\lambda) = (-a_n + \lambda + \xi((a_n - \lambda)^2 - 4d_n b_n))/2d_n$, and $z_n^-(\lambda) = (-a_n + \lambda - \xi((a_n - \lambda)^2 - 4d_n b_n))/2d_n$. If $a^2 - 4db \neq 0$, there exist a neighbourhood $\mathcal{W}$ of 0 and $n_0 \in \mathbb{N}$ such that $z^+(\lambda)$, $z^-(\lambda)$, $z_n^+(\lambda)$ and $z_n^-(\lambda)$ are well defined for any $n \geq n_0$ and for all $\lambda \in \mathcal{W}$. In particular, $z^+(\lambda)$ and $z^-(\lambda)$ are different roots of $dz^2 + (a-\lambda)z + b = 0$. An application of Lemma 0.5.3 for $w = (a - \lambda)/d$ and $r = b/d$ yields that, if $|b| < |d|$ and $|a| < |b + d|$, then $|z^{\pm}(\lambda)| < 1$ for any λ in a sufficiently small open ball $\mathcal{U} \subset B(0, \gamma)$.

If $a^2 = 4db$, then we can pick $n_0 \in \mathbb{N}$ and $\mathcal{U} = B(\varepsilon i, \delta)$ for sufficiently small $\varepsilon, \delta > 0$ such that $\mathcal{U} \subset B(0, \gamma)$, $(a - \lambda)^2 - 4db \in \mathcal{V}$, $(a_n - \lambda)^2 - 4d_n b_n \in \mathcal{V}$, and $|z^{\pm}(\lambda)| < 1$ for all $\lambda \in \mathcal{U}$ and for each $n \geq n_0$.

We clearly have $\lim_n z_n^+(\lambda) = z^+(\lambda)$ and $\lim_n z_n^-(\lambda) = z^-(\lambda)$ for all $\lambda \in \mathcal{U}$,

We set $f(\lambda)$ as the eigenvector of $\mathcal{L}$ to the eigenvalue λ constructed accordingly to (3.9), where $f_1 \neq 0$ is fixed (and does not depend on λ). By Lemma 3.2.2, $f(\lambda) \in X(\gamma)$ for every $\lambda \in \mathcal{U}$. The recurrent construction of the coordinates of $f(\lambda)$ shows that each of them is a polynomial on λ. Moreover, if $\phi \in (\ell^p)^* = \ell^q$, the following estimates are satisfied:

$$|\langle f(\lambda), \phi\rangle - \langle (f_1(\lambda), \dots, f_k(\lambda), 0, \dots), \phi\rangle| \leq \sum_{n>k} \|A_n \dots A_2\| \, \|(f_1, f_2(\lambda))\|_p \, \|\phi\|_q \,,$$

and $f : \mathcal{U} \to \ell^p$ is weakly holomorphic, therefore, holomorphic. That is, there exists

$$f'(\lambda) := \lim_{z \to 0} \frac{f(\lambda + z) - f(\lambda)}{z} \in \ell^p, \quad \forall \lambda \in \mathcal{U}.$$

We have seen some definitions and facts about this topic on subsection 0.1.3, we refer to [40] for more details. Also, it is easy to see that $\mathcal{L}^n f'(\lambda) = n\lambda^{n-1} f(\lambda) + \lambda^n f'(\lambda)$ for any $n \in \mathbb{N}$. In particular, $f'(\lambda) \in X(\gamma)$ since $\|\mathcal{L}^n f'(\lambda)\|_p \gamma^{-n} \leq \|f(\lambda)\|_p n|\lambda|^{n-1}\gamma^{-n} + \|f'(\lambda)\|_p \lambda|^n \gamma^{-n}$ for any $n \in \mathbb{N}$ and for all $\lambda \in \mathcal{U}$. Moreover, $f : \mathcal{U} \to X(\gamma)$ is holomorphic since

$$\begin{aligned}
&\left\| f'(\lambda) - \frac{f(\lambda+z)-f(\lambda)}{z} \right\| \\
&= \sum_{n=0}^{\infty} \left\| n\lambda^{n-1} f(\lambda) + \lambda^n f'(\lambda) - \frac{(\lambda+z)^n f(\lambda+z) - \lambda^n f(\lambda)}{z} \right\|_p \gamma^{-n} \\
&= \sum_{n=0}^{\infty} \left\| n\lambda^{n-1} f(\lambda) + \lambda^n f'(\lambda) - (\lambda+z)^n \frac{f(\lambda+z)-f(\lambda)}{z} - \frac{(\lambda+z)^n - \lambda^n}{z} f(\lambda) \right\|_p \gamma^{-n} \\
&\le \sum_{n=0}^{\infty} \left| n\lambda^{n-1} - \frac{(\lambda+z)^n - \lambda^n}{z} \right| \|f(\lambda)\|_p \gamma^{-n} \\
&\quad + \sum_{n=0}^{\infty} \left\| \lambda^n f'(\lambda) - (\lambda+z)^n \frac{f(\lambda+z)-f(\lambda)}{z} \right\|_p \gamma^{-n} \overset{z\to 0}{\longrightarrow} 0.
\end{aligned}$$

Now the assertion follows from Proposition 0.3.9. □

Based upon the above result, we are now able to prove a version of the previous theorem for unbounded coefficients.

Theorem 3.2.4 *Let* (a_n), $(b_n)_n$ *and* $(d_n)_n$ *be sequences of real numbers such that* $d_n \neq 0$ *for all* $n \in \mathbb{N}$, $1 \le p < \infty$, *and* $\gamma > 0$. *Assume that* $\lim_{n\to\infty}(a_n/d_n) = \alpha$, $\lim_{n\to\infty}(b_n/d_n) = \beta$, $\lim_{n\to\infty} d_n = \infty$, *with* $\alpha^2 \neq 4\beta$, $|\beta| < 1$ *and* $|\alpha| < |1+\beta|$.

Then the C_0*-semigroup* $\mathcal{T}_{X(\gamma)}$ *is sub-chaotic on* $X(\gamma)$.

Proof. Here, we set $z^+(\lambda) = (-\alpha + \xi(\alpha^2 - 4\beta))/2$, $z^-(\lambda) = (-\alpha - \xi(\alpha^2 - 4\beta))/2$, which are well defined since $\alpha^2 - 4\beta \in \mathcal{V}$, and $z_n^+(\lambda)$, $z_n^-(\lambda)$ as in Theorem 3.2.3. Since $z^{\pm}(\lambda)$ do not depend on λ, if we select $\mathcal{U} := B(0,\gamma)$ and apply again Lemma 0.5.3 with $w = \alpha$ and $r = \beta$, then $|z^{\pm}(\lambda)| < 1$ since $|\beta| < 1$ and $|\alpha| < |1+\beta|$. We follow the reasoning of the Theorem 3.2.3 to conclude the result. □

Open Problem 3.2.5 The Banach space considered in this chapter was constructed on purpose to ensure the generation of a norm continuous semigroup and to obtain sub-chaos. It would be interesting to study whether there exists chaos on the whole space for these models, that is, when the semigroup is not uniformly continuous.

Is it possible to give an explicit expression of the C_0-semigroup for the birth-and-death model with proliferation on the whole space?

Whatever the answer is, it could help to give conditions under which the C_0-semigroup is chaotic instead of sub-chaotic.

Finally, in case of success, we can improve these kind of models by assuming time dependent coefficients similar to the previous work of Banasiak in [6].

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